# Kalman Filter Divergence and Aircraft Motion Estimators

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**Kalman filters designed for many aerospace systems turn out to be unsatisfactory. The estimate errors become large compared to the errors predicted by the theory ("divergence"). One of the principal causes of this failure is that the system model contains states or modes that are undisturbed by the modeled process noise, and are neutrally stable (NS). One cure for such problems is periodic restarting of a time-varying Kalman filter. Other cures include minimum variance observers with eigenvalue constraints, added noise, pole-shifting, and destabilization. Several examples are given, including effective time-invariant estimators for the longitudinal and lateral motions of an airplane where several NS modes are undisturbed by wind gusts. An interpretation of these estimators as a "strapdown IMU" without accelerometers, gimbaled gyros, or servos is given.**

## Introduction

MANY aerospace systems are such that one or more neutrally stable (or almost neutrally stable) modes are unaffected (or affected only slightly) by the input disturbances. Obvious examples are a) no input disturbances at all, as in many spacecraft systems (see example 1) and b) unknown constant inputs or unknown constant errors in measurement (see example 2). Not so obvious examples are c) the neutrally stable heading mode of an airplane is undisturbed, and the almost neutrally stable spiral mode is almost undisturbed, by lateral wind gusts (see example 3), and d) the neutrally stable energy mode of an airplane is undisturbed by horizontal wind gusts (see example 4).

Undisturbed modes can, in principle, be estimated exactly by a time-varying Kalman filter (TKF); the uncertainty in these modes is eliminated during a short initial period of time-varying gains. However, the asymptotic steady-state gains of this filter, in modal coordinates, are *zero*. Hence, if only the steady-state Kalman filter (SKF) is used as a constant-gain state estimator (an "observer"), an undisturbed mode is estimated "open loop," i.e., there is no measurement feedback. This is quite unsatisfactory if the mode is neutrally stable (or nearly neutrally stable) since the initial estimate error will not attenuate (or will attenuate too slowly).

Undisturbed modes can be detected by testing the system for "controllability" by the process noise $w$ (disturbance inputs), i.e., $(F,\Gamma)$ must be a completely controllable pair where $\dot{x}=Fx+\Gamma w$. It would be more appropriate to say that the system must be completely disturbable by the process noise. Another way to detect undisturbed modes is to look for pole-zero cancellations (or near cancellations) in the transfer functions from the process noise $w(t)$ to the measured quantities $Hx(t)$. However, this method has the disadvantage that the mode corresponding to the cancelled pole may either be undisturbable by $w$ or it may be unobservable with $Hx$.

In the original, nonmodal coordinates, none of the SKF gains may be zero, so the problem is hidden. However, the presence of a neutrally stable eigenvalue in the SKF indicates the presence of an unobservable or undisturbable neutrally stable mode. Since constant-gain state estimators (observers) are easier to implement than the time-varying-gain filters, it is of interest to investigate modifications of the SKF to see if acceptable estimate errors can be achieved along with acceptable estimate-error eigenvalues.

## Time-Varying Kalman Filter (TKF)

To avoid divergence, Battin and Levine[1] suggest the periodic "restarting" of the time-varying-gain Kalman filter. Although somewhat complicated to implement, this scheme has the advantage of bringing the estimate-error variances to their lowest possible values without using "adaptive" filters. Restarting could be triggered, for example, when the magnitude of measurement residuals reaches a prespecified level.

## Eigenvalues and the Covariance Matrix of an Observer

Consider the stationary linear system

$$\dot{x}=Fx+\Gamma w \tag{1}$$

$$z=Hx+v \tag{2}$$

where $x$ is the state vector, $z$ is the measurement vector, and $w$ and $v$ are independent, zero-mean white noise processes with spectral density matrices $Q$ and $R$, respectively. An observer for this sytem (see e.g., Ref. 2) is of the form

$$\dot{\hat{x}}=F\hat{x}+K(z-H\hat{x}) \tag{3}$$

where $K$ is a matrix of constant gains. The estimate-error is defined as

$$\tilde{x}\triangleq\hat{x}-x \tag{4}$$

where

$$\dot{\tilde{x}}=(F-KH)\tilde{x}-\Gamma w+Kv \tag{5}$$

The eigenvalues of the observer are the roots of

$$|sI-F+KH|=0 \tag{6}$$

The estimate-error covariance matrix $P$ is defined as

$$P=E(\tilde{x}\tilde{x}^T) \tag{7}$$

The steady-state covariance matrix is obtained by solving the linear Lyapunov matrix equation

$$0=(F-KH)P+P(F-KH)'+\Gamma Q\Gamma^T+KRK^T \tag{8}$$

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### Constrained Minimum Variance (MV) Observer

The constrained minimum variance problem may be stated as follows: find a set of observer gains $K$ that minimize

$$J = \mathrm{tr}(2MP) \tag{9}$$

where $P=P(K)$ is found from Eq. (8), $M$ is a positive semidefinite matrix, and the gains are constrained so that

$$Rl(s_i) \leq -\sigma, \quad i=1,2,...n \tag{10}$$

where $s_1,...s_n$ are the eigenvalues of $F-KH$, i.e., the roots of Eq. (6), $\sigma$ is a specified positive number, and $Rl$ ( ) indicates "real part of ( )."

