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Resonant Wave Interactions on a Swept Wing

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Introduction

EXPERIMENTS on the growth of laminar instabilities in a Blasius boundary layer[1] show that near the end of the region of the linear growth of the two-dimensional Tollmien-Schlichting wave a three-dimensional pattern appears. The exact position and form of this pattern is a matter of controversy because more recent experiments[2] show that the three dimensionality is random with no distinguishable spanwise wavelength.

One of the analytical models that has been proposed to explain the three-dimensionality of the Blasius layer is resonant wave interactions. Craik[3] examined the pattern consisting of a two-dimensional Tollmien-Schlichting wave and two oblique waves symmetric about the flow direction. The oblique waves had half the frequency of the two-dimensional wave. The rapid growth of the oblique waves is explained by the rapid transfer of energy from the mean flow to the oblique waves.[3] The model has been re-examined by Lekoudis[4] using spatial stability theory, but no calculations were performed. Calculations of the amplitudes of resonant triads have been performed by Volodin and Zel'man.[5] They used Craik's model in a Blasius boundary layer and predicted rapid growth of the amplitudes of all the interacting waves; however, they also showed that the initial phasing and the spanwise periodicity could act as detuning and lessen the strength of the interaction. Nayfeh and Bozatli calculated the result of the modification of the Blasius layer from a Tollmien-Schlichting wave of large amplitude,[6] and examined nonlinear wave interactions using the resonance model.[7] They found that a rapid increase of the oblique waves can occur. They also found that the strength of the interaction is proportional to the initial amplitude of the waves.

Recent flat plate vibrating ribbon experiments[2] show the subharmonic growing in a region of Reynolds numbers where linear theory predicts it should decay. The same kind of experiments show the growth to be dependent on the initial conditions.[8] Moreover, another set of recent experiments[9] examined the evolution of the spectrum in a Blasius boundary layer excited by two vibrating ribbons. The spectrum is filled with frequencies that are combinations of the two excitation frequencies. This behavior can be explained by a generalization of the resonance model.

Next, the theory is extended to the case of three-dimensional boundary-layer flows, and applied to the case of the boundary layer on a transonic swept wing with laminar flow control.

Formulation of the Problem

We consider the nonlinear stability of a three-dimensional boundary-layer flow on a swept wing with laminar flow control. The Cartesian coordinate system used has the x axis in the direction of normal chord, the y axis normal to the surface, and the z axis along the span.

We assume a locally parallel, three-dimensional boundary-layer flow consisting of a steady mean part and an unsteady disturbance part. The disturbance part consists of wave triads:

$$\begin{Bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{p} \end{Bmatrix} = \epsilon \sum_{j=0}^{2} A_j(\epsilon x, \epsilon z, \epsilon t) \begin{Bmatrix} u_j(y) \\ v_j(y) \\ w_j(y) \\ p_j(y) \end{Bmatrix} \exp(i\theta_j) \tag{1}$$

where ϵ is a small number characteristic of the weak nonlinearity of the system we are examining, u, v, and w denote the velocity components in the x, y, and z directions, respectively, p is the pressure, and the tildes denote disturbance quantities. Also,

$$\theta_j = \alpha_j x + \beta_j z - \omega_j t \qquad j=0,1,2 \tag{2}$$

In Eq. (2), α_j and β_j are the wavenumbers in the x and z directions, respectively, and ω_j are frequencies. In Eq. (1), A_j is the slowly varying amplitude. We introduce the slow scales[10]

$$X=\epsilon x \qquad Z=\epsilon z \qquad T=\epsilon t \tag{3}$$

Substituting Eqs. (1-3) into the incompressible Navier-Stokes equations, linearizing and separating harmonics, we find that the disturbance is governed to first order in ϵ by the Orr-Sommerfeld problem for the case of a three-dimensional boundary layer. In order for the waves to satisfy resonance conditions, we require that

$$\theta = \theta_1 + \theta_2 - \theta_0$$

$$\frac{\partial\theta}{\partial x} = O(\epsilon) \quad \frac{\partial\theta}{\partial z} = O(\epsilon) \quad \frac{\partial\theta}{\partial t} = O(\epsilon) \tag{4}$$

