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Theory of Vibrating Diaphragm-Type Pressure Sensor

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In the present paper, we discuss the theory of vibrating diaphragm-type pressure sensors. Starting with the von Kármán equation, the series solution of the problem of nonlinear bending of circular thin plates under the action of uniform pressure and uniform initial edge tension (preloaded) at the periphery is obtained. Based on this solution, small vibrations in the vicinity of the static equilibrium configuration are discussed. Here, a series method is suggested in which the characteristic value problem is reduced to a third-order algebraic characteristic value problem with explicit expression, and thus the characteristic value problem of axisymmetric free vibration is solved precisely. The characteristic curve of natural frequency vs pressure is obtained, which is of great interest for practical engineering applications.

Nomenclature

a = radius
D = flexural rigidity, $Eh^3/12(1-\nu^2)$
E = Young's modulus
F = stress function
F_t = stress function referred to static equilibrium configuration
h = thickness
m = mass density per unit area
N_r = radial membrane force per unit length
N_θ = circumferential membrane force per unit length
$N_{r\theta}$ = membrane shearing force per unit length
$N_t(x)$ = reduced amplitude of radial membrane force per unit length, $12(1-\nu^2)(a^2T_t/Eh^3)$
$\bar{N}_0$ = reduced initial radial membrane force per unit length, $12(1-\nu^2)\,(a^2N_r^{(0)}/Eh^3)$
Q = reduced external transverse load, $\frac{3}{4}(1-\nu^2)\sqrt{3(1-\nu^2)}\,(q_na^4/Eh^4)$
$R_t(r)$ = amplitude of deflection referred to static equilibrium configuration
r, θ = polar coordinates
$S_0(y)$ = reduced radial membrane force per unit length, $[3(1-\nu^2)a^2N_r^{(0)}/Eh^3]\,y$
$S_0^*(y)$ = $S_0(y) - \delta y$
$T_t(r)$ = amplitude of radial membrane force per unit length referred to static equilibrium configuration
t_n = external initial radial tension
u_0 = initial radial displacement
$\tilde{u}_0$ = reduced initial radial displacement, $3(1+\nu)au_0/h^2$
$W_t(x)$ = reduced amplitude of deflection referred to static equilibrium configuration, $2\sqrt{3(1-\nu^2)}\,(R_t/h)$
$W(y)$ = dimensionless deflection function
W_0 = reduced initial deflection, $\sqrt{3(1-\nu^2)}\,(w_0/h$
$w(r)$ = deflection
w_0 = initial deflection
$\bar{w}_0$ = reduced initial deflection, $2\sqrt{3(1-\nu^2)}\,(w_0/h)$
x = reduced radial coordinate, r/a
y = reduced radial coordinate, $(r/a)^2$
α = reduced angular frequency, $(ma^4/D)\omega^2$
δ = reduced external initial radial tension, $3(1-\nu^2)a^2t_n/Eh^3$
λ = constant related to different boundary conditions
μ = constant related to different boundary conditions
ν = Poisson's ratio
ϕ_0 = reduced initial deflection slope, $y(\mathrm{d}W_0/\mathrm{d}y)$
ω = natural frequency
ω^* = dimensionless natural frequency

I. Introduction

IN this paper, we discuss the theory of vibrating diaphragm-type pressure sensors, consisting of a structure of high precision and simple design, as shown in Fig. 1. The variation of equilibrium configuration of a circular diaphragm (i.e., a circular thin plate) under the action of uniform external pressure and uniform initial edge tension (preloaded) at the periphery causes a variation of natural frequency. Making use of the corresponding relation between natural frequency and static load, and then measuring the natural frequency of the circular diaphragm by an arrangement of electromagnetically stimulated vibration and vibration pickup, one obtains the magnitude of the external pressure. Pressure sensors made of vibrating diaphragms have become the core element of digital-type atmosphere numerical data computers used in certain types of aircraft.[1]

According to Ref. 1, research into the mechanics of vibrating diaphragm-type pressure sensors is rather inadequate. Yamaki et al.[2,3] studied the large-amplitude nonlinear free-vibration problem of a circular thin plate with initial deflection and under initial tension, in which the theoretical analysis is first based on the third-order Galerkin method and therefore the method of harmonic wave equilibrium. Elishakoff et al.[4] studied nonlinear small vibrations in the vicinity of the static equilibrium configuration with initial imperfections of a curved plate. In the present paper, we use the von Kármán equation to treat the nonlinear static equilibrium problem of a

