Stiffness of Thin Pressurized Shells of Revolution

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A quasi-linear approach is developed. The strains and rotations are assumed to be small, but the equilibrium conditions are fulfilled on the deformed shell. In the resulting differential equation, bending stiffness and pressure stiffness appear side by side. The asymptotic form of this equation is discussed for the case of the circular torus. Considered separately, the two stiffness effects are fairly similar. Suitable superposition of the two limit solutions yields a close approximation to the composite solution. As a sample application, the transverse stiffness of torus shells is derived and is presented as a function of internal pressure p and shell thickness h. It is found in particular that membrane theory remains correct for h small but finite. Linear shell theory, on the other hand, involves an error that is proportional to p if p is small.

I. Introduction

THIS paper is concerned with the rotationally symmetric deformations that pressurized, thin elastic shells of revolution experience under axisymmetric external loads. The approach to this problem presented here is a quasi-linear approach: the shell strains and rotations are assumed to be small, but, contrary to the approach of linear shell theory, the shell equilibrium conditions will be fulfilled on the deformed shell. Thereby, the stiffening that a shell experiences when it is pressurized is taken into account. This "pressure stiffness" is neglected in linear shell theory.

This deficiency of linear theory has two aspects. Both are illustrated by the model, one half of which is shown in Fig. 1, a pressurized, complete circular torus shell S subject to an axisymmetric system P of external loads. Consider the equilibrium of the vertical forces on either one of the two toroidal shell segments that connect the two crowns of S. If we assume that S is a membrane shell, then the only vertical forces are P and the vertical components of p. These are not in equilibrium on the undeformed shell; according to linear theory, a membrane torus S cannot carry even the smallest load P. But this is contrary to experience. Clearly S will react to P by deforming, as indicated in Fig. 1, such that the upper crown moves outward, the lower crown inward, until equilibrium is achieved. We refer to this as the "load problem."

The "crown problem" arises already if the load P is zero. In this case, linear membrane theory predicts a system of pressure stresses $\sigma_{p\ \mathrm{linear}}$ which, on the undeformed shell, are in equilibrium with the internal pressure p. However, if one calculates the shell deformations that $\sigma_{p\ \mathrm{linear}}$ produces, one obtains discontinuities at the crowns. The linear result is thus inconsistent.

In a previous paper,[1] the author has developed a quasi-linear membrane shell analysis. This analysis led to a consistent solution σ_p of the crown problem. The differences $\sigma_p - \sigma_{p\ \mathrm{linear}}$ turned out to be fairly small. On the other hand, a consistent solution σ_p was a prerequisite for solving the load problem and, specifically, the engineering problem of predicting the transverse stiffness of S with respect to the load P.

The present paper generalizes Ref. 1 in two respects: shell bending stiffness is added, and the analysis is not confined to the circular torus. As a sample application, the transverse stiffness of S is determined as a function of both internal pressure p and shell thickness h.

Insofar as the present analysis can be considered a generalization of either quasi-linear membrane theory ($h \to 0$) or linear shell theory ($p \to 0$), it can be described as being concerned with the transition between two limiting cases. Reissner[2] discussed a conceptually similar problem, the cylindrical shell acted upon by longitudinal edge loads, within the framework of his finite deflection theory. (The intriguing aspect of the present problem is that the effect of shell rotation has to be taken into account even if the deflections are very small.)

Following the concepts thus developed,[2] Reissner[3] shows (for the circular torus) that the pressure stiffness can be introduced by adding supplementary terms to linear shell theory and makes an estimate of the relative magnitude of the two stiffness effects. The present approach, though closely related to that of Reissner,[2] differs in that the concept of a reference shape[1] is utilized. This leads to an unusually uncomplicated formulation, i.e., the Meissner operator $L(\ )$ in the bending moment equation becomes independent of the shell shape, and this equation no longer contains the Poisson ratio ν.

References follow to other previous work where pressure stiffness was taken into account. The classical paper in this respect is by Bromberg and Stoker[4] and is an essentially nonlinear membrane theory. It does not deal with the case of a free crown. Starting from nonlinear membrane theory, Sanders and Liepins[5] rederived the quasi-linear membrane equation of Ref. 1 which they solved by means of an asymptotic approximation. The formal solution of a general form of this equation is given in Ref. 6.

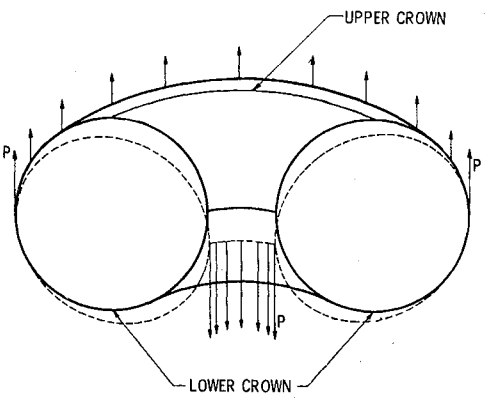

Fig. 1 Circular torus shell with transverse load P.

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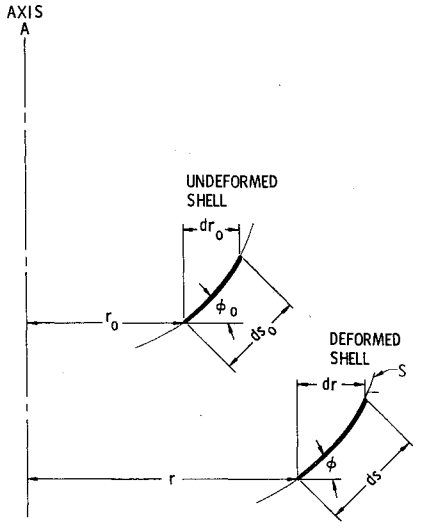

Fig. 2 Meridional view of shell element (ring).

II. Equilibrium Conditions

In the theory of thin shells of revolution, axisymmetrically loaded and axisymmetrically deformed, the shell element is an infinitesimally narrow axisymmetric (parallel) ring, the position of which is completely described by the meridional trace of its middle face. In Fig. 2, the traces of such a ring are shown before and after shell deformation. The suffix 0 refers to the undeformed shell; r is the distance from the axis A; ϕ is the angle between ring element and axisnormal; ds is the ring width; and dr is its projection. S is the trace of the deformed shell.

