A Theory for the Ablation of Non-Newtonian Liquids near the Stagnation Point

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Many liquids of practical importance for ablation have non-Newtonian flow characteristics that may be caused by flow-induced structural changes or by gas bubble concentrations in the liquid. The liquid-layer equations for viscous flow are integrated with a viscosity parameter that is a function of temperature and shear force. Near the stagnation point, non-Newtonian deviations are caused by shear as well as the pressure gradient. If the parameter responsible for non-Newtonian behavior approaches zero, the result of Adams and Bethe is obtained. Non-Newtonian flow may cause substantial changes of surface temperature and ablation effectiveness. Numerical results for an ablating model substance with the physical properties of Pyrex glass, but varying shear rate dependency, are given at several altitudes.

Introduction

IT was first shown by Sutton[1] that in liquid ablation the viscosity of the liquid must be high to obtain high cooling effectiveness. The reason for this is that, with a high viscosity, the material loss by flow to the back under the action of aerodynamic shear forces is small. The flow behavior of such highly viscous liquids is often non-Newtonian, which means that the viscosity is dependent upon the shear stress. As long as aerodynamic shear is not very large, only small deviations from normal Newtonian flow can be expected; however, at hypervelocity speeds in relatively dense atmospheres, non-Newtonian flow behavior of the ablating liquid may cause deviations in the temperature distribution of the liquid layer and influence the over-all cooling process. Because of the steep temperature gradient in the ablative liquid layer, isothermal solutions to this problem can be misleading, and the liquid-layer equations, consisting of the equations of the conservation of mass, momentum, and energy, must therefore be integrated simultaneously with a temperature and shear dependent viscosity.

Non-Newtonian behavior is in most cases of the shear-thinning type, which means that the apparent viscosity decreases with increasing shear rate. Substances with shear-thickening characteristics occur much less frequently. Shear-thinning can be strongly promoted by the presence of gas bubbles in the liquid. Because of the applied shear force, the originally spherical bubbles will be deformed to ellipsoids with the long axis in the direction of flow. The longer the ellipsoidal gas bubbles become under the action of the shear force, the greater the decrease in the apparent viscosity. Present ablation theories neglect non-Newtonian effects, and it is the purpose of this investigation to study the conditions under which the simplified theory remains correct as well as to examine the deviations from normal ablation characteristics when non-Newtonian effects become significant. For reasons of simplicity, the ablation mechanisms are calculated near the stagnation point, where both the shear stress and the pressure gradient contribute to non-Newtonian behavior.

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Shear Stress-Temperature Relationship for Non-Newtonian Flow

Non-Newtonian behavior is related to flow-induced structural changes within the liquid. In the case of a polymer, the long molecules will attain a transient, preferred orientation parallel to the flow direction at high shear rates. The macromolecules in glassy networks, such as the SiO_4 complex in silica glasses, may break up under applied shear stresses. Thixotropic flow behavior can result from either effect.

A theoretical treatment of the dependence on shear rate of macromolecular liquids was first presented by Eyring.[2] In more recent years, Bueche[3] developed a theory for structural viscosity. According to both theories, small deviations from Newtonian flow are proportional to the square of the shear stress. The following relationship between viscosity η_τ and shear stress was derived by Eyring:

$$1/\eta_\tau = (1/\eta)(1 + [(\varkappa\tau)^2/3!] + [(\varkappa\tau)^4/5!] + \dots) \quad (1)$$

where η is the viscosity at very low shear rates, $\varkappa$ is a constant, and τ is the shear stress. Under the low shear stresses that prevail in aerodynamic ablation, the power series may be terminated with the quadratic term.

In addition to its dependence on the shear stress, the viscosity is a strong function of the temperature. The following temperature-viscosity relationship is generally accepted:

$$\eta = e^{A/T-B} \quad (2)$$

where A and B are constants insensitive to temperature. The preceding formula refers to the temperature dependency of the strain-rate-independent η in Eq. (1). For a somewhat more useful analytical form, Eq. (2) may be approximated as

$$\eta/\eta_i = (T/T_i)^{-n'} \quad (3)$$

where the index i refers to a suitable reference state, and $n' = A/T$. For application later in this discussion, it is useful to write Eq. (3) in the following form:

