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Semi-implicit Hopscotch-Type Methods for the Time-Dependent Naiver-Stokes Equations

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Treatment of the time-dependent Navier-Stokes equations for the solution of the flowfield about airfoil configurations is generally performed using either explicit or implicit finite-difference methods. As a viable alternative, hopscotch-type methods combine the speed of the explicit and the favorable stability criteria of the implicit methods. Such methods are constructed here for the time-dependent Navier-Stokes equations in conservation law form; their implementation to a number of test cases (one of Euler's equations), including the stringent case of shock-wave/boundary-layer interaction with separation, indicates the possibility of competitive accuracy and more rapid computing time than is required by currently popular, fully implicit methods.

Nomenclature

A,B	= Jacobian matrices
e	= total enthalpy
F,F',G,G'	= defined in Eqs. (3) and (20), respectively
i,j	= index
I	= identity matrix
L	= (nonlinear) finite-difference operator
M	= Mach number
n	= index
P	= pressure
Pr	= Prandtl number
Q,Q',R,R'	= defined in Eqs. (4) and (21), respectively
Re	= Reynolds number
T	= temperature
T'	= "source" term
t	= time
u	= x-direction velocity component or dependent variable, Eq. (10)
U	= vector of dependent variables
v	= y-direction velocity component
x,y	= independent space variables
$\Delta x,\Delta y$	= increments in x and y, respectively
γ	= ratio of specific heats
δ	= central difference operator
θ	= switching function
μ	= viscosity
ρ	= density
σ_{ii}	= normal stresses
τ	= time increment
τ_{ij}	= shear stresses

Introduction

THE past decade has witnessed an increased use of high-speed large storage computers in aircraft design, partly due to the many different algorithms produced by computational aerodynamicists for solving various forms of the Navier-Stokes equations. Although it is the steady-state solution that is generally sought, the numerical methods adopted often consider the time-dependent equations and try to approach the desired solution via a transient (not necessarily numerically consistent) path.

Basically, there are two approaches that appear to be currently in use in this context. The first (also historically) consists of explicit methods of the sort suggested, for example, by MacCormack.[1,2] Although capable of handling a variety of high Reynolds number flow problems, including inviscid-viscous interaction,[1] the use of small time steps to insure stability is necessary and is sometimes prohibitively costly. The second approach makes use of alternating direction implicit (ADI) type methods[3-6] which, at least for the linear analysis, are unconditionally stable and hence require smaller computing times. Some of these latter methods rely upon local time linearization of nonlinear terms in the Navier-Stokes equations, together with approximate factorization of the difference operators in order to cast the finite-difference equations into a form suitable for ADI treatment.[5] The linearization enables fairly accurate solutions to be obtained with large time steps if a steady state is required or if the time evolution is slow (see Beam and Warming[5]). However, the block tridiagonal inversions that must be made at each step may be as expensive in terms of computer time as the many smaller time steps necessary with the use of an explicit method. Two recent state-of-the-art reviews[7,8] summarize and compare many of the variations of these two approaches and discuss their relative pros and cons (see also Balleur et al.[9]).

There exists, however, a sort of hybrid family of numerical methods that aims to combine the positive features of the explicit and implicit methods. These mixed-type algorithms were first proposed by Gordon[10] and subsequently implemented by Scala and Gordon[11] for solving the full time-dependent Navier-Stokes equations (in nonconservative form). The essence of the method was the use of explicit and implicit finite-difference formulas at alternate mesh points (combined with operator splitting). Later a formal hierarchy of these methods was discovered by Gourlay[12] and Gourlay and McGuire[13] and their positive features were thereafter exploited for solution of a variety of (mainly parabolic) partial differential equations. Greenberg[14] extended this family of so-called "hopscotch" methods and examined a number of two- and three-dimensional nonlinear time-dependent problems using extended Burger's equations. The ease of programming and fast computer times (relative to simple explicit or Peaceman-Rachford ADI methods) makes this third class of hybrid methods an attractive candidate for the solution of aerodynamic flow problems using the full Navier-Stokes equations. Indeed, Rudy et al.[15] have made a comparative study of several numerical techniques using mixed subsonic-supersonic flow problems (Reynolds numbers up to 8.1×10^4). For the steady-state solutions they computed, the simple hopscotch method (although not consistent in time) was superior to the ADI method (not time linearized). Furthermore, for the convection-diffusion equation it has been demonstrated[10] that, with upwind differencing for the convective terms, unconditional stability is achieved. The structure of the hopscotch-type method also makes it suitable

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for parallel processing computers which decrease the execution time required by conventional computers by several orders of magnitude (see Ballhaus and Bailey[16]).