The stresses in the deformed shell are described by stress resultants, i.e., the resultants N_ϕ and N_θ of the direct stresses, the transverse shear force Q, and the bending moments M_ϕ and M_θ. Of these, N_ϕ and Q can be combined to form radial (horizontal) and axial (vertical) stress resultants H and V, respectively,

$$H = N_\phi \cos\phi - Q \sin\phi \tag{1a}$$

$$V = N_\phi \sin\phi + Q \cos\phi \tag{1b}$$

In writing the local equilibrium conditions, we assume that the only local load arises from an internal pressure p. The mechanically interesting feature of this load, and the main subject of the present paper, is the fact that its local resultant is not fixed but rotates as the shell element rotates. Additional (fixed) loads, if such are present, will lead to additional inhomogeneous terms that are familiar from linear shell theory; these terms are not included here in order to avoid complicating the issue.

The local equilibrium conditions thus are[2]

$$(rV)' = rp \tag{2a}$$

$$(rH)' \cos\phi = N_\theta - rp \sin\phi \tag{2b}$$

$$(rM_\phi)' \cos\phi = M_\theta \cos\phi + rQ \tag{2c}$$

where the prime indicates differentiation with respect to r.

We next write

$$N_\phi \equiv N_\phi{}^m + N_\phi{}^b \tag{3}$$

etc., with m denoting the "membrane" part of the shell stresses, that is, the stresses† that result from the equilibrium conditions if $Q = 0$, and b denoting the "bending" part of the shell stresses. From Eqs. (1) and (2a)

$$H = N_\phi{}^m \cos\phi + H^b \tag{4a}$$

$$rV = rN_\phi{}^m \sin\phi + (rQ)_c \tag{4b}$$

† Note that these stresses are not identical with the stresses $\sigma_p{}^m$ linear.

The index c is used here, and henceforth, to denote the value of a function at the crown point $\phi = 0$. Further

$$rN_\phi = rN_\phi{}^m + rH^b \cos\phi + (rQ)_c \sin\phi \tag{5}$$

Eliminating p from Eqs. (2a, and (2b) and inserting Eq. (4)

$$N_\theta = (rN_\phi{}^m)' + (rH^b)' \cos\phi \tag{6}$$

Also

$$rQ = (rQ)_c \cos\phi - rH^b \sin\phi \tag{7}$$

Equations (5–7) express the two direct stresses N_ϕ and N_θ by two functions, $N_\phi{}^m$ and H^b, and the constant $(rQ)_c$, which represents the transverse shear force at the crown. H^b and $(rQ)_c$ jointly represent the bending moments according to Eqs. (2c) and (7). Abbreviating

$$\tilde{M} \equiv (rM_\phi)' - M_\theta \tag{8}$$

we have

$$(rQ)_c = \tilde{M}_c \tag{9}$$

$$rH^b \sin\phi = (\tilde{M}_c - \tilde{M}) \cos\phi \tag{10}$$

For $N_\phi{}^m$ we have from Eq. (2a)

$$rN_\phi{}^m \sin\phi = \int_{r_c}^{r} rp dr \tag{11}$$

If p is constant, thus

$$N_\phi{}^m = p \frac{r + r_c}{2r} \times \frac{r - r_c}{\sin\phi} \tag{12}$$

where the last factor describes the deformed shape S. Note that

$$\frac{r - r_c}{\sin\phi} = R = \text{const} \tag{12a}$$

if and insofar as S is a segment of a circle of radius R which includes the crown.

The integration in Eq. (11) can also be performed if p is not a constant but is a given function of r. The result can again be written in the form Eq. (12), with p replaced by a suitable reference pressure, and with a modified last factor that will then represent an equivalent (modified) meridional shell shape.

III. Reference Shape

Writing (Fig. 2)

$$\begin{aligned} dr &= (1 + \epsilon_r) dr_0 \\ ds &= (1 + \epsilon_\phi) ds_0 \\ r &= (1 + \epsilon_\theta) r_0 \\ \beta &= \phi - \phi_0 \end{aligned} \tag{13}$$

assuming

$$|\epsilon_\phi| \quad |\epsilon_\theta| \quad |\beta| \ll 1 \tag{13a}$$

and inserting into the compatibility condition

$$(1 + \epsilon_\phi) \cos\phi = (1 + \epsilon_r) \cos\phi_0 \tag{14}$$

results in

$$[\epsilon_\phi - u'] \cos\phi = \beta \sin\phi \tag{15}$$

where $u \equiv \epsilon_\theta r$, and where terms that are of small second order have been neglected. By the same rule, the unknown ϕ can be replaced by a known reference angle ϕ_r if $|\phi - \phi_r| \ll 1$ and $\phi_r \to \phi$ for $\cos\phi \to 0$.

Figure 3 illustrates the procedure in the case of a pressurized circular torus. There are three meridional shapes: the original circle S_0, the deformed shape S, and the reference

circle S_r, which is inscribed into S. To a given material point on the shell middle face belong three points in Fig. 3, B_0, B_r, and B, with the axis distances r_0, r, and r, and the tangent angles ϕ_0, ϕ_r, and ϕ, respectively. The relation between S_0 and S_r is determined by two strain terms u_1 and u_2, which are small by definition. In practice we can, and will, henceforth consider S_r as the given shell shape. (Consequently S_0 becomes an unknown.)

We define two new rotational quantities

$$s = \sin\phi - \sin\phi_r \qquad s_0 = \sin\phi_r - \sin\phi_0 \tag{16}$$

Equation (15) then transforms into our final compatibility equation

$$[\epsilon_\phi - u'] \cos^2\phi_r = (s + s_0) \sin\phi_r \tag{17}$$

where s describes the rotation that transforms S_r into S, whereas s_0 describes the relative positions of the points B_0 and B_r on S_0 and S_r, respectively, and is thus of the nature of a strain term rather than of a rotation term. One finds (Fig. 3)

$$s_0 R_r = u + \tfrac{1}{2}\{u_1 + u_2 - (u_2 - u_1)[(r - a_r)/R_r]\} \tag{18}$$