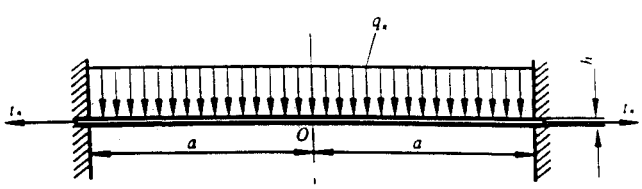

Fig. 1 Circular thin plate.

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circular thin plate under uniform initial tension at the periphery and uniform tranverse pressure. Based on its exact solution,[5] we discuss small free vibrations of the circular thin plate. For the design of harmonic vibration elastic elements, the fundamental natural frequency is of prime importance. To analyze precisely the frequency and the vibration modal function, we use a power series method to change the dynamic characteristic value problem into an algebraic characteristic value problem.[6] Introducing certain transformations, the order of the homogeneous algebraic characteristic equation is only related to the boundary conditions and can be expressed in explicit form. Therefore, the amount of calculation is decreased greatly. The computational method and results obtained can be applied conveniently in the design of this kind of instrument.

II. Fundamental Equations

A. Geometric Nonlinear Static Boundary Value Problem

Considering a circular thin plate (Fig. 1) with radius a, thickness h under preloaded uniform external tension t_n, which produces a radial displacement u_0 at the edge, as the plate is subjected to a uniformly distributed transverse pressure q_n, we have, from von Kármán's thin plate theory, the following nonlinear differential equations in dimensionless form:

$$y^2 \frac{d^2\phi_0}{dy^2} = \phi_0(y)[\delta y + S_0^*(y)] + y^2 Q \tag{1}$$

$$y^2 \frac{d^2 S_0^*}{dy^2} = -\tfrac{1}{2}\phi_0^2(y) \qquad 0<y<1 \tag{2}$$

The boundary conditions are, where $y = 0$,

$$\phi_0 = 0 \tag{3a}$$

$$S_0^*(y) = 0 \tag{3b}$$

and where $y = 1$,

$$\phi_0(y) = \frac{\lambda}{\lambda - 1}\frac{d\phi_0(y)}{dy} \tag{4a}$$

$$S_0^*(y) = \frac{\mu}{\mu - 1}\frac{dS_0^*(y)}{dy} \tag{4b}$$

where λ and μ are constants related to different boundary conditions,[7] and the dimensionless quantities used are

$$y = \left(\frac{r}{a}\right)^2, \qquad W_0 = \sqrt{3(1-\nu^2)}\,\frac{w_0}{h}, \qquad \phi_0 = y\frac{dw_0}{dy}$$

$$\delta = \frac{3(1-\nu^2)a^2 t_n}{Eh^3}, \qquad S_0(y) = \frac{3(1-\nu^2)a^2 N_r^{(0)}}{Eh^3}\,y$$

$$Q = \tfrac{3}{4}(1-\nu^2)\sqrt{3(1-\nu^2)}\,\frac{q_n a^3}{Eh^4}, \qquad \tilde{u}_0 = \frac{3(1+\nu)au_0}{h^2}$$

where w is the deflection and

$$S_0(y) = S_0^*(y) + \delta y \tag{5}$$

$$\delta = \tilde{u}_0 \tag{6}$$

Obviously, $S_0^*(y)$ corresponds to the internal membrane force during the thin plate in bending deformation. The subscript 0 denotes quantities in static nonlinear bending.