Using Eq. (4) and applying solvability conditions to the $0(\epsilon^2)$ equations, we arrive at the following quasilinear, first-order, partial differential equations that govern the slowly varying

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Table 1 Resonant unstable waves on a swept wing

$\omega_0 = 300$ Hz, $\omega_1 = 200$ Hz, $\omega_2 = 100$ Hz								
x/c	α_{1r}	β_{1r}	α_{2r}	β_{2r}	α_{0r}	β_{0r}	$\alpha_{1r}+\alpha_{2r}-\alpha_{0r}$	$\beta_{1r}+\beta_{2r}-\beta_{0r}$
0.01637	−0.05107	0.10645	−0.05133	0.10633	−0.10779	0.21010	−0.0053	0.0026
0.02116	−0.05709	0.12362	−0.05738	0.12349	−0.12000	0.24449	−0.0055	0.0026
0.02660	−0.06462	0.14415	−0.06494	0.14400	−0.13526	0.28553	−0.0057	0.0026
0.03942	−0.08067	0.18682	−0.08106	0.18665	−0.16731	0.37101	−0.0055	0.0024
0.05491	−0.10111	0.23403	−0.10156	0.23383	−0.20737	0.46580	−0.0047	0.0020
0.08317	−0.14234	0.31692	−0.14292	0.31666	−0.28775	0.63246	−0.0024	0.0011
0.11716	−0.18268	0.39219	−0.18336	0.39187	−0.36654	0.78384	−0.0005	0.0002
0.15621	−0.21771	0.45708	−0.21847	0.45671	−0.45671	0.91404	0.0004	−0.0002
0.20007	−0.24718	0.51136	−0.24801	0.51096	−0.49425	1.02200	0.0009	0.0003
0.24777	−0.27347	0.56049	−0.27436	0.56006	−0.54675	1.12100	0.0010	−0.0004

amplitudes:

$$\frac{\partial A_j}{\partial T} + V_{\alpha_j}\frac{\partial A_j}{\partial X} + V_{\beta_j}\frac{\partial A_j}{\partial Z} = Q_j \qquad j=0,1,2 \tag{5}$$

$V_{\alpha j}$ and $V_{\beta j}$ are the group velocities in the x and z directions, respectively, and the Q_j are the nonlinear terms, quadratic in the amplitudes, that arise from the resonance conditions. If the spatial theory is used, the equations can be simplified by the following:

$$\frac{\partial}{\partial t} \equiv 0, \quad \omega_1 + \omega_2 - \omega_0 = 0$$

$$A_j^* = A_j \exp(-\alpha_{ij}x - \beta_{ij}z) \qquad j=0,1,2 \tag{6}$$

Then Eq. (5) reduces to

$$V_{\alpha_j}\frac{\partial A_j^*}{\partial x} + V_{\beta_j}\frac{\partial A_j^*}{\partial z} + (\alpha_{ij}V_{\alpha j} + \beta_{ij}V_{\beta j})A_j^* = \epsilon Q_j^*, \quad j=0,1,2 \tag{7}$$

where

$$Q_0^* = \left(\int_0^\infty F_0 \mathrm{d}y\right) A_1^* A_2^* \exp(i\Gamma)$$

$$Q_1^* = \left(\int_0^\infty F_1 \mathrm{d}y\right) A_0^* \bar{A}_2^* \exp(-i\Gamma)$$

$$Q_2^* = \left(\int_0^\infty F_2 \mathrm{d}y\right) A_0^* \bar{A}_1^* \exp(-i\Gamma) \tag{8}$$

and

$$\Gamma = (\alpha_{1r} + \alpha_{2r} - \alpha_{0r})x + (\beta_{1r} + \beta_{2r} - \beta_{0r})z \tag{9}$$

The subscripts r and i denote real and imaginary parts of the subscripted quantity. The dispersion relation for three-dimensional, boundary-layer flows with wall suction shows that the imaginary parts of the wavenumbers are usually of $0(\epsilon)$ compared with their real parts. The coefficients F_0, F_1, and F_2 are functions of the adjoint problem and the solution of the linear problem and are defined in the Appendix together with the group velocities.