IV. Stress-Strain Relations

The stress-strain relations as derived on the basis of the Kirchhoff hypothesis are

$$\begin{aligned} C\epsilon_\phi &= N_\phi - \nu N_\theta \\ C\epsilon_\theta &= N_\theta - \nu N_\phi \end{aligned} \tag{19}$$

where $C \equiv Eh$ and

$$\begin{aligned} M_\phi &= D[\kappa_\phi + \nu\kappa_\theta - (\sin\phi/r)(\epsilon_\phi + \nu\epsilon_\theta)] \\ M_\theta &= D[\kappa_\theta + \nu\kappa_\phi - (1/R)(\epsilon_\theta + \nu\epsilon_\phi)] \end{aligned} \tag{20}$$

where $D \equiv Eh^3/[12(1 - \nu^2)]$ and

$$\kappa_\phi \equiv (1/R_0) - (1/R) \qquad \kappa_\theta \equiv (\sin\phi_0/r_0) - (\sin\phi/r)$$

According to Koiter,[7] the ϵ terms in Eq. (20) can usually be neglected, or replaced by terms of the same order of magnitude. Reissner[2] uses

$$\begin{aligned} M_\phi{}^R &= -D\left[\frac{d\beta}{dr_0} + \nu\frac{\beta}{r_0}\right]\cos\phi_0 = M_\phi{}^K - D\epsilon_\phi\left(\frac{1}{R} - \frac{\sin\phi}{r}\right) \\ M_\theta{}^R &= -D\left[\frac{\beta}{r_0} + \nu\frac{d\beta}{dr_0}\right]\cos\phi_0 = M_\theta{}^K + D\epsilon_\theta\left(\frac{1}{R} - \frac{\sin\phi}{r}\right) \end{aligned} \tag{21}$$

where K denotes the moments of Eq. (20). We are going to use instead

$$\begin{aligned} M_\phi &= -D\left[s' + \nu\frac{s}{r}\right] \approx M_\phi{}^R + \frac{D}{R}(\epsilon_\phi + \nu\epsilon_\theta) \\ M_\theta &= -D\left[\frac{s}{r} + \nu s'\right] \approx M_\theta{}^R + \frac{D}{R}(\epsilon_\theta + \nu\epsilon_\phi) \end{aligned} \tag{22}$$

At the right hand of Eq. (22), terms with $(R_r - R_0)$ and $(a_r - a_0)$ are omitted; these differences are not only small by definition, but are either constants (in the case of a circular torus) or vary only slowly with r. In either case their order of magnitude is not increased by differentiation.

It remains to check how well Koiter's assertion is justified in a problem that is so sensitive that one of the standard assumptions of linear theory becomes unacceptable. Indeed, our problem is to determine relatively small quasi-linear corrections to a relatively large linear stress system $\sigma_{p\ \mathrm{linear}}$, with correspondingly large strains $\epsilon_{\mathrm{linear}}$. However, the moments M_ϕ and M_θ enter the shell equation only in form of the combination $\tilde{M}$, Eq. (8), and the effect on $\tilde{M}$ of choosing

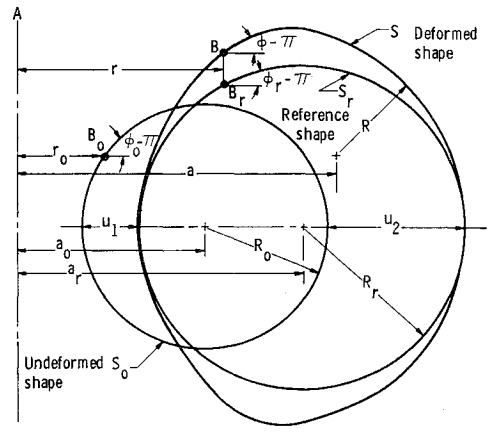

Fig. 3 Reference shape S_r for circular torus.

between Eqs. (20–22) is much smaller than the differences between the three definitions of the moments themselves.

The difference $\Delta\tilde{M}$ between using Eq. (21) or Eq. (22) is

$$\Delta\tilde{M} = (h^2/12R)[(rN_\phi)' - N_\theta]$$

if h^2/R is considered a constant. Then from Eqs. (5, 6, 9, and 10)

$$\Delta\tilde{M} = \tfrac{1}{12}(h/R)^2\tilde{M} \tag{23}$$

The difference between using Eq. (20) or Eq. (21) can be shown to be similarly small by calculating $\tilde{M}$ for the solution of quasi-linear membrane theory. We omit the details here. Briefly, differentiation increases the order of magnitude of the quasi-linear corrections but not of the linear result.

V. Bending Equation

Linear shell theory is usually written in terms of two simultaneous differential equations; these arise from inserting equilibrium relations and stress-strain relations into the bending moment equation and the compatibility equation, respectively. In discussing next the corresponding equations of the present analysis, we simplify our formulas, without losing the essence of the shell mechanics to be investigated, by treating the stiffness coefficients C and D as constants.

The bending moment equation becomes from Eqs. (2c, 8, and 22)

$$(rQ/\cos\phi) = \tilde{M} = -DL(s) \tag{24}$$

where the "Meissner operator" $L(\ \ )$ has the form

$$L(s) \equiv rs'' + s' - (s/r) \equiv r[(rs)'/r]' \tag{25}$$

Note that Eq. (25) is quite general insofar as the shell shape does not appear; nor does Poisson's ratio ν. This is a considerable simplification.

VI. Direct Stress Equation

Inserting the stress-strain relation Eq. (19) into the compatibility condition Eq. (17), and considering first in Eqs. (5) and (6) only the membrane part of the shell stresses, one obtains

$$-(1/C)L(rN_\phi{}^m)\cos^2\phi_r = [s^m + s_0{}^m]\sin\phi_r \tag{26}$$

again independent of ν. The operator $L(\ \ )$ is defined in Eq. (25). Equation (20) is, in essence, the differential equation of quasi-linear membrane theory. In it, $s_0{}^m$, by nature a strain term according to Eq. (18), can be expected to play

a minor role because of the factor $\sin\phi_r$. Whereas $s_0{}^m$ has been retained in Ref. (1), later recalculations of the same numerical samples showed that the effect of omitting $s_0{}^m$ was negligible. On this basis, we simplify our present formulation by omitting $s_0{}^m$ henceforth.