This problem was formulated, and an algorithm was given for finding the optimal gains $K$ in Ref. 3. Here we suggest a way to write the constraints in Eq. (10) explicitly in terms of $K$ and present a new necessary condition for the optimal $K$.

The eigenvalue constraints [Eq. (10)], may be written explicitly in terms of the gains using the Routh stability criteria (see e.g., Ref. 4) for the augmented-stability characteristic equation

$$|(s-\sigma)I - F + KH| = 0 \tag{11}$$

If only one real eigenvalue of the SKF does not satisfy the constraint in Eq. (10), then the constraint may be written simply as

$$|-\sigma I - F + KH| = 0 \tag{12}$$

which insures that one eigenvalue of $F-KH$ is equal to $-\sigma$.

Let the Routh inequality constraints on the gains be written symbolically as

$$C(K) \geq 0 \tag{13}$$

where $C$ is a vector of functions. Necessary conditions for minimizing Eq. (9) with constraints (8) and (13) are derived in Appendix A; they are Eqs. (8), (13), and

$$\frac{\partial J}{\partial K} \triangleq (HP - RK^T)M(F-KH)^{-1} + \mu^T \frac{\partial C}{\partial K} = 0 \tag{14}$$

where

$$\mu_i = 0 \text{ if } C_i(K) > 0 \tag{15}$$

and

$$\mu_i \leq 0 \text{ if } C_i(K) = 0 \tag{16}$$

are necessary conditions for a constrained minimum.

### Added Noise (AN) Observer

Schmidt[5] presented a method for modifying unsatisfactory Kalman filters by introducing additional noise (AN) into the system model, Eq. (1). The spectral densities of the added noise may be varied until the filter has desired stability.

### Pole-Shifted (PS) Observer

Reference 3 presents a "pole-shifting" (PS) method. First, the steady-state Kalman filter is designed; if the stability is unsatisfactory, the slow eigenvalues are shifted to specified, faster eigenvalues by changing the gains so as to leave the other eigenvalues unchanged. The method does not give a unique solution if there are two or more measurements.

### Destabilization (DS) Observers

Paradoxically, the steady-state Kalman filter for a system with an undisturbed *unstable* mode is stable. Thus, another method for designing observers for systems with undisturbed, neutrally stable (UNS) modes is to "destabilize" the system model, Eq. (1). The amount of destabilization may be varied until the resulting observer has specified stability. Anderson and Moore[6] presented a "total destabilization" (TDS) method in connection with designing regulators. A modification of this scheme is to destabilize only the UNS modes in the system model ["modal destabilization" (MDS)].

### Example 1: Estimating an Undisturbed Constant State

The simplest example of a UNS mode is

$$\dot{x} = 0, \quad z = x + v \tag{17}$$

where there is no input disturbance ($Q=0$), but the measurement has added white noise, $v$ with spectral density $R \neq 0$. The initial state $x(0)$ has zero mean value and variance $P_0$. The TKF for estimating $x(t)$ is (see e.g., Ref. 7, p. 367).

$$\dot{\hat{x}} = K(z-\hat{x}), \quad \hat{x}(0) = 0 \tag{18}$$

where

$$K(t) = \frac{P_0/R}{1+(p_0/R)t} \tag{19}$$

Clearly for $t \gg (R/P_0)$, $K(t) \to 0$, and the estimator ignores the measurement $z$. In practice, such a filter will almost certainly "diverge" for $(P_0/R)t \gg 1$.

The MV, AN, PS, and DS observers for this problem are all the same and amount to choosing

$$K = \sigma \tag{20}$$

where $\sigma$ is the eigenvalue constraint level [see Eq. (10)].

The variance of the estimate-error of the filter [Eqs. (18) and (19)] is predicted to be

$$P(t) = \frac{P_0}{1+(P_0/R)t} \tag{21}$$

i.e., $P(t) \to 0$ for $(P_0 t/R) \gg 1$. The variance of the observer, Eq. (18), plus Eq. (20) is predicted to be

$$P(t) = P_0 e^{-2\sigma t} + \frac{\sigma R}{2}(1-e^{-2\sigma t}) \tag{22}$$

so that $P \to (\sigma R/2)$ in steady state, but the observer will *not* diverge.