B. Small Vibration in the Vicinity of the Nonlinear Static Equilibrium Configuration

In von Kármán's plate equations, considering only the transverse inertial force, the differential equations of large-amplitude vibration are[8]

$$\nabla^2\nabla^2 w(r,\theta,t) = \frac{q_n}{D} - \frac{m}{D}\frac{\partial^2 w(r,\theta,t)}{\partial t^2} + \frac{h}{D}L(w,F) \tag{7a}$$

$$\nabla^2\nabla^2 F(r,\theta,t) = -\frac{E}{2}L(w,w) \tag{7b}$$

where $\nabla^2 = (\partial^2/\partial r^2) + (1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial\theta^2)$ is the Laplace operator in polar coordinates. We then have

$$N_r = h\left(\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2} + \frac{1}{r}\frac{\partial F}{\partial r}\right)$$
$$N_\theta = h\frac{\partial^2 F}{\partial r^2}, \qquad N_{r\theta} = -h\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right) \tag{8}$$

and

$$L(w,F) = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\right) + \left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\right)\frac{\partial^2 F}{\partial r^2}$$
$$- 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right) \tag{9}$$

Let

$$w(r,\theta,t) = w_0(r) + w_t(r,\theta,t) \tag{10a}$$

$$F(r,\theta,t) = F_0(r) + F_t(r,\theta,t) \tag{10b}$$

where $w_0(r)$ and $F_0(r)$ are static state solutions, which are satisfied by the dimensionless equations (1–4). Substituting Eqs. (10) into Eqs. (7), we obtain the large-amplitude free-vibration equations in the vicinity of the static equilibrium configuration, namely,

$$\nabla^2\nabla^2 w_t(r,\theta,t) = -\frac{m}{D}\frac{\partial^2 w_t}{\partial t^2} + \frac{h}{D}[L(w_0,F_t)$$
$$+ L(w_t,F_0) + L(w_t,F_t)] \tag{11a}$$

$$\nabla^2\nabla^2 F_t(r,\theta,t) = -\frac{E}{2}[2L(w_0,w_t) + L(w_t,w_t)] \tag{11b}$$

For small-amplitude vibration, i.e., for $|w_t| \ll 1$, the nonlinear terms can be neglected, and we obtain the following free-vibration equations in the vicinity of the static equilibrium configuration:

$$\nabla^2\nabla^2 w_t(r,\theta,t) = -\frac{m}{D}\frac{\partial^2 w_t}{\partial t^2} + \frac{h}{D}[L(w_0,F_t) + L(w_t,F_0)] \tag{12a}$$

$$\nabla^2\nabla^2 F_t(r,\theta,t) = -EL(w_0,w_t) \tag{12b}$$

Here we are only interested in the fundamental natural frequency. Thus we study an axisymmetrical free vibration whose dynamic characteristic equations can be reduced into

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{dR_t(r)}{dr} = \frac{m\omega^2}{D}R_t(r)$$
$$+ \frac{h}{D}\frac{1}{r}\frac{d}{dr}\left[rT_t(r)\frac{dw_0(r)}{dr} + rN_r^{(0)}\frac{dR_t(r)}{dr}\right] \tag{13}$$

$$r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}[r^2 T_t(r)] = -Eh\frac{dw_0(r)}{dr}\frac{dR_t(r)}{dr} \tag{14}$$

where

$$w_t(r, t) = R_t(r)\sin \omega t \tag{15a}$$

$$N_r^{(t)} = \frac{1}{r}\frac{\partial F_t(r, t)}{\partial r} = T_t(r)\sin \omega t \tag{15b}$$

Introducing the following dimensionless variables

$$x = \frac{r}{a}, \qquad W_t(x) = 2\sqrt{3(1-\nu^2)}\,\frac{R_t}{h}, \qquad \bar{w}_0 = 2\sqrt{3(1-\nu^2)}\,\frac{w_0}{h}$$
$$N_t(x) = 12(1-\nu^2)\frac{a^2 T_t}{Eh^3}, \qquad \bar{N}_0 = 12(1-\nu^2)\frac{a^2 N_r^{(0)}}{Eh^3}$$

we obtain characteristic equations of axisymmetrical small vibration in dimensionless form

$$L_x[W_t(x)] = \alpha W_t(x) + \frac{1}{x}\frac{d}{dx}\left(xN_t\frac{d\bar{w}_0}{dx} + x\bar{N}_0\frac{dW_t}{dx}\right) \tag{16}$$

$$x\frac{d}{dx}\frac{1}{x}\frac{d}{dx}[x^2 N_t(x)] = -\frac{d\bar{w}_0}{dx}\frac{dW_t}{dx} \qquad 0<x<1 \tag{17}$$