To Eq. (26) we have to add the contributions that arise from bending stiffness. In Eq. (5) we omit the constant $(rQ)_c$ [the argument here corresponding to the one used in justifying Eq. (22)] and, noting that in the case of Fig. 3 H^b does not contribute to the displacements u_1 and u_2 because of the factor $\cos\phi$ in Eqs. (5) and (6), omit also u_1 and u_2. Equation (18) then yields

$$s_0{}^b \sin\phi_r = (u^b/R_r) \sin\phi_r = -u^b(\cos\phi_r)' \cos\phi_r$$

so that, combining with u' on the left of Eq. (17)

$$u^{b\prime} \cos^2\phi_r + s_0{}^b \sin\phi_r = (u^b/\cos\phi_r)' \cos^3\phi_r$$

Thus, supplementing Eq. (26)

$$-(1/C)[L(rN_\phi{}^m) + L(rH^b) \cos\phi_r] \cos^2\phi_r = s \sin\phi_r \quad (27)$$

Membrane and bending contributions appear side by side in nearly identical form.

Only two unknown functions H^b and s appear in the bending equation, Eq. (7) with Eq. (24). Equation (27) still contains a third unknown function, $N_\phi{}^m$. We establish next a relation between $N_\phi{}^m$ and s by means of some further notation. Introduce the axis distance $\bar{a} \equiv a_{rc} \equiv r_{rc}$ of the crown of the reference shape S_r and the horizontal distance $x \equiv r - \bar{a}$ from this reference crown and define the crown shift x_0 by writing

$$r_c = \bar{a} + x_0$$

Describing the reference shape S_r by a shape function F_r

$$x/\sin\phi_r = R_{rc}F_r \quad (28)$$

such that $F_r = 1$ for $x = 0$, write in Eq. (12)

$$\frac{r - r_c}{\sin\phi} \equiv \frac{x - x_0}{\sin\phi} = R_{rc}F_r(1 - f) \quad (29)$$

We treat the two quantities, crown shift x_0 and function f, which describe the deviation of the deformed shape S from the reference shape S_r, as small quantities.[1] Thus in Eq. (27)

$$-(2/C)L(rN_\phi{}^m) = \epsilon_0 L[(2\bar{a} + x)F_r(f - 1)] \quad (30)$$

where ϵ_0 is a reference strain

$$\epsilon_0 \equiv pR_{rc}/Eh \quad (31)$$

In Eq. (30), the term with f is retained because the derivatives of f will be large compared with f. On the other hand, F_r is assumed to vary only slowly with x, and in consequence the term with x_0 has been dropped.

From Eqs. (28) and (29)

$$s \equiv \sin\phi - \sin\phi_r = [(fx - x_0)/R_{rc}F_r] \quad (32)$$

If S_r is a circle, $F_r \equiv 1$, and Eq. (30) becomes

$$-\frac{2}{C} L(rN_\phi{}^m) = \epsilon_0 \left\{\frac{\bar{a}}{\bar{a} + x} + L[(2\bar{a} + x)f]\right\} \quad (30a)$$

VII. Shell Equation

Combining Eqs. (21, 38, 41, and 47), we obtain the following two simultaneous second-order differential equations

$$\begin{aligned} rH^b \sin\phi_r &= D[L(s) \cos\phi_r - L(s)_{x=x_0}] \\ 2s \sin\phi_r &= \{\epsilon_0 L[(r + \bar{a})F_r(1 - f)] - \\ &\qquad (2/C)L(rH^b) \cos\phi_r\} \cos^2\phi_r \end{aligned} \quad (33)$$

This set of equations represents the composite problem of a pressurized shell of revolution with finite bending stiffness. In the case of zero pressure, $\epsilon_0 = 0$, it reduces to the equivalent of the equations of linear shell theory, with s and H^b as the unknown functions. The opposite limit case is the membrane shell, that is, the case where rH^b can be neglected in the second equation of Eqs. (33). In this case, the first equation is no longer required; the second equation, with Eq. (32) inserted, becomes a second-order equation‡ for the single unknown function f.

In the case of zero pressure, the transverse shear force at the crown

$$-DL(s)_{x=x_0} = (rQ)_c \approx (rQ)_{x=0}$$

is given by its equilibrium with the external load. However, $(rQ)_c$ is an unknown quantity if the pressure p is not zero. $(rQ)_c$ then depends upon the unknown position of the crown and (Fig. 1) the unknown magnitude of the stresses that go through the horizontal plane of symmetry.

The two Eqs. (33) can be combined with Eq. (32) to form a single fourth-order equation for the single unknown function f. All stresses and deformations are determined by this function f. We write here the fourth-order equation only for the special case that S_r is a circle (or part of a circle) and, in what follows, deal with this case only. In this case, $R_{rc} = R_r = \text{const}$; we make all coordinate lengths (r, $\bar{a}$, x, and x_0) nondimensional with R_r. For example, below x replaces $x/R_r = \sin\phi_r$. In addition, we introduce the following abbreviations:

$$\bar{h} \equiv h/R_r \qquad \bar{\nu} \equiv 12(1 - \nu^2) \quad (34)$$

Then [note Eq. (30a)]

$$fx^2 - \left\{\epsilon_0 m(f) - \frac{\bar{h}^2}{\bar{\nu}} b(xf)\right\}(1 - x^2) = \frac{\epsilon_0\bar{a}}{2(\bar{a} + x)}(1 - x^2) + xx_0 \quad (35)$$

Here $m(f)$ and $b(xf)$ are the contributions of pressure (membrane) stiffness and bending stiffness, respectively:

$$2m(f) \equiv L[(2\bar{a} + x)f] \quad (35a)$$

$$b(xf) \equiv L\{[L(xf) - L(xf)_{x=0}][(1 - x^2)^{1/2}/x]\}(1 - x^2)^{1/2} \quad (35b)$$

Equation (35) is a linear inhomogeneous differential equation of fourth order for the unknown function f, with x_0 as an unknown parameter.§

In order to determine x_0, one writes the solution of Eq. (35) in the form

$$f = \epsilon_0 f_0 + x_0 f_1 \quad (36)$$

The two functions f_0 and f_1 can then be determined separately, and x_0 follows from the boundary conditions. As an example, in the case of zero external load P, Fig. 1, this boundary condition is the condition of zero relative vertical deflection Δw between the two circles $x = \pm 1$ in the horizontal plane of symmetry

$$\Delta w = \int_{-1}^{+1} w'dx = \int_{-1}^{+1} \frac{sdx}{\cos^3\phi_r} = \int_{-1}^{+1} \frac{xf - x_0}{(1 - x^2)^{3/2}} dx \quad (37)$$

Here w is the vertical displacement between S and S_r, positive upwards on the lower half of the shell.

