Dilation-Free Solutions for the Incompressible Flow Equations on Nonstaggered Grids

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Introduction

RESEARCHERS using primitive variable formulations for the incompressible Navier–Stokes equations on nonstaggered grids have long been frustrated with spatial odd–even decoupling of the pressure field. The odd–even decoupling is caused when second-order central difference approximations are implemented for both the pressure gradient operator in the momentum equation and the divergence operator in the continuity equation. The combined divergence–gradient operator acting on the pressure in the pressure Poisson equation suffers from a lack of communication between directly adjacent grid points, which in turn produces an oscillatory pressure field as recognized by Ghia et al.[1] Despite this drawback, the simplicity of nonstaggered grid implementation continues to captivate the focus of many groups.[2,3] The problem of odd–even decoupling can be treated by explicitly adding a dissipative, fourth-order derivative term for the pressure to the continuity equation.[4] The discrete form of the fourth-order derivative contains both odd and even grid points and thus provides spatial coupling of the pressure field. An implicit technique for eliminating odd–even decoupling involves the derivation of the pressure Poisson equation in continuum form and then discretizing the Laplace operator with the classical five-point approximation, which contains both odd and even mesh points.[5,6] Although these modifications are successful in smoothing the pressure field, they compromise the second-order accuracy of the implied discrete continuity equation. Aksoy and Chen[2] provide an excellent review of the varying levels of dilation resulting from several popular methods that utilized nonstaggered grids.

The present method employs a pressure Poisson formulation that has the unique capability of satisfying the discrete continuity equation exactly while producing smooth pressure on nonstaggered grids. A fourth-order accurate discretization is employed for the pressure gradient operator in both the momentum equation and the pressure Poisson equation. The use of higher-order accurate approximations for the pressure derivatives results in the enlargement of the Laplace operator, which incorporates both odd and even grid points and thus spatially couples the pressure field. The numerical solutions are compared to solutions obtained from the classic pressure Poisson approach, which uses artificial dissipation to couple the pressure field, as well as benchmark results of a stream function–vorticity formulation (which automatically produces dilation-free solutions).

Analysis

The incompressible Navier–Stokes equations for two-dimensional laminar flow of a Newtonian fluid can be written in discrete form as follows:

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \Delta t\left(\gamma_x p_{i,j}^{n+1} + R_{i,j}^{n}\right) \tag{1a}$$

$$v_{i,j}^{n+1} = v_{i,j}^{n} - \Delta t\left(\gamma_y p_{i,j}^{n+1} + S_{i,j}^{n}\right) \tag{1b}$$

$$\lambda_x u_{i,j}^{n+1} + \lambda_y v_{i,j}^{n+1} = 0 \tag{2}$$

where

$$R_{i,j}^{n} = \left[u_{i,j}^{n}\delta_x + v_{i,j}^{n}\delta_y - (1/Re)(\delta_{xx} + \delta_{yy})\right]u_{i,j}^{n} \tag{3a}$$

$$S_{i,j}^{n} = \left[u_{i,j}^{n}\delta_x + v_{i,j}^{n}\delta_y - (1/Re)(\delta_{xx} + \delta_{yy})\right]v_{i,j}^{n} \tag{3b}$$

In Eqs. (1–3), u and v are the velocity components in the x and y directions, respectively; p is the static pressure divided by the density; and Re is the Reynolds number. The operators (δ_x, δ_y) and $(\delta_{xx}, \delta_{yy})$ are the central second-order finite difference approximations for the first- and second-order partial derivatives, respectively. Here, the gradient operator (γ_x, γ_y) and the divergence operator (λ_x, λ_y) are represented with individual symbols to facilitate their independent finite difference representations. The pressure Poisson equation is formulated by applying the divergence operator to the momentum equations:

$$\lambda_x(\gamma_x p_{i,j}^{n+1}) + \lambda_y(\gamma_y p_{i,j}^{n+1}) = -(\lambda_x R_{i,j}^{n} + \lambda_y S_{i,j}^{n}) + (1/\Delta t)(\lambda_x u_{i,j}^{n} + \lambda_y v_{i,j}^{n}) \tag{4}$$