The corresponding boundary conditions are, where $x = 0$,

$$\frac{dW_t}{dx} = 0 \tag{18a}$$

$$\bar{N}_t \text{ finite} \tag{18b}$$

and where $x = 1$,

$$W_t = 0 \tag{19a}$$

$$(\lambda - 2)\frac{dW_t}{dx} - \lambda\frac{d^2 W_t}{dx^2} = 0 \tag{19b}$$

$$N_t(x) + \frac{\mu}{2}\frac{dN_t}{dx} = 0 \tag{19c}$$

where

$$\alpha = \frac{ma^4}{D}\omega^2$$
$$L_x = \frac{1}{x}\frac{d}{dx}x\frac{d}{dx}\frac{1}{x}\frac{d}{dx}x\frac{d}{dx} \tag{20}$$

Comparing dimensionless quantities in the previously mentioned two sets, we have

$$y = x^2, \qquad \bar{w}_0 = 2W_0, \qquad \bar{N}_0 = \frac{4S_0(y)}{y} \tag{21}$$

III. Series Method Solution of Nonlinear Static Bending Problem

Here we use the series method to solve the static boundary value problem, i.e., we choose, where $y \in [0, 1]$,

$$\phi_0(y) = \sum_{i=1}^{\infty} A_i y^i \tag{22a}$$

$$S_0^*(y) = \sum_{i=1}^{\infty} B_i y^i \tag{22b}$$

Substituting Eqs. (22) into Eqs. (1) and (2), we obtain the following recurrence formulas for the coefficients A_i and B_i to be determined:

$$A_i = \frac{1}{i(i-1)}\left[\delta A_{i-1} + \sum_{j=1}^{i-1} A_j B_{i-j} + Q\delta^*(i-2)\right] \tag{23a}$$

$$B_i = -\frac{1}{2i(i-1)}\sum_{j=1}^{i-1} A_j A_{i-j} \qquad i = 2, 3, 4, \ldots \tag{23b}$$

where

$$\delta^*(i-2) = \begin{cases} 1 & i = 2 \\ 0 & i \neq 2 \end{cases}$$

Obviously, only A_1 and B_1 are independent constants, which can be determined from boundary conditions of Eqs. (4) as

$$\sum_{i=1}^{\infty}(\lambda - 1 - \lambda i)A_i = 0 \tag{24a}$$

$$\sum_{i=1}^{\infty}(\mu - 1 - \mu i)B_i = 0 \tag{24b}$$

Equations (24) are a set of nonlinear algebraic equations with respect to A_1 and B_1 under given Q and δ, and it is not difficult to find their roots by the Newton iteration method.

After getting solutions for $\phi_0(y)$ and $S_0^*(y)$, we obtain

$$\frac{d\bar{w}_0}{dx} = \sum_{i=1}^{\infty} A_i^* x^{2i-1} \tag{25}$$

$$N_0(x) = \sum_{i=0}^{\infty} B_i^* x^{2i} \tag{26}$$

where

$$A_i^* = 4A_i, \qquad i = 1, 2, \ldots \tag{27}$$

$$B_0^* = 4(\delta + B_1), \qquad B_i^* = 4B_{i+1}, \qquad i = 1, 2, \ldots \tag{28}$$

The deflection function is

$$W(y) = -\int_y^1 \frac{1}{\xi}\phi(\xi)\,d\xi \tag{29}$$

IV. Solution for the Dynamic Characteristic Value

A. Series Method

According to the series form of the static state solutions (25) and (26), it is not difficult to confirm the series form of the characteristic functions $W_t(x)$ and $N_t(x)$, which are

$$W_t(x) = \sum_{i=0}^{\infty} a_i x^{2i} \tag{30}$$

$$N_t(x) = \sum_{i=0}^{\infty} b_i x^{2i} \tag{31}$$

where a_i and b_i are constants to be determined. Substituting Eqs. (30) and (31) as well as (25) and (26) into the characteristic equations (16) and (17), and comparing coefficients of x of the same power, we obtain the recurrence formulas for the coefficients a_i and b_i:

$$b_i = -\frac{1}{2i(i+1)}\sum_{j=1}^{\infty} j a_j A_{i-j+1}^* \qquad i = 1, 2, \ldots \tag{32}$$