Let x_{00} be the crown shift that is obtained from setting $\Delta w = 0$ and set $x_0 = x_{00} + x_{01}$ on the upper half, $x_0 = x_{00} -$

‡ This equation is a generalized form of the quasi-linear membrane equation that was established in Ref. 1 for the special case of the circular torus. In Ref. 1, the present function f is written as $\varkappa f \equiv \epsilon_0 f$. (Also, the sign of x is reversed.) The present definition of f allows a direct transition ($\epsilon_0 \to 0$) to linear shell theory.

§ The relative simplicity of Eq. (35), and in particular the invariant form of $L(\ )$, Eq. (25), is due in part to the choice of r (respectively, x) as the independent variable. Tao[8] claims advantages from using a variable that he denotes by x and that he introduces by two consecutive transformations: $[\omega = (\pi/2) - \phi_r]$ and $x_{\text{Tao}} = \sin^2(\omega/2)$. It is readily seen that this is again in fact a horizontal distance $2\, x_{\text{Tao}}\, 1 - x$ and $2(\eta - x)_{\text{Tao}} = r$.

x_{01} on the lower half of the torus. This will represent the deformed shell of Fig. 1, that is, two torus half shells with $\Delta w \pm 0$. The two half shells will fit together vertically, but will have different forces N_ϕ at their respective boundary circles. These differences correspond to the external load system P. The two half shells will not fit together exactly in horizontal direction,[1] and, strictly speaking, these gaps will have to be closed by an additional system of horizontal forces and moments. However, these local effects are secondary in nature and will not be considered in the present analysis.

VIII. Limit Cases; Asymptotic Consideration

Owing to the factor x^2 to f in Eq. (35), we are dealing with an equation with a double transition point at $x = 0$. At this point, the differential term (enclosed within braces) has to equal the right-hand side, in spite of its small factors, ϵ_0 and $\bar{h}^2$. On the other hand, the differential term may be negligible if $|x|$ is sufficiently large; in this "outer range," the function f^0, which is given by the algebraic relation

$$f^0 x^2 = \frac{\epsilon_0 \bar{a}}{2(\bar{a} + x)} (1 - x^2) + x x_0 \tag{38}$$

should be a close approximation to a possible solution f of Eq. (35). This function f^0 represents the result of linear membrane theory. We discuss next "convergent" solutions only, that is, solutions which approach f^0 in the outer range.

Apply a stretching transformation by setting

$$kx \equiv \bar{x} \tag{39}$$

where k is a suitable constant. If we omit in each one of the two stiffness terms all but the term of highest order in k, there remains the asymptotic form

$$f\bar{x}^2 - Af'' + B\bar{f}^{\mathrm{IV}} = \frac{k^2\epsilon_0}{2(1 + \bar{x}/\bar{a}k)} + k\bar{x}x_0 \tag{40}$$

where

$$A \equiv k^4\epsilon_0\bar{a}^2 \qquad B \equiv k^6\bar{h}^2\bar{a}^2/\bar{\nu} \tag{40a}$$

and where differentiation now occurs with respect to $\bar{x}$. The function $\bar{f}$ may be defined by

$$\bar{f}^{\mathrm{IV}} \equiv [(1/\bar{x})(\bar{x}f)'']'' - (2/\bar{x}^3)(\bar{x}f)''_{\bar{x}=0} \tag{40b}$$

An equivalent definition is

$$\bar{f} \equiv \sum_4^\infty \frac{\nu + 1}{\nu - 1} c_\nu \bar{x}^\nu \qquad \text{if} \qquad f \equiv \sum_0^\infty c_\nu \bar{x}^\nu \tag{40c}$$

In each one of the two limit cases, zero pressure ($A = 0$) and membrane shell ($B = 0$), Eq. (40) reduces to an inhomogeneous Bessel equation, either exactly ($B = 0$) or essentially ($A = 0$). In the case $B = 0$, we can define k by setting $A = 1$ and solve Eq. (40) by establishing a set of convergent solutions T_n of

$$M(T_n) \equiv T_n\bar{x}^2 - T_n'' = \bar{x}^n \qquad (n = 0, 1, 2 \ldots) \tag{41}$$

In the other limit, $A = 0$, and setting $B = 1$

$$B(S_n) \equiv S_n\bar{x}^2 + \bar{S}_n^{\mathrm{IV}} = \bar{x}^n \qquad (n = 0, 1, 2 \ldots) \tag{42}$$

The set S_n is related to the (complex valued) Lommel functions $S_{2n/3,\, 1/3}(\frac{2}{3}(i\bar{x})^{3/2})$. The set T_n is a modification[6] of the Lommel functions $S_{(2n-1)/4,\, 1/4}(i\bar{x}^2/2)$. The two sets have a remarkable similarity. A brief description follows.

Because of the obvious relations

$$S_2 \equiv T_2 \equiv 1 \qquad S_3 \equiv T_3 \equiv \bar{x}$$

$$S_4 \equiv \bar{x}^2 \qquad S_5 \equiv \bar{x}^3$$

$$S_n = \bar{x}^{n-2} - (n-1)(n-2)(n-4)(n-5)S_{n-6}$$

$$T_n = \bar{x}^{n-2} - (n-2)(n-3)T_{n-4}$$

we have only four basic solutions, S_0, S_1, T_0, and T_1. Consider the functions

$$s_\mu = \bar{x}^\mu \left\{ 1 - \frac{\bar{x}^6}{(\mu+3)(\mu+4)(\mu+6)(\mu+7)} \times \left[1 - \frac{\bar{x}^6}{(\mu+9)(\mu+10)(\mu+12)(\mu+13)} (1 - \cdots) \right] \right\}$$

$$t_\mu = \bar{x}^\mu \left\{ 1 + \frac{\bar{x}^4}{(\mu+3)(\mu+4)} \times \left[1 + \frac{\bar{x}^4}{(\mu+7)(\mu+8)} (1 + \cdots) \right] \right\}$$