In the present work the solution of the time-dependent compressible Navier-Stokes equations in conservative form is attempted using more sophisticated hopscotch-type methods than have hitherto been applied in this context. Some numerical examples isolate certain important features of time-dependent and/or transonic flows, showing that accurate and rapid results can be obtained using these methods.

Equations of Flow

The two-dimensional compressible Navier-Stokes equations (CNS) for laminar flow are written in the following conservative form:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=\frac{\partial Q}{\partial x}+\frac{\partial R}{\partial y} \tag{1}$$

where the vector of dependent variables is given by

$$U=[\rho,\rho u,\rho v,e]^T \tag{2}$$

and

$$[F,G]=\begin{bmatrix} \rho u & , & \rho v \\ P+\rho u^2 & , & \rho uv \\ \rho uv & , & P+\rho v^2 \\ u(e+P) & , & v(e+P) \end{bmatrix} \tag{3}$$

$$[Q,R]=\begin{bmatrix} 0 & , & 0 \\ \sigma_{xx} & , & \tau_{xy} \\ \tau_{xy} & , & \sigma_{yy} \\ \frac{\mu}{Pr}\frac{\partial T}{\partial x}+u\sigma_{xx}+v\tau_{xy} & , & \frac{\mu}{Pr}\frac{\partial T}{\partial y}+u\tau_{xy}+v\sigma_{yy} \end{bmatrix} \tag{4}$$

where

$$\sigma_{xx}=\frac{2}{3}\mu\left(2\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right) \tag{5}$$

$$\sigma_{yy}=\frac{2}{3}\mu\left(2\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right) \tag{6}$$

$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \tag{7}$$

and the total enthalpy is

$$e=\rho\left[\frac{T}{\gamma}+\frac{1}{2}(u^2+v^2)\right] \tag{8}$$

The equation of state must be added to the above set of equations, viz.,

$$P=[(\gamma-1)/\gamma]\rho T \tag{9}$$

Equations (1-9) are expressed completely in terms of nondimensional quantities (with appropriate reference quantities). A further relationship, Sutherland's law, expresses the viscosity as a function of the temperature.

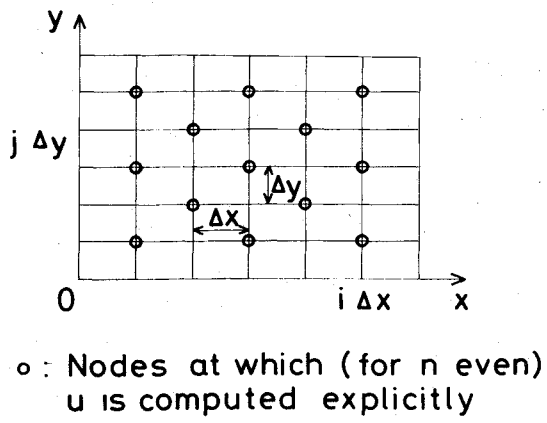

Fig. 1 Finite-difference mesh.

The advantage to be accrued from use of the strong conservation law form of the governing equations is that this form is retained if a transformation is performed from regular Cartesian coordinates to some arbitrary curvilinear coordinates (e.g., coordinates that fit an airfoil shape exactly) and, hence, shock capturing is insured (see Viviand[17]). Thus, with a view to future practical transonic computations in mind, the use of the conservative form here is desirable.

Numerical Approach

Hopscotch Methods

Before proceeding to the numerical treatment of the CNS it will be useful to illustrate the original hopscotch-type method by a simple example. Let $u^n_{i,j}$ be the value of a dependent variable u at the point $(i\Delta x, j\Delta y, n\tau)$ in space-time, where Δx, Δy, and τ are increments in the x, y, and time directions, respectively. Now consider the diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} \tag{10}$$

which is to be solved numerically in some region of the x-y plane, part of which is shown in Fig. 1 covered with a finite-difference mesh. The value of $u^{n+1}_{i,j}$ is computed with the following two sweeps through the mesh. For all points at which $i+j+n$ is even, use

$$u^{n+1}_{i,j}=u^n_{i,j}+\tau\frac{[u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}]}{(\Delta x)^2}+\tau\frac{[u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}]}{(\Delta y)^2} \tag{11}$$

i.e., an explicit step. The second sweep is performed for all the remaining mesh points but with the following formally *implicit* formula

$$u^{n+1}_{i,j}=u^n_{i,j}+\tau\frac{[u^{n+1}_{i+1,j}-2u^{n+1}_{i,j}+u^{n+1}_{i-1,j}]}{(\Delta x)^2}+\tau\frac{[u^{n+1}_{i,j+1}-2u^{n+1}_{i,j}+u^{n+1}_{i,j-1}]}{(\Delta y)^2} \tag{12}$$