$$\eta/\eta_i = [(T - T_0)/(T_i - T_0)]^{-n} \quad (3a)$$

where T_i now defines the temperature at the gas-liquid interface and T_0 the temperature far inside the body, and the exponent n is slightly smaller than n'. Inserting Eq. (3a) into Eq. (1) and abbreviating $\varkappa^2/3!$ by k yields a relationship that expresses the dependency of the viscosity on both temperature and shear stress:

$$1/\eta_\tau = (1/\eta_i)[(T - T_0)/(T_i - T_0)]^n(1 + k\tau^2) \quad (4)$$

This formula will be used to study the effects of non-Newtonian behavior on ablation. Equation (4) may also be applied to viscous liquids with gas bubble concentrations. However, k is then a function of bubble concentration and bubble size. If this relationship is known, probably from experimental data, one can use a pro forma k value. The property values of the liquid layer, the thermal conductivity, the thermal capacity, etc., must then be treated as properties of the gas-liquid mixture. Since the subsequent calculations are based on constant thermal diffusivity α, the results obtained do not apply to liquids charged with gas bubbles. For such a case, α would be dependent not only on the gas-liquid mixture ratio but possibly also on size and degree of ellipticity of the bubbles.

Liquid-Layer Equations

When aerodynamic heating becomes severe enough, the surface layer of the exposed body will melt. The thickness of the molten layer will always be small because of the steep temperature gradient and aerodynamic shear. It is assumed that the liquid wets the surface and does not coagulate into droplets. Part of the liquid is removed by flow, and part of the liquid may evaporate. The highly viscous glassy or polymeric melts flow in laminar fashion and so slowly in comparison with the gas velocity that the motion of the film has no influence on the heat transfer and shear relations.

The continuous removal of matter produces a very steep temperature gradient inside the body. It can easily be shown that, despite the increase in thermal conductivity with temperature, the heat-shielding capacity of a thermal insulator may be virtually improved by ablation. The very steep temperature gradient that is associated with a good insulator is responsible for a decided change in the viscosity of the liquid layer, so that calculations cannot be based on a constant viscosity. Because of the high viscosity of non-Newtonian liquids, the inertia terms in the momentum equation may be disregarded. The equations of mass, momentum, and energy conservation may then be written as

$$(\partial/\partial x)ru + (\partial/\partial y)rv = 0 \tag{5}$$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left[\eta_i\left(\frac{T - T_0}{T_i - T_0}\right)^{-n}\frac{1}{1 + k\tau^2}\frac{\partial u}{\partial y}\right] \tag{6}$$

$$u(\partial T/\partial x) + v(\partial T/\partial y) = \alpha(\partial^2 T/\partial x^2) \tag{7}$$

where u and v are the velocity components tangential and normal to the body surface, and r is the body radius measured normal to the axis. The pressure p is independent of the y coordinate and is a function of the outer flow conditions. A constant thermal diffusivity α was assumed in Eq. (7). The y axis of the coordinate system was chosen as perpendicular to the body surface with its origin at the liquid-gas interface. The x axis is parallel to the body surface with its origin at the stagnation point. The shear force τ is defined by

$$\tau = \eta_\tau(\partial u/\partial y) \tag{8}$$

where η_τ is the non-Newtonian or structural viscosity as defined by Eq. (4).

A first integration of Eq. (6) under consideration of Eq. (4) yields

$$\tau = (\partial p/\partial x)y + \tau_i \tag{9}$$

where τ_i is the shear stress at the liquid-gas interface. Inserting Eq. (8) into Eq. (9) and integrating a second time gives the velocity distribution in the liquid layer:

$$u = \frac{\partial p}{\partial x}\int_{-\infty}^{0}\frac{y\,dy}{\eta_\tau} + \tau_i\int_{-\infty}^{0}\frac{dy}{\eta_\tau} \tag{10}$$

If η_τ as a function of y were known, the integrals could be evaluated. But at this stage η_τ is known only as a function of the temperature distribution in the liquid layer. Therefore, the next step is the solution of Eq. (7). Near the stagnation point, the first term of Eq. (7) becomes zero, and the first integration gives a value for the temperature gradient of

$$\frac{\partial T}{\partial y} = \left(\frac{\partial T}{\partial y}\right)_i \exp\frac{1}{\alpha}\int_0^y v\,dy \tag{11}$$

where $(\partial T/\partial y)_i$ is the thermal gradient at the gas-liquid interface. Because n in Eqs. (3) and (4) is a large number (larger than 10 in most cases), the thickness of the thermal layer is large in comparison with the thickness of the liquid layer represented by η or η . At a depth greater than the liquid-layer thickness, the v in Eq. (11) may be replaced by v_w, which is the velocity by which the solid-liquid interface moves toward the interior of the body. With the proper boundary conditions, a second integration of Eq. (11) will yield