Results and Discussion

In order for the interaction of the type described in the previous section to exist in the three-dimensional boundary layer, the coefficients of x and z in Eq. (9) have to be of $0(\epsilon)$. Because the interest in the present study is on transition on wings, we need to examine if the previous condition exists for crossflow instability waves on a typical LFC wing. Table 1 shows the variation of the wavenumbers and the coefficients of x and z in Eq. (9) for a certain triad of waves with $\omega_0 = 300$ Hz, $\omega_1 = 200$ Hz, and $\omega_2 = 100$ Hz. It is clear that the coefficients of x and z in Eq. (9) are very small. The preceding results mean that conditions that favor resonant interactions are present. There are differences between the case of a Blasius layer and a three-dimensional boundary layer. The most important is that all three waves are linearly unstable in the present case, and this could make the interaction much stronger. Moreover, there is no need for the waves to satisfy the symmetry conditions as prescribed in the original models.

One easily can compute many combinations of frequencies so that resonance conditions are satisfied, and it seems to be an abundance of conditions that favor resonant interactions on a swept wing with laminar flow control. If the rapid growth of the low-frequency waves, computed for the case of the Blasius layer,[5,7] exists in the case a three-dimensional boundary layer, then the filtering mechanism, which favors growth of certain frequencies, might not be operative. This is quite possible because all three of the interacting waves grow linearly throughout the chordwise region examined. If that is the case, then it becomes difficult to justify using an e^N criterion to predict transition on swept wings.

Appendix

The adjoint problem is defined by the following system for the q_{1j}, q_{2j}, q_{3j}, and q_{4j}:

$$Dq_{3_j} - i\alpha_j q_{2_j} - i\beta q_{4_j} = 0 \tag{A1}$$

$$i\alpha_j q_{1_j} + i(\alpha_j U + \beta_j W - \omega_j)q_{2_j} - R^{-1}(D^2 - \alpha_j^2 - \beta_j^2)q_{2_j} = 0 \tag{A2}$$

$$i\beta_j q_{1_j} + i(\alpha_j U + \beta_j W - \omega_j)q_{4_j} - R^{-1}(D^2 - \alpha_j^2 - \beta_j^2)q_{4_j} = 0 \tag{A3}$$

$$-Dq_{1_j} + q_{2_j}DU + i(\alpha_j U + \beta_j W - \omega_j)q_{3_j} + q_{4_j}DW - R^{-1}(D^2 - \alpha_j^2 - \beta_j^2)q_{3_j} = 0 \tag{A4}$$

where q_{2j}, q_{3j}, and q_{4j} vanish at $y=0$, all q_j's decay at $y=\infty$, R is the Reynolds number, and D denotes derivative with respect to y.

The group velocities are:

$$V_{\alpha_j} = \int_0^\infty [(U - 2i\alpha_j R^{-1})(u_j q_{2_j} + v_j q_{3_j} + w_j q_{4_j}) + q_{1_j}p_j + q_{1_j}u_j]\mathrm{d}y \Big/ \int_0^\infty (u_j q_{2_j} + v_j q_{3_j} + w_j q_{4_j})\mathrm{d}y \tag{A5}$$

$$V_{\beta_j} = \int_0^\infty [(W - 2i\beta_j R^{-1})(u_j q_{2_j} + v_j q_{3_j} + w_j q_{4_j})$$

$$+ q_{4_j} p_j + q_{1_j} w_j] \mathrm{d}y \Big/ \int_0^\infty (u_j q_{2_j} + v_j q_{3_j} + w_j q_{4_j}) \mathrm{d}y \tag{A6}$$