As

$$B(s_\mu) = (\mu+1)\mu(\mu-2)(\mu-3)\bar{x}^{\mu-4} \qquad (\mu \neq 1)$$

$$M(t_\mu) = -\mu(\mu-1)\bar{x}^{\mu-2}$$

and as the functions s_0, s_1, s_2, and s_3 and t_0 and t_1 are the respective homogeneous solutions, we have

$$\begin{aligned} S_0 &= \frac{(-\frac{1}{3})!}{3^{1/3}} s_0 - \frac{3^{1/3}(\frac{1}{3})!}{6} s_2 + \frac{1}{40} s_4 \\ S_1 &= \frac{1}{2\,(3^{1/3})(\frac{1}{3})!} s_1 - \frac{3^{1/3}(-\frac{1}{3})!}{36(\frac{1}{3})!} s_3 + \frac{1}{180} s_5 \\ T_0 &= \frac{(\frac{1}{4})!}{(-\frac{1}{4})!} \pi^{1/2} t_0 - \frac{1}{2} t_2 \\ T_1 &= \frac{(-\frac{1}{4})!}{4(\frac{1}{4})!} \pi^{1/2} t_1 - \frac{1}{6} t_3 \end{aligned} \tag{43}$$

The transcendental factors here follow simply from inspecting the series coefficients.¶

The four basic functions are shown in Fig. 4 together with their respective f^0 functions, that is, $1/\bar{x}^2$ and $1/\bar{x}$. For practical purposes, f and f^0 are undistinguishable in the

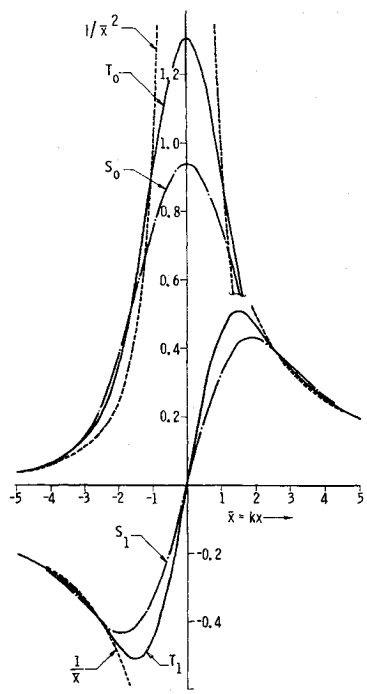

Fig. 4 Basic solutions S_0, S_1, T_0, and T_1.

¶ Because of the condition of convergency, the νth term in each power series Eq. (43) has to converge, as $\nu \to \infty$, toward the negative geometric mean of its two neighboring terms.

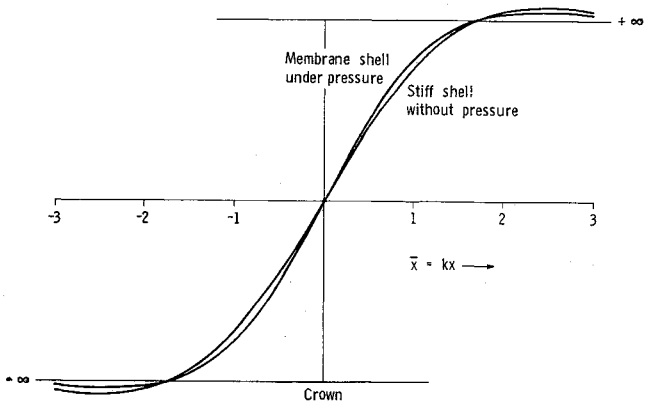

Fig. 5 Vertical deflection $\bar{w}$ under transverse loading.

outer ranges $|\bar{x}| > 4$. Furthermore, a simple transformation would make T_0 almost coincide with S_0 in the inner range also, and T_1 with S_1. This is demonstrated for T_1 and S_1 in Fig. 5.

Tabulations of the two sets of basic solutions are available in the literature. Clark,[9, 10] in dealing with the limit $A = 0$, introduced the Lommel function $S_{0,\,1/3}$ and tabulated it in the form $T_1 + iT_1$. (Another tabulation is given by Hetényi and Timms).[11] To these functions, the present functions S_0 and S_1 are related by $T_r = 3^{1/3}(\frac{1}{3})!(\bar{x}S_1 - 1)$; $T_i = \bar{x}S_0$. The present functions T_0 and T_1 are identical with the functions T_1 and T_2, respectively, that were defined and tabulated by Sanders and Liepins.[5]

It remains to discuss how well the limit solutions S_n and T_n of the asymptotic Eq. (40), where terms of lower order in k have been omitted, approximate the solutions of Eq. (35). It is usual to consider such lower-order terms approximately by means of transcendental transformations. In Ref. 6, where the formal solution for $B = 0$ is given, it is shown that such approximate procedures require considerable caution. However, the corrections to the asymptotic solutions T_n are small enough to be negligible in the present context. A numerical example that follows refers to the problem that is the subject of the remainder of this paper, namely, the transverse stiffness of the torus, Fig. 1, under the load P. The deflection Δw that arises from the crown shift $\pm x_{01}$ is determined by the function f_1 [Eq. (36)] according to Eq. (37). Accurate values, obtained from accurate solutions for $B = 0$, are given in Ref. 1, Eq. (29).** The asymptotic formula for Δw is in this case

$$\frac{\Delta w}{x_{01}R_r} = \frac{1}{k}\int_{-k}^{+k} \frac{(1 - \bar{x}T_1)d\bar{x}}{[1 - (\bar{x}/k)^2]^{3/2}} \rightarrow \frac{2}{k}\int_0^{\infty} (1 - \bar{x}T_1)d\bar{x} = \frac{2c_t}{k} \quad (44)$$

Now $c_t = (-\frac{1}{4})!^2/2^{1/2} \approx 1.062$; from this Table 1 follows. The relative error should be $0(1/k^2)$ as indeed it is. Equation (44) yields a good approximation if $k \geq 4$, say, and is useful for $k \approx 3$.

Table 1 Δw, accurate and asymptotic

			$\Delta w/x_{01}R_r$		
ϵ_0	$\bar{a}$	k	Eq. (35)	Eq. (44)	Error
0.002	3	2.730	0.862	0.778	−10%
0.002	1.5	3.861	0.583	0.550	− 6%

** Although Ref. 1, Eq. (29), contains an error, i.e., 0.433/1.305 ≈ 0.331 should be replaced by 0.433/0.862 ≈ 0.502, the diagrams of Ref. 1 are not materially affected.