However, since $u^{n+1}_{i\pm1,j}$ and $u^{n+1}_{i,j\pm1}$ have already been computed from Eq. (11) $u^{n+1}_{i,j}$ can be isolated in Eq. (12) and calculated *directly* from known quantities without the necessity for matrix inversion that use of Eq. (12) at all mesh points would entail. The entire calculation is repeated to $(n+2)\tau$. Gourlay[12] considers this method a two-step process. The solution is thus attained *in practice* in an explicit fashion, despite the formal

implicit nature of Eq. (12), and is therefore very rapid. The stability of this method is discussed by Gordon[10]; Gourlay[12] has described a family of such methods, some of which do involve matrix inversions but only of tridiagonal systems; and Greenberg[14] has further extended the family. The basic underlying structure of the method has been shown to be akin to ADI methods[12,13] with their favorable stability criteria—a fact that, coupled with the fast execution times, makes them attractive alternatives to current finite-difference methods.

Application to Euler's Equations

As an initial application of the hopscotch method, Euler's equations are considered so that the right-hand side of Eq. (1) is deleted. Beam and Warming[18] presented an implicit finite-difference algorithm for hyperbolic systems in conservation law form. To arrive at the final implicit algorithm, time-differencing according to the trapezoidal rule was performed to give $0(\tau^3)$ accuracy and F and G were expanded in a Taylor series about U^n in order to linearize the equations. The latter were factorized and with the aid of Euler's theorem on homogeneous functions the final factored scheme was of the form

$$\left(I+\frac{\tau}{2}\frac{\partial}{\partial x}A^n\right)\left(I+\frac{\tau}{2}\frac{\partial}{\partial y}B^n\right)U^{n+1}$$

$$=\left(I-\frac{\tau}{2}\frac{\partial}{\partial x}A^n\right)\left(I-\frac{\tau}{2}\frac{\partial}{\partial y}B^n\right)U^n+0(\tau^3) \tag{13}$$

where A and B are the Jacobian matrices of F and G with respect to U.

In order to accommodate the use of a hopscotch method, Eq. (13) is split in a Peaceman-Rachford fashion to yield the following algorithm:

$$\left(I+\frac{\tau}{2}\frac{\partial B^n}{\partial y}\right)U^{N+1}=\left(I-\frac{\tau}{2}\frac{\partial A^n}{\partial x}\right)U^n \tag{14a}$$

$$\left(I+\frac{\tau}{2}\frac{\partial A^n}{\partial x}\right)U^{n+1}=\left(I-\frac{\tau}{2}\frac{\partial B^n}{\partial y}\right)U^{N+1} \tag{14b}$$

The matrices A^n and B^n remain constant during this two-step process. This algorithm is similar to that mentioned by Mitchell.[19] It can be compactly written using the following switch function:

$$\begin{aligned}\theta^n &= 1 \quad n \text{ even}\\ &= 0 \quad n \text{ odd}\end{aligned} \tag{15}$$

as

$$\left[I+\frac{\tau}{2}\left(\theta^{n+1}\frac{\partial A}{\partial x}+\theta^n\frac{\partial B}{\partial y}\right)\right]U^{n+1}$$

$$=\left[I-\frac{\tau}{2}\left(\theta^n\frac{\partial A}{\partial x}+\theta^{n+1}\frac{\partial B}{\partial y}\right)\right]U^n \tag{16}$$

where the suffix over A and B has been dropped for clarity. Calculation of the U^{n+1} in this fashion involves the inversion of tridiagonal matrices at each stage, if second-order central differences are used for the space derivative. The form of the splitting Eq. (14) is unlike that of Beam and Warming[18] and has the consequence of reducing the time-wise accuracy to $0(\tau^2)$. To introduce a hopscotch version of Eq. (16) use is made of the switch function

$$\begin{aligned}\theta_i^n &= 1 \quad \text{if } i+n \text{ is even}\\ &= 0 \quad \text{if } i+n \text{ is odd}\end{aligned}$$

whence

$$\left[I+\frac{\tau}{2}\left(\theta_i^{n+1}\frac{\partial A}{\partial x}+\theta_i^n\frac{\partial B}{\partial y}\right)\right]U^{n+1}$$

$$=\left[I-\frac{\tau}{2}\left(\theta_i^n\frac{\partial A}{\partial x}+\theta_i^{n+1}\frac{\partial B}{\partial y}\right)\right]U^n \tag{17}$$

Now, during the first sweep of the mesh, Eq. (14a) is solved along alternate constant i lines by tridiagonal matrix inversions. During the second sweep, which covers the remaining constant i lines, the following must be solved:

$$U_{i,j}^{n+1}+\frac{\tau}{4\Delta x}\left[A_{i+1,j}^n U_{i+1,j}^{n+1}-A_{i-1,j}^n U_{i-1,j}^{n+1}\right]=\text{RHS} \tag{18}$$

Since both the right-hand side (RHS) and the expression in brackets in Eq. (18) are known quantities, calculation of $U_{i,j}^{n+1}$ is explicit. Note that half the number of tridiagonal inversions required by Peaceman-Rachford is necessary here. The sacrifice of the order of magnitude of timewise accuracy of the algorithm of Beam and Warming[18] should be weighed against the fast, easy way in which the above method is executed. Finally, A and B are updated after the entire mesh has been covered twice. (This is in keeping with Gourlay[12] who strictly considers hopscotch as a two-step process.) It is anticipated that for problems in which shocks occur artificial viscosity will need to be introduced.