$$a_2 = \frac{1}{64}[\alpha a_0 + 2(2a_1 B_0^* + A_1^* b_0)] \tag{33a}$$

$$a_3 = \frac{1}{24^2}[\alpha a_1 + 4(2a_1 B_1^* + A_1^* b_1 + 4a_2 B_0^* + A_2^* b_0)] \tag{33b}$$

$$a_{i+2} = \frac{1}{[16(i+1)^2(i+2)^2]}\left[\alpha a_i + 2(i+1)\sum_{j=1}^{i+1}(2ja_j B_{i-j+1}^* + A_j^* b_{i-j+1})\right], \qquad i = 2, 3, \ldots \tag{33c}$$

Obviously in the recurrence formulas (17) and (18) only the unknowns a_0, a_1, b_0, and α are independent, which must satisfy boundary conditions (18) and (19). Since Eqs. (18) are identically satisfied, we obtain from Eqs. (19)

$$a_0 + a_1 + \sum_{i=2}^{\infty} a_i = 0 \tag{34}$$

$$a_1 + \sum_{i=2}^{\infty}(\lambda i + 1 - \lambda)ia_i = 0 \tag{35}$$

$$b_0 + \sum_{i=1}^{\infty}(1 + i\mu)b_i = 0 \tag{36}$$

From Eqs. (32–36), we see that, as $a_0 = a_1 = b_0 = 0$, Eqs. (34–36) are identically satisfied, i.e., zero solutions $W_t(x) \equiv 0$ and $N_t(x) \equiv 0$ are solutions of the dynamic characteristic value problem. But we are interested in their nonzero solutions, i.e., solutions of $W_t(x) \not\equiv 0$ and $N_t(x) \not\equiv 0$, which correspond to a_0, a_1, and b_0 not being zero simultaneously. Even though Eqs. (32–36) construct the algebraic characteristic value problem, it will require a lot of labor and trouble, searching for the characteristic value of α by directly using these equations, because if one uses $[a_0, a_1, b_0]$ as a characteristic vector, one must make repeated adjustments of a_0, a_1, and b_0 and of the characteristic value α, i.e., simultaneous adjustments of the characteristic vector $[a_0, a_1, b_0]$ and the characteristic value α, to find nonzero solutions to satisfy Eqs. (34–36). Obviously, this is difficult to do in this fashion. Another way is to place all a_i and b_i as elements of a characteristic vector, together with Eqs. (32–36), and to construct a linear characteristic value problem of infinite order; then according to the condition that the determinant of its coefficients matrix (square matrix) of infinite order must vanish, we can construct the algebraic equation for determining the characteristic value α. Obviously, the amount of calculation work to be done using this method is increasing enormously with increasing numbers of series terms. To conquer this inadequateness, we shall use a simple transformation to render the aforementioned characteristic value problem in explicit form.

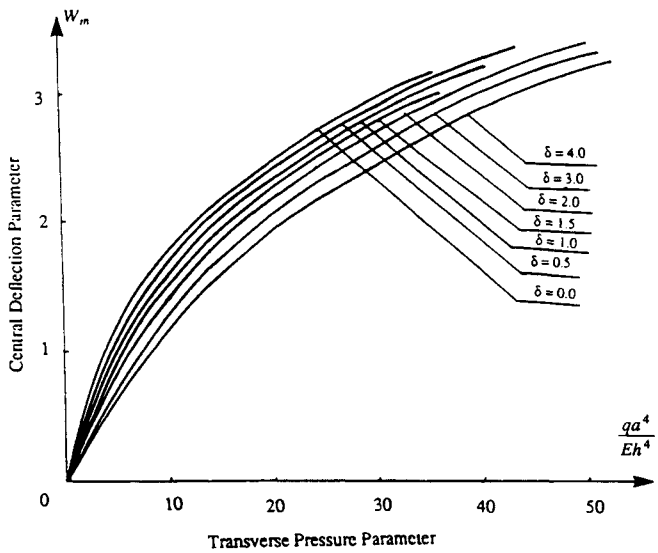

Fig. 2 Characteristic curves of central deflection vs transverse pressure for the circular thin plate.