Equation (4) is then incorporated in the numerical scheme to enforce continuity as a replacement for Eq. (2). When second-order central difference approximations are used for both the gradient operator (γ_x, γ_y) and the divergence operator (λ_x, λ_y), the resulting x component of the discrete operator acting on the pressure in the pressure Poisson equation (4) takes the form

$$\lambda_x(\gamma_x p_{i,j}^{n+1}) = \left(\frac{p_{i+2,j} - 2p_{i,j} + p_{i-2,j}}{4\Delta x^2}\right)^{n+1} \tag{5}$$

where Δx is the grid spacing in the x direction and (i, j) refers to a grid location in the (x, y) directions, respectively. It is evident that numerical solutions to Eq. (4) with the discrete operator (5) suffer from odd–even decoupling.

Coupling of the pressure field may be achieved by using higher-order central difference approximations for the gradient operator (γ_x, γ_y), the divergence operator (λ_x, λ_y), or both. However, the use of higher-order approximations for both the divergence and gradient operators is probably not feasible because of the excessive enlargement of the resulting discrete operator acting on the pressure in Eq. (4). In the present study, second-order central difference approximations are used for the divergence operator (λ_x, λ_y), whereas the gradient operator is approximated with fourth-order central differencing. For example, the x component is

$$\gamma_x p_{i,j}^{n+1} = \left(\frac{-p_{i+2,j} + 8p_{i+1,j} - 8p_{i-1,j} + p_{i-2,j}}{12\Delta x}\right)^{n+1} \tag{6}$$

The resulting x component of the discrete operator acting on the pressure in the pressure Poisson equation (4) can then be written as

$$\lambda_x(\gamma_x p_{i,j}^{n+1}) = \left(\frac{-p_{i+3,j} + 8p_{i+2,j} + p_{i+1,j} - 16p_{i,j} + p_{i-1,j} + 8p_{i-2,j} - p_{i-3,j}}{24\Delta x^2}\right)^{n+1} \tag{7}$$

A similar set of discrete operators may be derived for the y component. The discrete operator (7) allows communication between odd and even grid points when used to solve the pressure Poisson equation (4) and thus couples the pressure field. Note that the discrete form of the gradient operator (6) must be consistent in both the momentum equation (1a) and the pressure Poisson equation (4). Any inconsistency will result in solutions with nonzero dilation.

We apply the boundary conditions to the discrete form of the continuity equation (2) at interior points adjacent to the boundaries prior to formulating the pressure Poisson equation. This approach eliminates the task of developing complex boundary conditions for the pressure that satisfy both local and global mass conservation.

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For example, the discrete continuity equation at interior grid points ($i = 2$) directly adjacent to the boundary takes the form

$$\frac{u_{3,j}^{n+1} - u_{1,j}^{n+1}}{2\Delta x} + \lambda_y v_{2,j}^{n+1} = 0 \tag{8}$$

Now the discrete momentum equations (1a) and (1b) are substituted into Eq. (8) for the interior velocity components only (assuming the velocity on the boundary at $i = 1$ is known):

$$\frac{\left[u_{3,j}^{n} - \Delta t\left(\gamma_x p_{3,j}^{n+1} + R_{3,j}^{n}\right)\right] - u_{1,j}^{n+1}}{2\Delta x} + \lambda_y\left[v_{2,j}^{n} - \Delta t\left(\gamma_y p_{2,j}^{n+1} + S_{2,j}^{n}\right)\right] = 0 \tag{9}$$

The fourth-order gradient operator defined in Eq. (6) then can be employed to discretize the remaining pressure terms in Eq. (9) to promote coupling of the pressure. This approach automatically enforces the implied discrete continuity equation (2) at grid points adjacent to the boundaries. Therefore, after the pressure is computed at the interior points, the boundary values can be determined simply by third-order accurate extrapolation formulas to preserve accuracy.

Results and Discussion

Numerical results are obtained by solving Eq. (4) for the pressure using the successive overrelaxation method. Then the velocity components u^{n+1} and v^{n+1} are obtained from Eqs. (1a) and (1b), respectively, using an explicit marching procedure. Steady-state flow in a two-dimensional lid-driven cavity is used as a test case and solutions are obtained for a Reynolds number of 10^3 on a uniform, nonstaggered (129×129) grid. Results are acquired by the present method as well as by the classic pressure Poisson method.[5,6] The maximum dilation converges to machine accuracy ($\approx 1.15 \times 10^{-12}$) for the present method and to a constant value of 7.731 (located in the corner adjacent to the moving wall) for the classic pressure Poisson method. Thus, the relevance of the present study resides in the effect of a dilation-free solution on the accuracy of the calculated primitive variables.