Application to the CNS: ADI-Hopscotch

Returning to the original form of Eq. (1): implicit methods for solving Eq. (1) have either involved linearization[5] of the sort described in the previous section or simply lagging of the nonlinear terms.[20] The operator splitting in the former case enables high temporal accuracy to be attained (see also Ref. 21 where *stability* is discussed in the context of ordinary differential equations). In both instances it is possible to construct ADI-hopscotch schemes, but only from factorization in the manner of Peaceman-Rachford. Once again, some temporal accuracy may have to be conceded in order that a rapid algorithm be produced. (The Douglas-Gunn formulation of splitting is not readily amenable to this treatment.) To illustrate the line of attack the ADI-hopscotch algorithm will be constructed here for the second nonlinearized type of implicit method. Following Samarski and Andreev[22] a slight rearrangement of Eq. (1) is first of all necessary. After manipulation

$$\frac{\partial U}{\partial t}+\frac{\partial F'}{\partial x}+\frac{\partial G'}{\partial y}=\frac{\partial}{\partial x}\left[Q'\frac{\partial}{\partial x}(U/\rho)\right]$$

$$+\frac{\partial}{\partial y}\left[R'\frac{\partial}{\partial y}(U/\rho)\right]+T' \tag{19}$$

$$[F',G']=\begin{bmatrix}\rho & , \rho\\ \rho u & , \rho u\\ \rho v & , \rho v\\ e & , e\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} \tag{20}$$

Table 1 Test cases for hopscotch methods

Case	Comments
1) Flow over wavy wall	Regular square mesh 50×50, boundary conditions specified from analytic solution, $M=0.7$, Courant number 50.
2) Couette flow	Regular square mesh 6×11, $U_0=30.48$ m/s, distance between plates 3.048×10^{-5} m, $\tau=1.16\times 10^{-9}$ s, $M=0.09$, $Re_h=6.2$, periodic boundary conditions in x direction, Courant number 1.
3) Flow between oscillating plate and stationary plate	Regular square mesh 6×11, velocity of oscillating wall $100\sin wt$, distance between plates 3.048×10^{-5} m, $\tau=1.16\times 10^{-8}$ s, $Re_h=6.2$, $M=0.09$, $\omega=2\pi/40$.
4) Shock-wave/boundary-layer interaction, no separation	Pressure ratio 1.2, distance from leading edge to shock $x_s=4.88\times 10^{-2}$ m, $Re_{xs}=2.84\times 10^5$, mesh size (see data of MacCormack[1]) freestream $M=2$, Courant number of 150 possible.
5) Shock-wave/boundary-layer interaction, with separation	Shock angle 32.6 deg, distance from leading edge to shock $x_s=4.88\times 10^{-2}$ m, $Re_{xs}=2.96\times 10^5$, mesh size 32×45, $\Delta x=3.048\times 10^{-3}$ m, $\Delta y_j=1.95\times 10^{-3}\times(1.56\times 10^{-2})^{(33-j)/32}$ for $1\le j\le 33$, $\Delta y_j=1.95\times 10^{-3}$ for $33<j\le 44$, freestream $M=2$, Courant number of 150 possible.

$$[Q',R']=\frac{1}{Re}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4\mu}{3} & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 & \frac{4\mu}{3} & 0 \\ 0 & 0 & 0 & \frac{\gamma\mu}{Pr} & 0 & 0 & 0 & \frac{\gamma\mu}{Pr} \end{bmatrix} \tag{21}$$

and T' contains all the remaining terms. Written as an ADI scheme this becomes

$$\left(I-\frac{\tau}{2}L_y^{(n)}\right)\left(I-\frac{\tau}{2}L_x^{(n)}\right)U^{n+1} = \left(I+\frac{\tau}{2}L_y^{(n)}\right)\left(I+\frac{\tau}{2}L_x^{(n)}\right)U^n+T^{(n)} \tag{22}$$