B. Explicit Form of the Algebric Characteristic Value Problem

From Eqs. (32) and (33), one can readily see that all coefficients of a_i and b_i ($i = 0, 1, 2, \ldots$) can be expressed by homogeneous linear functions of a_0, a_1, and b_0, i.e.,

$$\begin{aligned} a_i &= e_{i1}(\alpha)a_0 + e_{i2}(\alpha)a_1 + e_{i3}(\alpha)b_0 \\ b_i &= g_{i1}(\alpha)a_0 + g_{i2}(\alpha)a_1 + g_{i3}(\alpha)b_0 \quad (i = 0, 1, 2, \ldots) \end{aligned} \tag{37}$$

Here $e_{ir} = e_{ir}(\alpha)$ and $g_{ir} = g_{ir}(\alpha)$ ($r = 1, 2, 3$) are only functions of α and are independent of a_0, a_1, and b_0. According to this property of independence, we obtain

$$\begin{aligned} &e_{01} = 1, && e_{02} = e_{03} = 0 \\ &e_{11} = 0, && e_{12} = 1, && e_{13} = 0 \\ &g_{01} = 0, && g_{02} = 0, && g_{03} = 1 \end{aligned} \tag{38}$$

Substituting Eqs. (37) into Eqs. (32) and (33), from the property of independence of a_0, a_1, and b_0, we obtain the following recurrence formulas for e_{ir} and g_{ir}:

$$g_{ir} = -\frac{1}{2i(i+1)}\sum_{j=1}^{i} je_{jr}A^*_{i-j+1}, \quad i = 1, 2, \ldots, \quad r = 1, 2, 3 \tag{39}$$

$$\begin{aligned} &e_{21} = \frac{\alpha}{64}, \qquad e_{22} = \frac{B_0^*}{16}, \qquad e_{23} = \frac{A_1^*}{32} \\ &e_{31} = (A_1^* g_{11} + 4B_0^* e_{21})/(6 \times 24) \\ &e_{32} = [\alpha + 8B_1^* + 4(A_1^* g_{12} + 4B_0^* e_{22})]/24^2 \\ &e_{33} = (A_1^* g_{13} + 4e_{23}B_0^* + A_2^*)/(24 \times 6) \end{aligned} \tag{40}$$

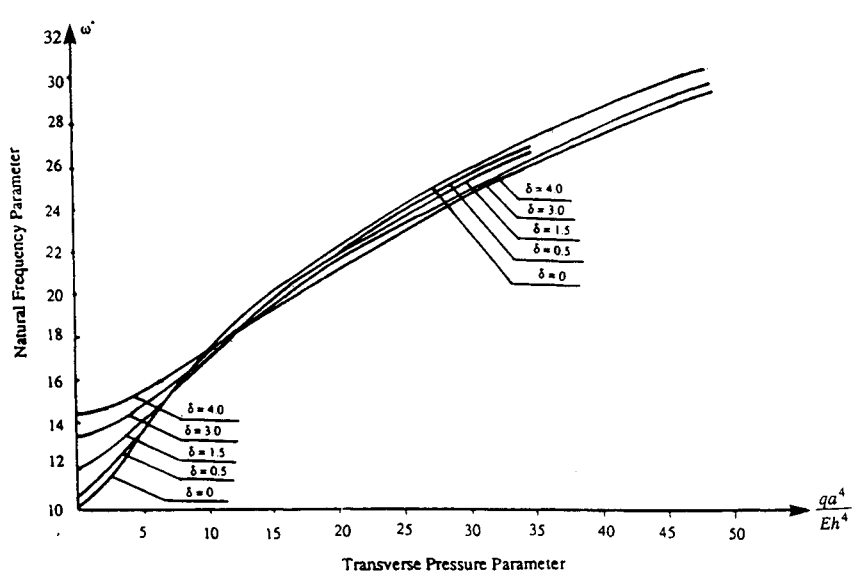

Fig. 3 Characteristic curves of fundamental natural frequency vs pressure for the circular thin plate.