In the opposite limit ($A = 0$) correspondingly

$$\frac{\Delta w}{x_{01}R_r} \rightarrow \frac{2}{k}\int_0^{\infty} (1 - \bar{x}S_1)d\bar{x} = \frac{2c_s}{k} \quad (45)$$

with $c_s = \pi/2\ (3^{1/3})(\frac{1}{3})! \approx 1.220$. Equation (45) has about the same limit of validity as Eq. (44) (see also Sec. X).

The vertical deflections w under the external load P, Fig. 1 are shown in Fig. 5, again for the two limit cases and in normalized form

$$\bar{w} = \begin{cases} \dfrac{1}{c_s}\displaystyle\int_0^{\bar{x}} (1 - \bar{x}S_1)d\bar{x} & (A = 0) \\ \dfrac{1}{c_t}\displaystyle\int_0^{\bar{x}} (1 - \bar{x}T_1)d\bar{x} & (B = 0) \end{cases}$$

The two normalized deflection curves are almost identical. Another observation is that most of the deformation occurs near the crown if k is sufficiently large.

IX. Composite Problem

One consequence of the similarity between the two limit solutions, that is, between the two stiffness effects, is that the composite problem, the transition between the two limits can be approximated in a simple manner that illustrates the mechanics involved and is briefly described next. (The numerical results given below, Figs. 6–8, have been obtained by a more accurate method which will be described elsewhere.)

We obtain an approximate solution of Eq. (40) by establishing solutions f of the equation

$$f\bar{x}^2 - Af'' + B\bar{f}^{\mathrm{IV}} = \bar{x}^n + e_n(\bar{x}) \quad (46)$$

with error terms $e_n(\bar{x})$ sufficiently small. Writing

$$f = (1 - \alpha)S_n + \alpha T_n \quad (47)$$

we achieve convergency for $|\bar{x}| \rightarrow \infty$ automatically, and, for any assumed value α, can determine the amplitudes A and B such that $e_n(\bar{x}) = 0(\bar{x}^{n+4})$ in the neighborhood of the transition point. The set of pressurized shells to which these values of A and B refer can then be read from Eq. (40a) where k is now an open parameter.

As an application, we determine the stiffness of the complete pressurized torus, Fig. 1, with respect to the external load system P. We do not need to know the initial crown shift x_{00} but have to determine the effect of adding a shift $\pm x_{01}$. From Eqs. (40, 44, 45, and 47)

$$\Delta w_1 = \frac{2Rx_{01}}{k}\,[1.227(1 - \alpha) + 1.062\alpha] \equiv \frac{2Rx_{01}}{k}\,c(\alpha)$$

where R_r is now replaced by R for brevity. We write $P = P_p + P_b$, with P_p carried by pressure stiffness and P_b carried by bending stiffness. P_p equals the pressure on the ring $-x_{01} \leq x \leq x_{01}$, whereas P_b equals the sum of the two transverse shear forces at the crowns of the two half shells [compare Eq. (5)]. Note that in Eqs. (25, 32, 40, and 43)

$$L(s)_{x=0} = 2\bar{a}\left(\frac{df}{dx}\right)_{x=0} = 2\bar{a}k\left(\frac{df}{d\bar{x}}\right)_{x=0} = 2\bar{a}x_{01}k^2b(\alpha)$$

where $b(\alpha) = 0.3882(1 - \alpha) + 0.5991\alpha$. The total stiffness $S = S_p + S_b = P/\Delta w_1$ becomes

$$S = \frac{\pi}{3^{1/2}c(\alpha)}\left(\frac{A}{B^{1/2}} + 2b(\alpha)B^{1/2}\right)\frac{Eh^2}{R(1 - \nu^2)^{1/2}} \quad (48)$$

For a given shell, S is a function of the pressure p. A suitable reference pressure is

$$p_0 = \lambda E\bar{h}^{7/3}/(\bar{a}\bar{\nu})^{2/3} \quad (49)$$

with the numerical factor λ so determined that $S_b = S_p$ for $p = p_0$. Then

$$(p/p_0) = \nu(1/\lambda)(A/B^{2/3}) \quad (49a)$$

The stiffness S is found, as a function of p/p_0, by determining the amplitudes A and B in Eq. (46), with $n = 1$, for a number of values α. Numerically††

$$\lambda \approx 0.700 \tag{49b}$$

and thus

$$p_0 = 0.134\{E/[\bar{a}(1 - \nu^2)]^{2/3}\}(h/R)^{7/3} \tag{50}$$

Figure 6 shows curves of S_p, S_b, and S over p/p_0.

X. Discussion

In Fig. 6, the (asymptotic) transverse stiffness S of pressurized circular torus shells of uniform wall thickness is given by a single curve that combines all the parameters of the problem. For the limit $p = 0$, Eq. (48) yields

$$S_b = \frac{2}{3^{1/2}} \times \frac{Eh^2}{R(1 - \nu^2)^{1/2}} \qquad (p = 0) \tag{51}$$

independent of the torus opening ratio $\bar{a}$ (as already noted by Clark[10]). The range of validity of Eq. (51) is determined by the parameter

$$k = k_b = [12(1 - \nu^2)]^{1/6}(R/\bar{a}h)^{1/3} \qquad (p = 0)$$

(As we should have $k \geq 4$ or ≥ 3, say, Eq. (51) does not allow the transition $\bar{a} \to \infty$ to a cylinder of infinite length.)

Hetényi and Timms[11] measured S on an unpressurized partial torus shell that extended from $x \approx -0.7$ to $x \approx +0.7$. With $h/R = \frac{1}{60}$, $\bar{a} = 2.23$, $k \approx 4.5$, the $\bar{x}$ range was $-3.1 \leq \bar{x} \leq 3.1$; it thus approached the limit of applicability of Eq. (45). The measured values were $S = 6920$ lb/in. with the shell edges rigidly restrained and $S = 5800$ lb/in. with the edges simply supported. The value obtained from Eq. (51) for a single half-shell is $S = 6300$ lb/in. It properly lies between the two measured values.