The difference operators $L_y^{(n)}$ and $L_x^{(n)}$ are central-difference replacements of the convection and viscous/diffusion-type terms, while $T^{(n)}$ replaces the source term T', the index n denoting that time n values of the nonlinear coefficients be utilized in the calculation (i.e., each step is noniterative). From Eq. (22) the building of the ADI-hopscotch equivalent follows the same lines as above: namely, for $j+n$ even solve

$$\left[I-\frac{\tau}{2}L_x^{(n)}\right]U_{i,j}^{n+1}=\left[I+\frac{\tau}{2}L_y^{(n)}\right]U_{i,j}^{(n)}+T_{i,j}^{(n)} \tag{23}$$

and for $j+n$ odd solve

$$\left[I-\frac{\tau}{2}L_y^{(n)}\right]U_{i,j}^{n+1}=\left[I+\frac{\tau}{2}L_x^{(n)}\right]U_{i,j}^{(n)}+T_{i,j}^{(n)} \tag{24}$$

In practice, Eq. (24) is implicit involving solution of tridiagonal systems. Equation (24) becomes explicit; the number of implicit matrix inversions is thus halved.

Application to CNS: A Line-Hopscotch Method

A further possibility of solving Eq. (1) or (19), using another combination of implicit and explicit methods, also arises. Assume that the central differences approximate all spatial derivatives and forward differences the time derivatives. The first traversal of the mesh will use purely explicit formulas only and will be performed in the following way:

1) For $j+n$ even, solve the continuity equation

$$\rho_{i,j}^{n+1}=\rho_{i,j}^{n}-\frac{\tau}{\Delta x}\delta_x(\rho u^{(n)})-\frac{\tau}{\Delta y}\delta_y(\rho v^{(n)}) \tag{25}$$

2) For $j+n$ odd, solve

$$\rho u_{i,j}^{n+1}=\rho u_{i,j}^{(n)}-\frac{\tau\delta_x}{\Delta x}(u\rho u^{(n)})-\frac{\tau}{\Delta y}\delta_y(v\rho u^{(n)}) + \frac{\tau}{\Delta x Re}\delta_x\left[\frac{4}{3}\frac{\mu}{\Delta x}\delta_x\left(\frac{\rho u}{\rho}\right)^n\right] + \frac{\tau}{\Delta y Re}\delta_y\left[\mu\frac{\delta_y}{\Delta y}\left(\frac{\rho u}{\rho}\right)^{(n)}\right]+T_{ij}^{(n)} \tag{26}$$

with a similar equation for ρv and e. δ_x and δ_y are the standard difference operators.

The next stage leads to tridiagonal systems along the remaining $j+n$ lines:

3) For $j+n$ even, solve for ρu (and similarly ρv) using

$$\rho u_{i,j}^{n+1}=\rho u_{i,j}^{(n)}-\frac{\tau}{\Delta x}\delta_x[\tilde{u}(\rho u)^{n+1}]-\frac{\tau}{\Delta y}\delta_y[\tilde{v}(\rho u)]^{n+1} + \frac{\tau}{Re\Delta x}\delta_x\left[\frac{4}{3}\frac{\mu}{\Delta x}\delta_x\left(\frac{\rho u}{\rho}\right)^{n+1}\right] + \frac{\tau}{Re\Delta y}\delta_y\left[\frac{\mu}{\Delta y}\delta_y\left(\frac{\rho u}{\rho}\right)^{(n+1)}\right]+T_{i,j}^{(n)} \tag{27}$$

This formula is implicit and the following points should be noted: $\tilde{u}$ and $\tilde{v}$ should be lagged (i.e., assume the index n). Since ρ^{n+1} is known along $j+n$ even from Eq. (25) it may be used directly in the first viscous term. In the second viscous term, $(\rho u)^{n+1}$ is known along $(j+n)$ odd from Eq. (26), and, as an approximation, ρ may be *estimated* there using an average of the surrounding known density values. (Lagging ρ in this derivative also causes no appreciable error.)

4) Since ρu^{n+1} and ρv^{n+1} are now completely known everywhere in the mesh, the missing density values are updated using

$$\rho_{i,j}^{n+1}=\rho_{i,j}^{n}-\frac{\tau}{\Delta x}\delta_x((\rho u)^{n+1})-\frac{\tau}{\Delta y}\delta_y((\rho v)^{n+1}) \tag{28}$$

for $j+n$ odd, which, in practice, is now completely explicit.

5) Finally, the energy equation is solved along lines of $j+n$ even using updated values of ρu, ρv, and ρ wherever possible. This results in an equation similar to Eq. (27) which is tridiagonal in form.