$$e_{(1+2),r} = \left[\alpha e_{ir} + 2(i+1)\sum_{j=1}^{i+1}(2jB^*_{i-r+1}e_{ir} + A^*_j g_{i-j+1,r}) \right] \Big/ [4(i+1)(i+2)]^2 \qquad i = 2, 3, 4, \ldots, \quad r = 1, 2, 3 \tag{41}$$

For a given α, we can find out the whole e_{ir} and g_{ir} by recurrence formulas (38–41). Let

$$\begin{aligned} &f_{11} = 1 + \sum_{i=2}^{\infty} e_{i1}, \quad f_{12} = 1 + \sum_{i=2}^{\infty} e_{i2}, \quad f_{13} = \sum_{i=2}^{\infty} e_{i3} \\ &f_{21} = \sum_{i=2}^{\infty}(\lambda i - \lambda + 1)ie_{i1}, \quad f_{22} = 1 + \sum_{i=2}^{\infty}(\lambda i - \lambda + 1)ie_{i2} \\ &f_{23} = \sum_{i=2}^{\infty}(\lambda i - \lambda + 1) + ie_{i3}, \quad f_{31} = \sum_{i=1}^{\infty}(1 + i\mu)g_{i1} \\ &f_{32} = \sum_{i=1}^{\infty}(1 + i\mu)g_{i2}, \quad f_{33} = 1 + \sum_{i=1}^{\infty}(1 + i\mu)g_{i3} \end{aligned} \tag{42}$$

Obviously $f_{ij} = f_{ij}(\alpha)$ for $i, j = 1, 2, 3$, and the third-order algebraic characteristic value proplem in explicit form, which takes the place of Eqs. (34–36), is

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_0 \end{bmatrix} = 0 \tag{43}$$

Then the necessary and sufficient condition for a nonzero solution is

$$R(\alpha) = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} = 0 \tag{44}$$

This is the characteristic value equation for determining α, which is a nonlinear algebraic equation with respect to α. Using the Newton iteration method, one can find the corresponding characteristic value α under the action of a static load. Then from Eq. (43), one can confirm the characteristic vector $[a_0, a_1, b_0]^T$; moreover, one can obtain the characteristic functions $W_t(x)$ and $N_t(x)$. Here the derivative $R'(\alpha)$ can be obtained by calculating the derivative with respect to α of recurrence formulas (38–41) and (43).

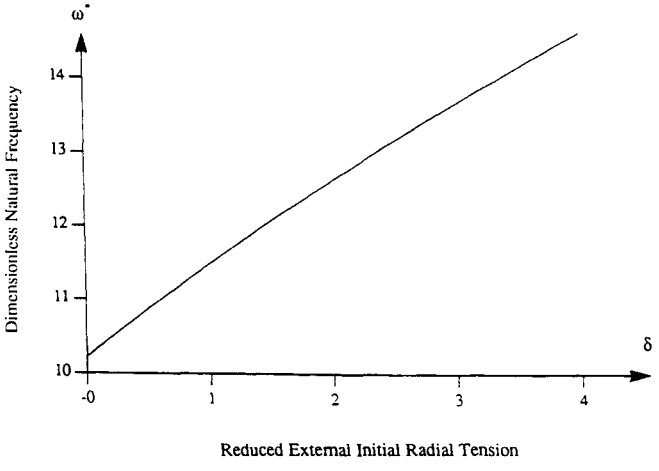

Fig. 4 Characteristic curves of fundamental natural frequency vs reduced external initial radial tension ($Q = 0$).

Table 1 Characteristic relations between δ and ω^* of a plate under initial radial tension