The two methods produce smooth pressure fields because of the coupling of odd and even grid points as seen in Fig. 1. Although both pressure fields exhibit similar general trends, notable differences occur because the classic pressure Poisson method incorporates the fourth-order dissipative term[4] used to couple the pressure at the expense of nonzero dilation. A quantitative assessment of the accuracies is achieved by comparing the results of the two methods with the benchmark results of Ghia et al.[7] for the driven cavity. Note that Ghia et al.[7] incorporate the stream function–vorticity formulation, which automatically ensures dilation-free results for two-dimensional computations. Figure 2 shows the u component of velocity along the vertical centerline of the cavity for $Re = 10^3$. The results of the present method show a favorable comparison with the results of Ghia et al.,[7] whereas the classic pressure Poisson results differ because of the nonzero dilation.

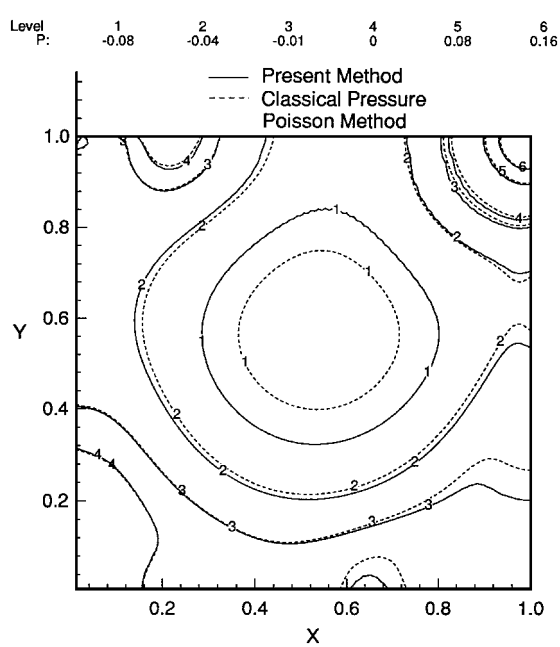

Fig. 1 Static pressure contours for the driven cavity problem at $Re = 10^3$.

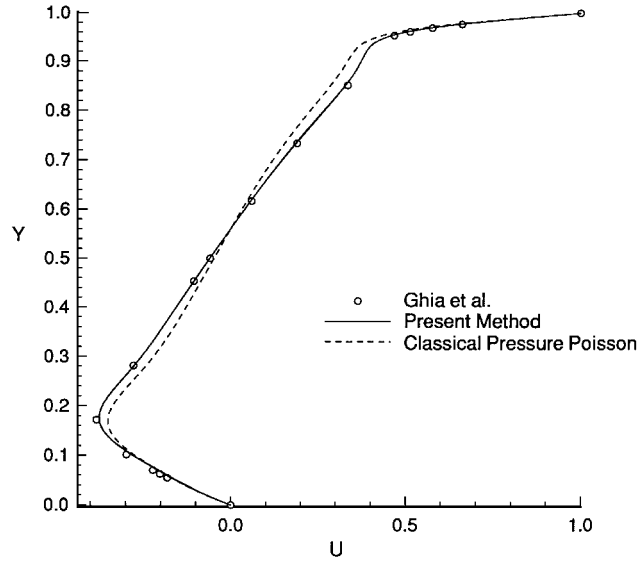

Fig. 2 The u velocity profile at the vertical centerline of the cavity for $Re = 10^3$.

Conclusions

The use of primitive variables formulations for the incompressible Navier–Stokes equations on nonstaggered grids notoriously produces spatial odd–even decoupling of the pressure field. Past efforts to remove these oscillations resulted in solutions with residual errors in the implied discrete continuity equation. The method introduced in this study has the capability of producing smooth pressure and dilation-free solutions. Further, it is demonstrated that the residual dilation that occurs in the classic pressure Poisson method can significantly affect the accuracy of the computed velocity field.

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