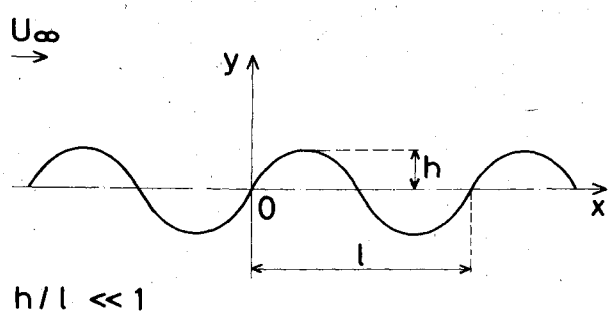

Fig. 2 Subsonic compressible flow over a wave-shaped wall.

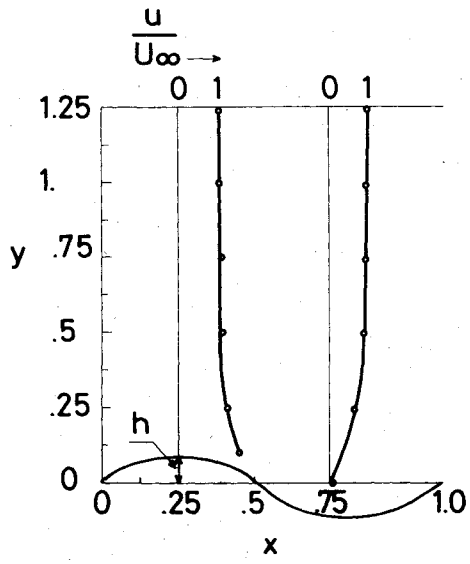

Fig. 3 Velocity profile for inviscid compressible flow over a wavy wall ($M_\infty = 0.7$, $h = 0.1$): --- analytic solution, o o computed results using ADI-hopscotch.

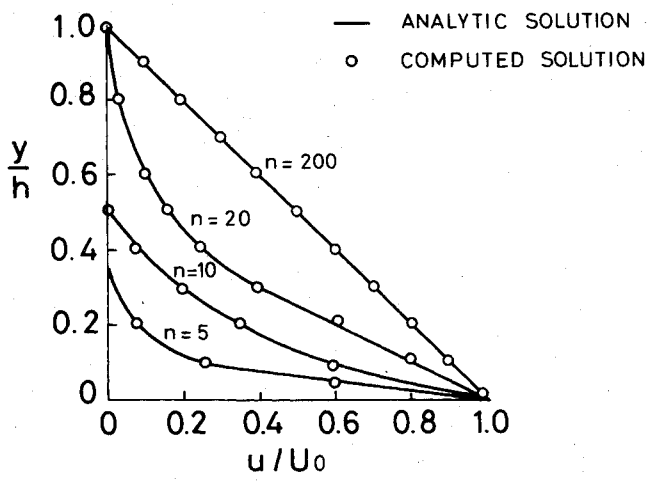

Fig. 4 Development of Couette flow.

This new method is a variation on the line-hopscotch method (see Ref. 12) and, whereas a regular ADI method will require $4(NX+NY)$ tridiagonal inversions (where NX and NY are the number of i and j lines, respectively), the current method asks for $3NY$ (or $3NX$)—a considerable improvement. Although the description above has implied that this method is noniterative, the possibility of iterating should not be discounted. Thus, line iteration in which the nonlinear occurrences of u, v, and ρ [e.g., in Eq. (27)] are updated until convergence is attained *for a given time* can be carried out. While ostensibly increasing accuracy, a payoff should be made relative to computing time.

Numerical Experiments

A number of numerical experiments were carried out to test the above-described methods and to get some measure of their performance capability. Comparisons between the results computed here and those available in the literature are used to estimate accuracy. Table 1 summarizes the relevant data and comments.

Euler's Equations

As most nonpotential calculations of the flowfield about airfoil configurations appear to make use of the CNS, only a single simple example of a solution of Euler's equations was selected for illustrative purposes. The problem concerns two-dimensional flow over a wave-shaped wall (see Fig. 2). The "waviness" of the wall is slight ($h/l \ll 1$) and, for simplicity, the case of subsonic flow was selected. Using linearized potential theory it can be shown that, for the steady-state problem, the velocity field is given by

$$\left(\frac{u}{U_\infty}, \frac{v}{U_\infty}\right) = (1,0) - \nabla\left(\frac{h}{\sqrt{1-M_\infty^2}} \cos\frac{2\pi x}{l} \exp\left(\frac{-2\pi y}{l}\sqrt{1-M_\infty^2}\right)\right) \tag{29}$$

where M_∞ is freestream Mach number and U_∞ the freestream velocity. The steady-state computed solution using the ADI-hopscotch method [Eq. (17)] is shown and compared to the analytic solution [Eq. (29)] in Fig. 3 and indicates good agreement. The solution obtained using an ordinary ADI method, as in Eq. (14), gave identical results but required almost twice as much computer time to reach the steady state. In addition, it is interesting to note that it was possible to work with a Courant number some 50 times larger than would be allowable if an *explicit* scheme had been used with the same spatial grid. Of course, this does not necessarily imply that the correct *transient* solution is thus obtained. Considerations of temporal accuracy would still necessitate use of a "reasonable" time step (see also MacCormack and Lomax[7]).