$$T - T_0 = (\alpha/v_w)(\partial T/\partial y)_i\, e^{v_w y/\alpha} \tag{12}$$

An expression for $(\partial T/\partial y)_i$ may be derived from a simple heat balance

$$(\partial T/\partial y)_i = [(T_i - T_0)v_w]/\alpha$$

With this expression and with $\delta_T = \alpha/v_w$, Eq. (12) may then be written as

$$T - T_0 = (T_i - T_0)\, e^{y/\delta_T} \tag{13}$$

where δ_T is the thickness of the thermal layer. Now T is expressed as a function of y, and Eq. (10) may be integrated, because the viscosity distribution through the liquid layer is given by combining Eqs. (4) and (13):

$$\frac{1}{\eta_\tau} = \frac{\{1 + k[(\partial p/\partial x) + \tau_i]^2\}e^{yn/\delta_T}}{\eta_i} \tag{14}$$

Inserting Eq. (14) into Eq. (10) and using the definition $\delta_T = n\delta$ yields the tangential velocity distribution in the liquid layer

$$u = \frac{\partial p}{\partial x}\int_{-\infty}^{y}\frac{\{1 + k[(\partial p/\partial x)y + \tau_i]^2\}ye^{y/\delta}dy}{\eta_i} + \tau_i\int_{-\infty}^{y}\frac{\{1 + k[(\partial p/\partial x)y + \tau_i]^2\}e^{y/\delta}dy}{\eta_i} \tag{15}$$

Abbreviating

$$N = 1 + k\tau_i^2 \tag{16a}$$

$$M = 2k(\partial p/\partial x)\tau_i \tag{16b}$$

$$L = k(\partial p/\partial x)^2 \tag{16c}$$

we obtain from Eq. (15), after integration,

$$u = (\partial p/\partial x)(e^{y/\delta}/\eta_i)[-\delta^2 N + 2\delta^3 M - 6\delta^4 L + (\delta^2 N - 2\delta^3 M + 6\delta^4 L)y/\delta + (\delta^3 M - 3\delta^4 L)y^2/\delta^2 + \delta^4 L y^3/\delta^3] + (\tau_i e^{y/\delta}/\eta_i)[N\delta - \delta^2 M + 2\delta^3 L + (\delta^2 M - 2\delta^3 L)y/\delta + \delta^3 L y^2/\delta^2] \tag{17}$$

The vertical velocity component v may now be obtained by an integration of Eq. (5):

$$v_i = -\frac{1}{r}\int_{-\infty}^{0}\frac{\partial}{\partial x}(ru)\,dy + v_w \tag{18}$$

where $v_i\rho_L$ is the rate of evaporation. The velocity component u from Eq. (17) has to be inserted into Eq. (18) to

obtain an expression for the ablation rates. Changing the sequence of integration and differentiation, solving the integrals, and setting $x = r$ yields

$$\eta_i(v_i - v_w) = (1/x)(\partial/\partial x)[x(\partial p/\partial x)(2N\delta^3 - 6M\delta^4 + 24L\delta^5)] + (1/x)(\partial/\partial x)[x\tau_i(-N\delta^2 + 2M\delta^3 - 6L\delta^4)] \quad (19)$$

In the vicinity of the stagnation point, $\partial p/\partial x$ and τ_i can be considered linear functions of the distance; however, in retaining the terms that depend on k, terms of the order of x^2 are introduced [see Eq. (15)]. In order to estimate the effect of non-Newtonian flow deviations, it is therefore necessary to take into account the higher-order terms of pressure and shear stress along the body surface. The pressure gradient $\partial p/\partial x$ and the shear stress τ_i are both uneven functions with respect to the stagnation point. For the present purpose, it is sufficient to disregard the fifth-order and higher terms so that we may write

$$\tau_i = a_1x + a_2x^3 \quad (20a)$$

$$\partial p/\partial x = b_1x + b_2x^3 \quad (20b)$$

where a_i and b_i are independent of x and y and are only functions of the outer stream conditions. Introducing these terms into Eq. (19), executing the differentiation, and neglecting terms of fourth and higher orders in x yields

$$\mu_i(v_i - v_w) = 4b_1\delta^3 - 2a_1\delta^2 + x^2[8b_2\delta^3 - 4a_2\delta^2 + k(-4a_1^3\delta^2 + 24b_1a_1^2\delta^3 - 72b_1^2a_1\delta^4 + 96b_1^3\delta^5)] \quad (21)$$