δ	ω^*	δ	ω^*	δ	ω^*	δ	ω^*
0.00	10.215830	1.05	11.545620	2.05	12.672560	3.05	13.699270
0.05	10.283400	1.10	11.604830	2.10	12.726140	3.10	13.748390
0.10	10.350500	1.15	11.663690	2.15	12.779400	3.15	13.797320
0.15	10.417130	1.20	11.722240	2.20	12.832420	3.20	13.846060
0.20	10.483290	1.25	11.780480	2.25	12.885170	3.25	13.896430
0.25	10.549010	1.30	11.838400	2.30	12.937690	3.30	13.942980
0.30	10.614290	1.35	11.896020	2.35	12.989990	3.35	13.991170
0.35	10.679130	1.40	11.953320	2.40	13.042060	3.40	14.039100
0.40	10.743550	1.45	12.010330	2.45	13.093900	3.45	14.087000
0.45	10.807560	1.50	12.067050	2.50	13.145520	3.50	14.134650
0.50	10.871150	1.55	12.123470	2.55	13.196910	3.55	14.182120
0.55	10.934340	1.60	12.179610	2.60	13.248100	3.60	14.229420
0.60	10.997130	1.65	12.235470	2.65	13.299050	3.65	14.276530
0.65	11.059550	1.70	12.291050	2.70	13.349790	3.70	14.323500
0.70	11.121560	1.75	12.346350	2.75	13.400340	3.75	14.370290
0.75	11.183220	1.80	12.401390	2.80	13.450670	3.80	14.416900
0.80	11.244500	1.85	12.456160	2.85	13.500780	3.85	14.463370
0.85	11.305440	1.90	12.510680	2.90	13.550700	3.90	14.509640
0.90	11.365990	1.95	12.564920	2.95	13.600420	3.95	14.555700
0.95	11.426210	2.00	12.618920	3.00	13.649930	4.00	14.601740
1.00	11.486090						

After obtaining α, the natural frequency of a circular thin plate under static load is

$$\omega = \frac{\omega^*}{a^2}\sqrt{\frac{D}{m}} \tag{45}$$

where

$$\omega^* = \sqrt{\alpha} \tag{46}$$

is the dimensionless natural frequency. To obtain the minimum characteristic value α (or fundamental natural frequency), in the process of searching for α by the Newton iteration method, the initial value of iteration used is obtained from an approximation by Galerkin method whose vibration model function (i.e., test function) corresponds to a single wave solution, i.e., corresponding to the form of the linear solution of uniform load.

V. Results and Discussion

According to the calculation program just described, we obtain the static solution of the nonlinear bending problem of the clamped circular thin plate with initial tension under uniform tranverse load and the natural frequency of small vibration in the vicinity of the static equilibrium configuration. The stiffness curve as function of central deflection is shown in Fig. 2. From it, we see that initial tension makes the central deflection decrease. The curve of fundamental natural frequency vs transverse pressure is shown in Fig. 3, from which we find that the natural frequency is increasing as the external load is increasing and the effect of external edge tension on the frequency is different in the range of different transverse loads. As $(qa^4/Eh^4) < 8$, the fundamental natural frequency increases when the initial tension δ rises, which is because an increase of initial tension causes the stiffness of the plate to increase. In the range $(qa^4/Eh^4) > 15$, the fundamental natural frequency decreases as the initial tension rises, which is because of the nonlinear terms in the static problem of the plate. The increase of initial tension makes the deflection of the transverse bending decrease, which causes a greater effect on the natural frequency than the effect of deformation in the plane, which produces the initial tension. In the range $8 < (qa^4/Eh^4) < 15$, an effect of δ on the natural frequency is not evident.

As $Q = 0$, we can obtain the curve of δ vs ω^* as shown in Fig. 4 whose numerical data are shown in Table 1, which can

be used to determine initial tension measured by frequency response. It is very important for the design use, since we cannot use any other measurement, such as strain gauges, to do this with the precision required.

VI. Conclusions

In this paper, von Kármán's equations are used. Based on their nonlinear bending solution of a circular thin plate under the action of uniform pressure and uniform initial edge tension (preloaded), the characteristic relation between the fundamental natural frequency and external load of the vibrating diaphragm-type pressure sensor is obtained. Through the introduction of the transformation of coefficients, the original characteristic value problem is reduced to a third-order algebraic characteristic value problem with explicit expression, and thus the labor of computation is much decreased.

1) The relation of stiffness of a circular thin plate under uniform transverse pressure and uniform initial tension is nonlinear, the deflection increases as the pressure increases and decreases as initial tension increases.

2) The curve of the fundamental natural frequency varies with external load and is nonlinear. As $(qa^4/Eh^4)<8$, the natural frequency increases as the initial tension rises. As $(qa^4/Eh^4)>15$, the fundamental natural frequency decreases as the initial tension rises.

3) This paper is of engineering significance. All of the data presented here can be directly used for the engineering design of the vibrating diaphragm-type pressure sensor.

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