CNS Equations

The test cases used by Beam and Warming[5] for their implicit factored scheme were investigated here using the various hopscotch methods described above.

Consider, first of all, the (incompressible) flow between two infinite parallel plates, distance h apart, the lower of which has a velocity U_0 parallel to itself (Couette flow). Figure 4 shows computed transient velocity profiles as well as the analytic solution given by Schlichting.[23] For this case it was found that if the Courant number exceeded unity the transient solution was incorrect, although the steady-state solution gave good agreement. For the given conditions (i.e., Courant number 1) the temporal and spatial correlation is very satisfactory.

Although almost identical results were yielded by both the ADI-hopscotch and the regular ADI method, the former was superior in terms of execution time—a factor of almost two being involved.

Now suppose that the lower wall oscillates in its own plane in a sinusoidal fashion: $U_0 \sin wt$. Both hopscotch methods yielded identical results (see Fig. 5, where the solution reported by Schlichting[23] is also plotted). This calculation serves to illustrate that *under certain circumstances* an accurate temporal solution can be computed using Courant numbers greater than unity (here 10). Beam and Warming[5] were also able to establish this conclusion. However, the faster performance of the hopscotch methods together with their ADI-like capabilities lends more confidence to their usefulness.

The final, rather more stringent, test cases involved the computation of the interaction of an oblique shock wave with a laminar boundary layer that has developed on a flat plate.

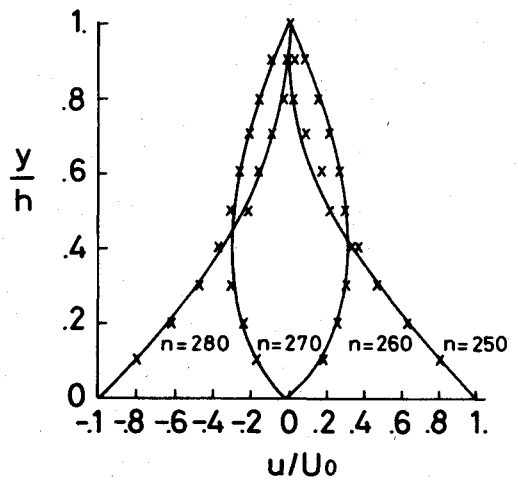

Fig. 5 Velocity profiles between a sinusoidally oscillating wall and a stationary wall.

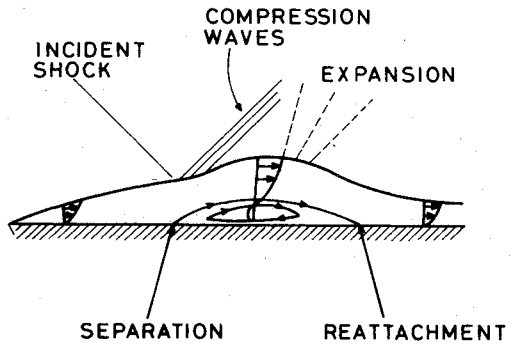

Fig. 6 Sketch of shock-wave/boundary-layer interaction.

For this more difficult case only the new line hopscotch method was utilized (although, in view of the other test cases, it might be anticipated that ADI-hopscotch would behave similarly). The first calculation performed paralleled that of MacCormack[1] in which the domain of solution is extended far out into the essentially inviscid freestream. For a sufficiently strong shock separation will occur in the boundary layer on the wall (see Fig. 6). The boundary conditions imposed at the wall were those of no-slip and zero-normal gradients of temperature and pressure. The shock angle is set by appropriate control of postshock conditions. Note that the only comparison possible in this case is for the steady-state solution. Considering MacCormack's calculation in which no separation occurs (see Ref. 1): a comparison between the experimental,[24] MacCormack's, and the line hopscotch results are shown in Fig. 7. On the whole, the agreement with the experiments is fair for the skin friction coefficient and excellent for the surface pressure.

The second case of shock/boundary-layer interaction was that of Beam and Warming[5] in which separation occurs. The mesh size was uniform in the x direction but stretched normal to the wall so as to be tighter in the boundary-layer region where more accurate resolution is necessary. A comparison with profiles given in Ref. 5 is presented in Figs. 8 and 9. There is little significant difference between the results, although the current method somewhat diminishes the back flow velocity.

For both the latter calculations a Courant number of 150 was possible and any spurious temporal oscillations died out as the steady state was achieved. However, as mentioned earlier, if rapid temporal changes are to be computed the Courant number *must* be judiciously lower.