When the second-order term in x can be disregarded, as is possible for Newtonian liquids in the immediate vicinity of the stagnation point, Eq. (21) is identical to the result reported by Bethe and Adams for the ablation of glasses.[4] The last equation shows that in the very immediate vicinity of the stagnation point the effects of non-Newtonian behavior are negligible. At greater distances from the stagnation point, non-Newtonian effects must be compared with second-order variations of τ_i and $\partial p/\partial x$. An estimate of the influence that non-Newtonian behavior has on the flow characteristics can be expressed simply by the condition

$$|8b_2\delta - 4a_2| < |k(-4a_1^3 + 24b_1a_1^2\delta - 72b_1^2a_1\delta^2 + 96b_1^3\delta^3)| \quad (22)$$

The a_i and b_i are given by the following relations for spherical nose caps[5]:

$$a_1 = \tfrac{3}{2}(U_\infty/R)(3U_\infty\eta_s\rho_s/R)^{1/2} \quad (22a)$$

$$a_2 = -(U_\infty/2R^3)(3U_\infty\eta_s\rho_s/R)^{1/2} \quad (22b)$$

$$b_1 = -2p_s/R^2 \quad (22c)$$

$$b_2 = \tfrac{4}{3}(p_s/R^4) \quad (22d)$$

where R = radius of nose cap, U_∞ = velocity of the outer stream, p_s = stagnation pressure, ρ_s = gas density at the stagnation point, and η_s = gas viscosity at the stagnation point. By introducing these terms we can write x^2 in the dimensionless form x^2/R^2. By introducing the terms of Eqs. (22a–22d) and setting $p_s = U_\infty^2\rho_0$, we obtain for the inequality (22)

$$\left|\frac{32}{3}\frac{\rho_0U_\infty}{R}\delta + 2\left(\frac{3U_\infty\eta_s\rho_s}{R}\right)^{1/2}\right| < \left|k \times \left[\frac{27}{2}U_\infty^2\left(\frac{3U_\infty\eta_s\rho_s}{R}\right)^{3/2} + \frac{324\,U_\infty^4\rho_0\eta_s\rho_s}{R^2}\delta + \frac{432\,\rho_0^2U_\infty^4}{R^2}\left(\frac{3U_\infty\rho_s\eta_s}{R}\right)^{1/2}\delta^2 + \frac{768\,\rho_0^3U_\infty^5}{R^3}\delta^3\right]\right| \quad (23)$$

To evaluate this inequality, we must know η_s and ρ_s as functions of the outer stream conditions. These relationships are

$$\eta_s = \eta_0(T_s/T_0)^{3/2}[410/(T_s + 110)]\text{(Sutherland's Formula)}$$

$$\rho_s = p_s/R^*T_s$$

where R^* is the gas constant, and T_s is the stagnation temperature. For very high flight velocities, ρ_s has to be calculated by considering the dissociation of air molecules. Now the ratio Q, of the terms of the left- and right-hand sides of inequality (23), may be evaluated as a function of k, δ, R, and U_∞. The ratio Q then becomes

$$Q = \frac{\dfrac{32\,\rho_0U_\infty\delta}{3R} + 2\left(\dfrac{3U_\infty\eta_s\rho_s}{R}\right)^{1/2}}{k\left[\dfrac{27U_\infty^2}{2}\left(\dfrac{3U_\infty\eta_s\rho_s}{R}\right)^{3/2} + \dfrac{324\,U_\infty^4\rho_0\eta_s\rho_s\delta}{R^2} + \dfrac{432\,\rho_0^2U_\infty^4\delta^2}{R^2}\left(\dfrac{3U_\infty\rho_s\eta_s}{R}\right)^{1/2} + \dfrac{768\,\rho_0^3U_\infty^5\delta^3}{R^3}\right]} \quad (24)$$

Discussion of Results

If the ratio Q is considerably smaller than unity, the non-Newtonian terms cannot be neglected, but must be compared to second-order variations of τ_i and $\partial p/\partial x$ in the neighborhood of the stagnation point. It must be expected that, for such a case, the ablation behavior of a non-Newtonian liquid is different from that of a Newtonian liquid, and consequently the heat-transfer conditions will also be different.