If we define

$$M_0 = -v_1 D u_2 - v_2 D u_1 - i(\alpha_1 + \alpha_2) u_1 u_2$$

$$- i(\beta_1 w_2 u_1 + \beta_2 w_1 u_2) \tag{A7}$$

$$M_{1,2} = -v_0 D \bar{u}_{2,1} - \bar{v}_{2,1} D u_0 - i(\alpha_0 - \alpha_2) u_0 \bar{u}_{2,1}$$

$$+ i(\beta_2 w_0 \bar{u}_2 - \beta_0 \bar{w}_{2,1} u_0) \tag{A8}$$

$$N_0 = -v_1 D v_2 - v_2 D v_1 - i(\alpha_1 u_2 + \beta_1 w_2) v_1 - i(\alpha_2 u_1 + \beta_2 w_1) v_2 \tag{A9}$$

$$N_{1,2} = -v_0 D \bar{v}_{2,1} - \bar{v}_{2,1} D v_0 - i(\alpha_0 u_{2,1} v_0 - \alpha_2 u_0 \bar{v}_{2,1})$$

$$- i(\beta_0 \bar{w}_{2,1} v_0 - \beta_2 w_0 \bar{v}_{2,1}) \tag{A10}$$

$$P_0 = -v_0 D w_2 - v_2 D w_0 - i(\beta_1 + \beta_2) w_1 w_2$$

$$- i(\alpha_1 u_2 w_1 + \alpha_2 u_1 w_2) \tag{A11}$$

$$P_{1,2} = -v_0 D \bar{w}_{2,1} - \bar{v}_{2,1} D w_0 - i(\beta_0 - \beta_2) w_0 \bar{w}_{2,1}$$

$$+ (\alpha_{2,1} u_0 \bar{w}_{2,1} - \alpha_0 \bar{u}_{2,1} w_0) \tag{A12}$$

Then

$$F_j = \frac{M_j q_{2_j} + N_j q_{3_j} + P_j q_{4_j}}{\int_0^\infty (u_j q_{2_j} + v_j q_{3_j} + w_j q_{4_j}) \mathrm{d}y} \quad j = 0,1,2 \tag{A13}$$

In the preceding quantities, bars denote complex conjugates.

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Effect of Shear Deformation and Rotatory Inertia on the Stability of Beck's and Leipholz's Columns

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Introduction

STABILITY of nonconservative systems is discussed in the works of Bolotin[1] and Leipholz.[2] Stability of slender cantilever columns subjected to follower forces using the finite-element method has been reported in Refs. 3-5. In the present Note the effects of shear deformation and rotatory inertia on the stability of cantilever columns subjected to 1) a concentrated follower force, Beck's column, and 2) a uniformly distributed follower force, Leipholz's column, are studied with use of the finite-element method. The finite-element formulation including shear deformation and rotatory inertia is the same as that given in Ref. 6. As special cases, critical loads for slender columns are obtained from the present solution which agree very well with the published work.[5]

Finite-Element Formulation

The matrix equation governing the stability problem is obtained as[3]

$$\lambda^2 [M]\{q\} - [K]\{q\} + Q[G^c]\{q\} + Q[G^{Nc}]\{q\} = 0 \tag{1}$$

where $[K]$, $[M]$, $[G^c]$ and $[G^{Nc}]$ are the global elastic stiffness, mass and geometric stiffness for the conservative part of the load, and the geometric stiffness matrix for the nonconservative part of the load, respectively. In Eq. (1), $\lambda^2 = m\omega^2 L^4/EI$, where m is the mass per unit length, ω is the circular frequency, L is the length of the column, E is the Young's modulus, and I is the moment of inertia. For Beck's column, $Q = PL^2/\pi^2 EI$ and for Leipholz's column $Q = pL^3/\pi^2 EI$, where P is the tip load and p is the distributed load per unit length.

The element stiffness matrix $[k]$, the mass matrix $[m]$, geometric stiffness matrices $[g^c]$ and $[g^{Nc}]$ are obtained using the usual procedure[7] from the expressions:

$$U = \frac{1}{2} \int_0^l \left[EI\psi_x^2 + \frac{5}{6} \left(\frac{EA}{2(1+\nu)} \epsilon_{xz}^2 \right) \right] \mathrm{d}x \tag{2}$$

$$T = \frac{1}{2} \omega^2 \int_0^l m[w^2 + r^2 (w_x + \gamma^2)^2] \mathrm{d}x \tag{3}$$

$$W^c = \frac{P}{2} \int_0^l w_x^2 \mathrm{d}x \quad \text{for Beck's column} \tag{4}$$

$$= \frac{p}{2} \int_0^l (l - x) w_x^2 \mathrm{d}x \quad \text{for Leipholz's column} \tag{5}$$

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