With regard to the hopscotch methods, it was found in all cases that the tridiagonal inversions along lines of constant j were to be preferred to lines traversing the main direction of

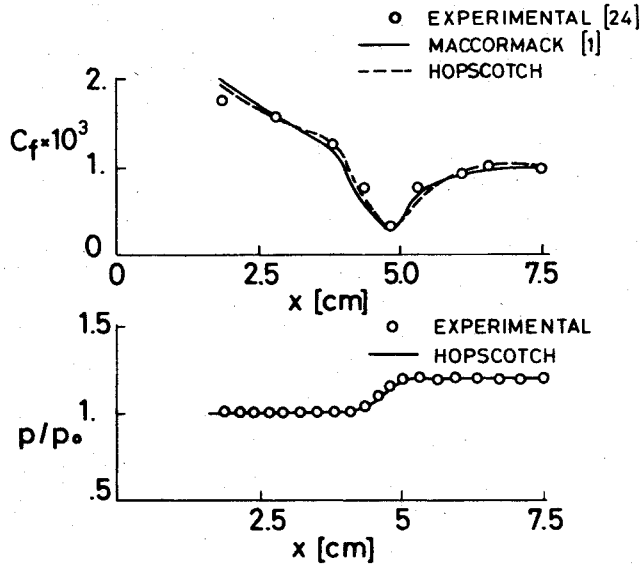

Fig. 7 Skin friction coefficient and surface pressure profiles.

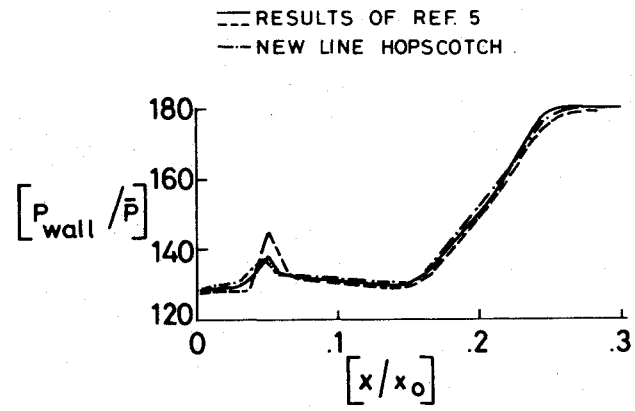

Fig. 8 Wall pressure distribution ($x_0 = 3.048 \times 10^{-1}$ m; $\bar{P} = 4.4$ N).

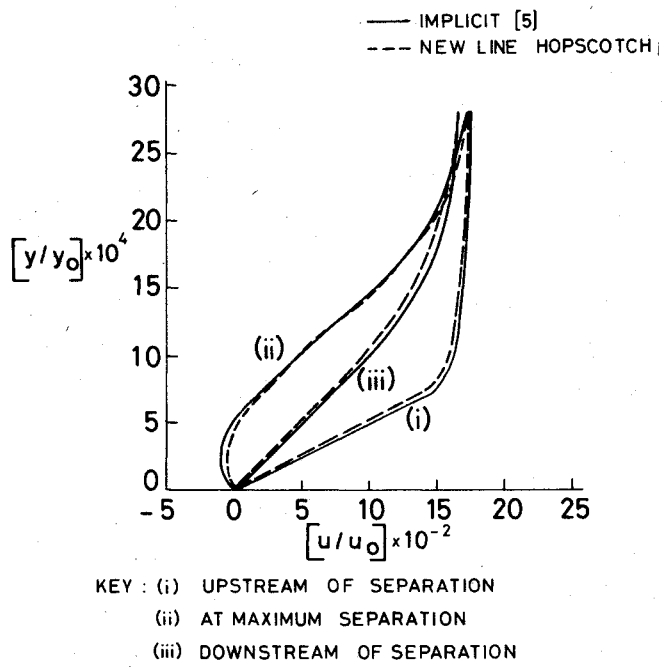

Fig. 9 Velocity profiles through the boundary layer for the case of shock boundary layer interaction ($y_0 = 3.048 \times 10^{-1}$ m, $u_0 = 3.048 \times 10^{-1}$ m/s).

flow. In the latter case resolution was not satisfactory, presumably a manifestation of the analysis of Morris and Nicoll[25] was present.

Finally, for the shock-wave/boundary-layer interaction no direct comparison of computing times was possible, but about 4-7 min of CPU time was necessary.

Conclusions

The construction and use of some hopscotch-type methods has been demonstrated in the context of the conservative law form of the compressible Navier-Stokes equations. Computed results highlight the reasonable accuracy and speed of these hybrid methods. Furthermore, good temporal results and the ability to handle the stringent problem of shock-wave/boundary-layer interaction augur well for application of the methods, coupled with automatic mesh generation techniques, to tackle transonic flow calculations over realistic airfoil configurations.

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