It is now of some interest to investigate the conditions under which deviations from normal behavior will occur as a function of the degree of non-Newtonianness expressed by the factor k. In Figs. 1–3, the factor Qk is plotted against the thickness of the liquid layer δ with the Mach number as a curve parameter. In Fig. 1 the radius of the spherical nose cap is 5 cm, and in Figs. 2 and 3 the respective radii are 10 and 30 cm. A 20-km alt was used for all three figures. The

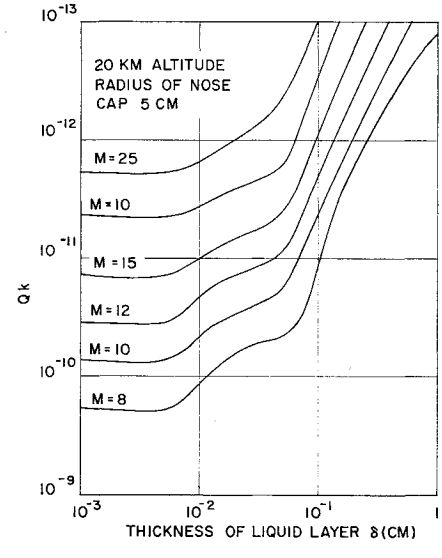

Fig. 1 The factor Qk as a function of the thickness of the liquid layer and Mach number for a nose cap radius of 5 cm at an altitude of 20 km.

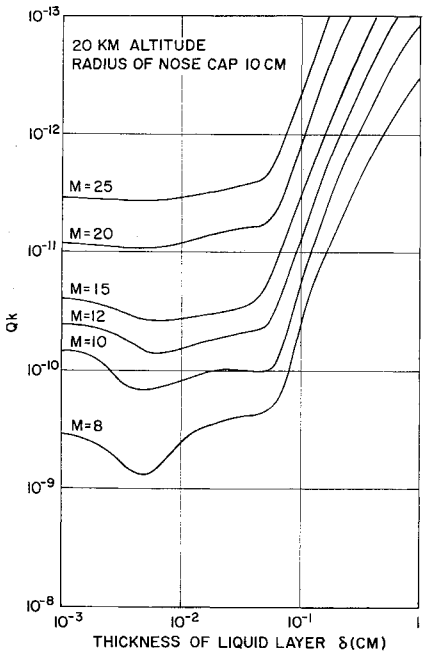

Fig. 2 The factor Qk as a function of the thickness of the liquid layer and Mach number for a nose cap radius of 10 cm at an altitude of 20 km.

following general conclusions may be drawn: 1) for constant liquid-layer thickness δ, non-Newtonian flow deviations increase with flight velocity; 2) except for some irregularities, the effect of non-Newtonianness becomes more severe as the thickness of the liquid layer increases; 3) a non-Newtonian liquid of a certain k value will show increasing deviation from normal flow behavior as the radius of the nose cap becomes smaller; and 4) the curves of Figs. 1–3 have minima; these minima shift to a larger thickness as the radius of the nose cap increases.

A liquid with a k value as small as 10^{-10} will show deviations from normal Newtonian flow behavior at an altitude of 20 km. Most glasses have k values of this order of magnitude, and many show a much higher degree of shear-thinning.

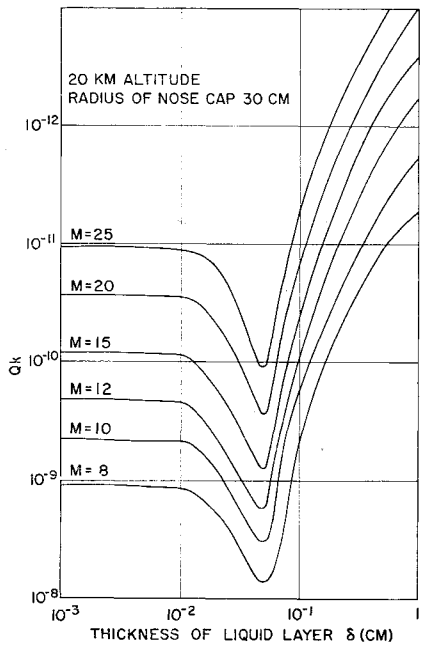

Fig. 3 The factor Qk as a function of the thickness of the liquid layer and Mach number for a nose cap radius of 30 cm at an altitude of 20 km.