According to Fig. 6, S_p increases at first almost linearly with p when a given torus is being pressurized; S_b decreases slowly. The two stiffness contributions become equal when p reaches p_0. Here

$$S \approx 1.94[Eh^2/R(1 - \nu^2)^{1/2}] \qquad (p = p_0) \tag{52}$$

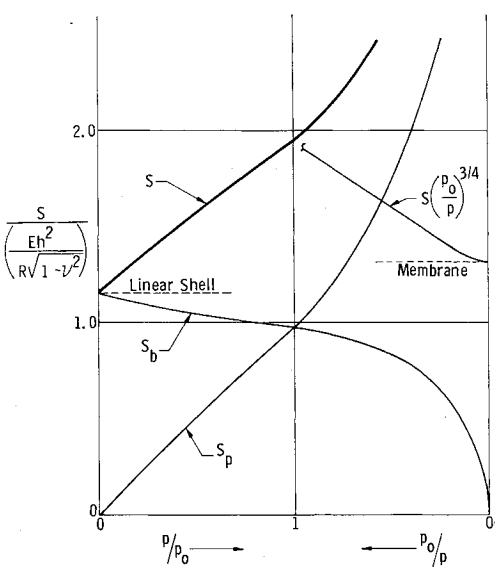

Fig. 6 Transverse torus stiffness S vs pressure.

†† Reissner,[3] estimating the relative magnitude of the two stiffness effects, obtained $\lambda = 0(1)$ in agreement with Eq. (49b).

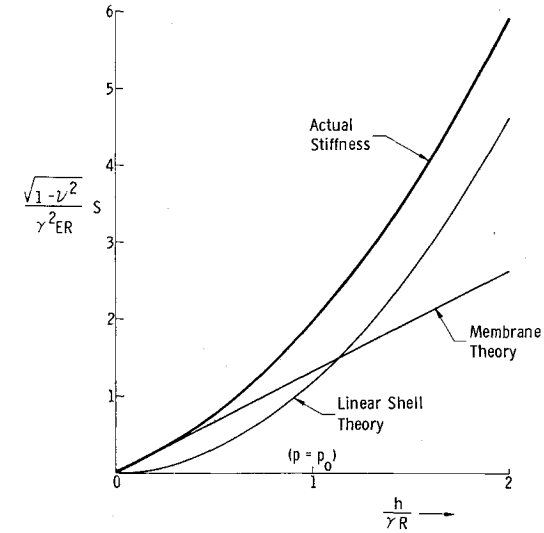

Fig. 7 Stiffness S vs shell thickness.

If the pressure p is increased far beyond p_0, the stiffness S tends toward the membrane solution

$$S \to (2\pi\bar{a}^{1/2}/c_t)(Eh)^{1/4}(Rp)^{3/4}$$

or

$$S(p_0/p)^{3/4} \to 1.307[Eh^2/R(1 - \nu^2)^{1/2}] \qquad (p_0/p \to 0) \tag{53}$$

The range of validity is here determined by

$$k \to k_p = (1/\bar{a}^{1/2})(Eh/pR)^{1/4} \qquad (p_0/p \to 0)$$

The curve $S(p_0/p)^{3/4}$ is also shown in Fig. 6. It has a horizontal tangent at the limit $p_0/p = 0$; the shell stiffness S that is predicted by membrane theory remains accurate for a range of finite shell thicknesses h.

In discussing the mechanical implications of Fig. 6, it is convenient to distinguish two ranges, $p \leq p_0$ and $p \geq p_0$. For the former, assume a given torus with a fixed shell thickness h; the curves then describe the shell behavior as p is varied. For the range $p \geq p_0$, Fig. 7 gives an alternative presentation in which the shell thickness h is considered as the independent variable. Assume here that the quantity

$$\gamma = \left(\frac{p}{p_0}\right)^{3/4}\frac{h}{R} = \left(\frac{\epsilon_0}{0.134}\right)^{3/4}[\bar{a}(1 - \nu^2)]^{1/2}$$

is kept constant; as h is varied, the pressure p is varied pro-

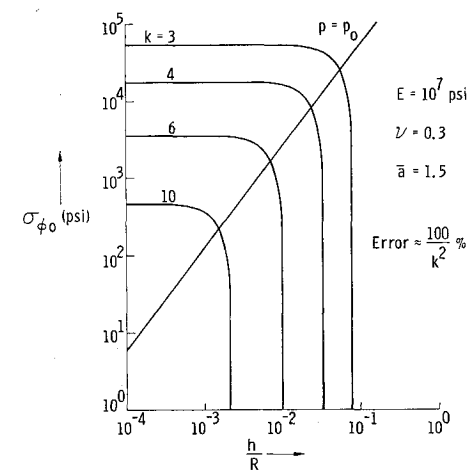

Fig. 8 Validity range of asymptotic results.

portionally such that the reference strain $\epsilon_0 = pR/Eh$ (that is a measure of the utilization of the torus as a pressure vessel) is kept constant. Linear shell theory then predicts $S \sim h^2$, whereas membrane theory predicts $S \sim h$. Figure 7 shows both results plus the actual stiffness.

Finally, Fig. 8 is an indication of the range of validity of Figs. 6 and 7. According to Sec. VIII, the error that is introduced by the asymptotic simplification is $\approx (100/k^2)\%$. Figure 8 shows curves of constant k in the (h/R) vs $\sigma_{\phi 0}$ plane, where $\sigma_{\phi 0} = E \cdot \epsilon_0$ is the mean stress due to pressurization. Aluminum is assumed to be the shell material, and $\bar{a} = 1.5$ the opening parameter. For this relatively small value of $\bar{a}$, characteristic for a torus-shaped fuel tank but not for a torus-shaped space station, the range of validity of the asymptotic approximation is seen to about coincide with the range of applicability of thin shell theory as such. This statement, of course, refers to the over-all transverse stiffness and does not necessarily apply to the details of the stress state of the torus shell.

XI. Conclusions

An approach that utilizes the concept of a reference shape has revealed certain simple shell relations. The resulting quasi-linear shell equation has been discussed in its asymptotic form for the specific case of circular torus shells of uniform wall thickness, and the over-all transverse stiffness of such shells has been determined as a function of internal pressure p and shell thickness h.

The asymptotic shell solution is described by a pair of real-valued functions. In the two limit cases, the membrane shell (pressure stiffness only) and the unpressurized shell (bending stiffness only), this pair is related to the Lommel functions of order $\pm\frac{1}{4}$ and of order $\pm\frac{1}{3}$, respectively. The two limit pairs are remarkably similar. Mechanically, this means that, excepting the local reactions at the shell edges, the two stiffnesses have remarkably similar effects.

The prediction of quasi-linear membrane theory remains correct for small but finite shell thicknesses. On the other hand, linear shell theory fails to predict the stiffness increase due to pressurization; this increase is proportional to p if p is small.

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