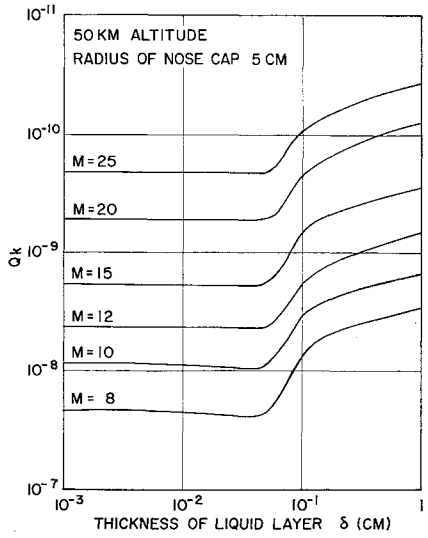

Fig. 4 The factor Qk as a function of the thickness of the liquid layer and Mach number for a nose cap radius of 5 cm at an altitude of 50 km.

According to Eq. (21), non-Newtonian behavior has no effect in the immediate vicinity of the stagnation point; however, at a distance where the third-order terms of τ and $\partial p/\partial x$ are no longer negligible, the simple theory for the ablation of glassy materials becomes incorrect to a certain degree. Since according to a simplified Reynolds analogy the shear force is proportional to the rate of heat transfer, not only the flow behavior but also the over-all ablation behavior will be influenced. In order to determine the ablation characteristics of non-Newtonian liquids, Eq. (21) has to be solved simultaneously with the heat-transfer conditions. The results of this calculation will be given in a later paper.

In Figs. 4–6, the factor Qk is again plotted in an analog manner but for an altitude of 50 km. In order to be of any effect at this altitude, the k value must be about two orders of magnitude larger than under equivalent conditions at 20 km. The minima of the curves are also shifted to larger thicknesses, and this position for the 30-cm nose cap is shifted so far to the right that the location of the minimum is out of range for purposes of practical application.

It should be emphasized that the results presented here are strictly valid only for nonevaporating liquids because τ decreases when vapors are injected into the boundary layer. However, in the vicinity of the stagnation point, the pressure gradient is large in comparison with τ, so that the preceding results are at least good approximations for ablating

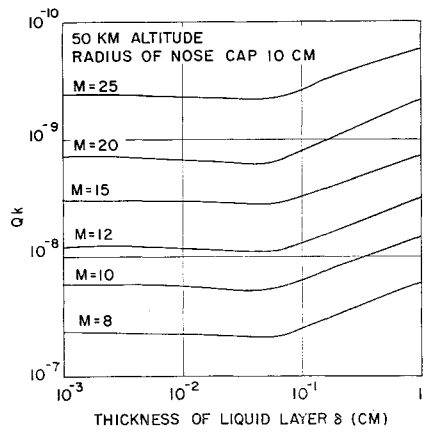

Fig. 5 The factor Qk as a function of the thickness of the liquid layer and Mach number for a nose cap radius of 10 cm at an altitude of 50 km.

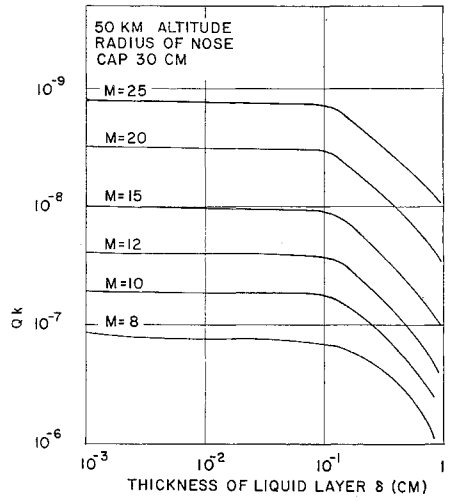

Fig. 6 The factor of Qk as a function of the thickness of the liquid layer and Mach number for a nose cap radius of 30 cm at an altitude of 50 km.

non-Newtonian liquids, which also evaporate. It should be mentioned also that a vapor addition to the boundary layer also will influence density and viscosity, but for most practical cases the overwhelming term in the denominator of Eq. (24) is the last term in which ρ_s and η_s do not appear.

It will now be interesting to blend into our figures some results for Newtonian liquids obtained from the literature in order to see under which conditions these results must be modified if the ablating liquid has some degree of non-Newtonianness. From the results of Bethe and Adams, the thickness of the liquid layer δ may be calculated by the following formula:

$$\delta = h_{\text{eff}}\rho_L\alpha/q_0 n \tag{25}$$

where h_{eff} is the effective heat of ablation that may be obtained from the previous reference as a function of flight velocity and altitude, q_0 is the rate of heat transfer without ablation, and ρ_L is the density of the liquid. The thickness of the liquid layer δ may therefore be plotted as a function of the Mach number. The results obtained for an altitude of 20 km and a nose cap radius of 30 cm, with Pyrex glass as the ablating liquid, are presented in Fig. 7. The curve in

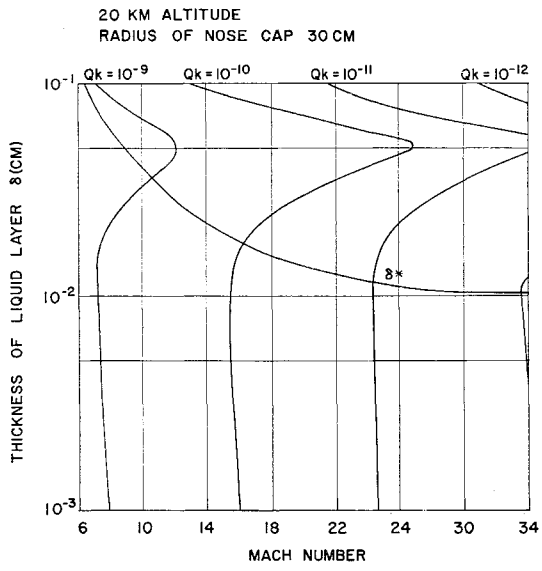

Fig. 7 The influence of various degrees of non-Newtonian behavior of a model substance with the physical properties of Pyrex glass on ablation characteristics (20-km alt).

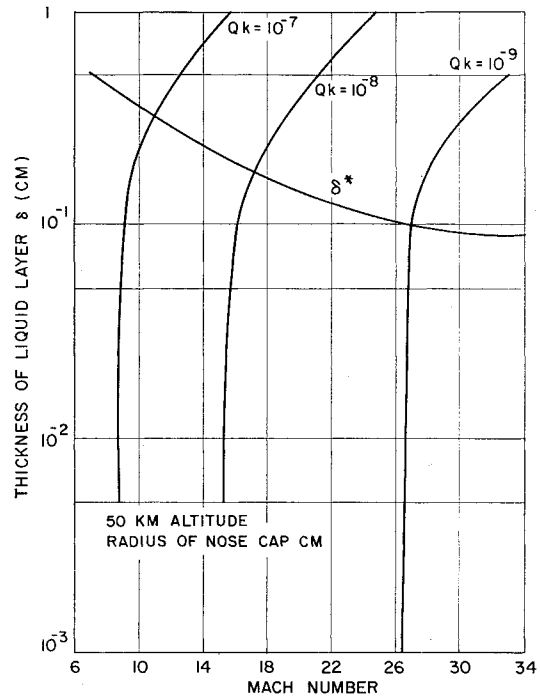

Fig. 8 The influence of various degrees of non-Newtonian behavior of a model substance with the physical properties of Pyrex glass on ablation characteristics (50-km alt).

this figure indicated by δ^* was derived by means of Eq. (25) taken from the previous reference. At relatively small flight velocities at about a Mach number of 10, the flow of the glass would not be affected if the k value of the Pyrex were smaller than 10^{-9}. However, at high speeds such as a Mach number of 30, the k value of the glass has only to be of the order of 10^{-12} for non-Newtonian flow deviations to occur. It is very likely that most glasses have this small degree of shear-thinning.[6] Similar results are given in Fig. 8 for an altitude of 50 km. It can be seen that the k value must be about two orders of magnitude larger for any substantial deviation to occur.

It should be mentioned that radiation from the body has been disregarded. If radiation were taken into consideration, the value of q_0 in Eq. (25) would become smaller and δ larger, and the δ^* curves in Figs. 7 and 8 would be lifted somewhat. At an altitude of 50 km the effect of non-Newtonianness is reduced by radiation; however, at a 20-km alt the circumstances are more complicated because the curves with a constant Qk value have a strong peak. Thus, by blending the results of investigations of other ablation materials with a potential degree of non-Newtonianness into the forementioned figures, we may determine up to what point these results are correct and what adjustment is needed beyond this point.

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