### Example 2: Estimating a Constant Disturbance (or Measurement Bias) for a System with One Real Eigenvalue

A first-order system with an unknown constant input† is modeled as

$$\dot{x}_1 = -x_1 + x_2 + w, \; \dot{x}_2 = 0, \; z = x_1 + v \tag{23}$$

The input $x_2$ is observable with $z$, but is undisturbed by $w$. The transfer function from $w$ to $x_1$ has a pole-zero cancellation at $s=0$:

$$\frac{x_1(s)}{w(s)} = \frac{s}{s(s+1)} \equiv \frac{1}{s+1} \tag{24}$$

Estimators for this system have the form

$$\dot{\hat{x}}_1 = -\hat{x}_1 + \hat{x}_2 + K_1(z-\hat{x}_1), \quad \hat{x}_1(0) = 0 \tag{25a}$$

†The system may be converted into one with a measurement bias by the transformation $x_1' = x_1 - x_2$, $x_2' = x_2$.

$$\dot{\hat{x}}_2 = K_2(z - \hat{x}_1), \quad \hat{x}_2(0) = 0 \tag{25b}$$

The gains $K_1$ and $K_2$ for the TKF are cumbersome to write down even for this simple example. However, the steady-state gains are

$$K_1 = \sqrt{1+(Q/R)} - 1, \; K_2 = 0 \tag{26}$$

i.e., the SKF does *not* use the measurement to estimate $x_2$. The eigenvalues and variances of the SKF are

$$s_1 = -\sqrt{1+(Q/R)}, \quad s_2 = 0 \tag{27a}$$

$$P_{11} = RK_1, \; P_{22} = 0 \tag{27b}$$

In practice, the TKF will "diverge" and the SKF, used as an observer, does not estimate $x_2$ (it will give $\hat{x}_2(0) = 0$ for all time).

Appendix B gives derivations of the MV observer (minimizing the variance of $\tilde{x}_1$, namely $P_{11}$) and the AN, PS, MDS, and TDS observers. Figures 1 through 4 show $P_{11}$, $s_1$, $s_2$, $K_1$, and $K_2$ for these various observers as functions of the eigenvalue constraint level $\sigma$ for $Q/R = 3$.

Figure 1 shows that the rms error in estimating $x_1$ increases as the eigenvalue constraint level $\sigma$ is increased. Of the five observers shown, the constrained minimum variance (MV) observer, of course, gives the smallest error. The Schmidt added-noise observer (SAN) is next best until the constraint level is moved beyond $\sigma = 1$ (the real eigenvalue of the open-loop system). The modal destabilization (MDS) observer is reasonably good over the whole range of $\sigma$ shown. The pole-shifted (PS) and total destabilization (TDS) observers are the poorest of those shown.

Figure 2 shows the second eigenvalue of the various observers. The SAN and MDS observers have complex eigenvalues (real part = $\sigma$) for large values of $\sigma$.

Figures 3 and 4 show the two observer gains $K_1$ and $K_2$ as functions of $\sigma$. The CMV observer has the smallest gains over the whole range of $\sigma$ shown.

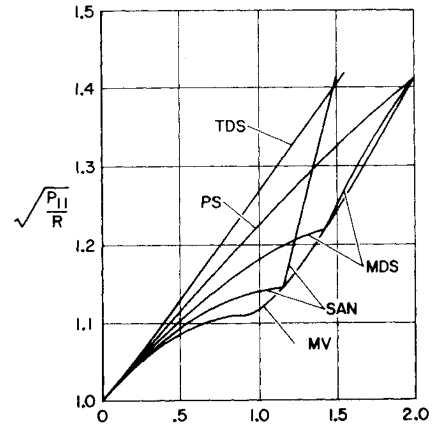


**Fig. 1 Root-mean-square estimate error for $x_1$ as a function of $\sigma$ for example 2 ($Q/R = 3$).**

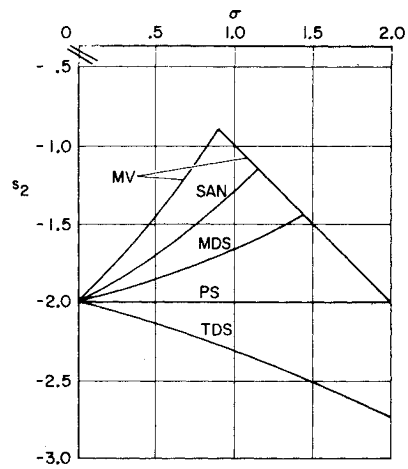


**Fig. 2 Second eigenvalue as a function of $\sigma$ for example 2 ($Q/R = 3$).**

## Example 3: Estimating the Lateral Motions of an Airplane

The lateral motions of an airplane are modeled quite well by a fifth-order system where the states are $\beta$, sideslip angle; $r$, yaw rate; $p$, roll rate; $\varphi$, roll angle; $\psi$, heading angle. The main disturbance input is lateral wind $v_g$; the fluctuating part of $v_g$ is modeled reasonably well (at high altitudes) by a first-order shaping filter with a white noise input[8]. From Ref. 9, the lateral motions of a DC-8 cruising at an altitude of 33,000 ft at a velocity $V = 824$ fps (weight 230,000 lb) are described by

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{p} \\ \dot{\varphi} \\ \dot{\psi} \\ \dot{\beta}_g \end{bmatrix} = \begin{bmatrix} -0.0868 & -1 & 0 & 0.03907 & 0 & -0.0868 \\ 2.14 & -0.228 & -0.0204 & 0 & 0 & 2.14 \\ -4.41 & 0.334 & -1.181 & 0 & 0 & -4.41 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.853 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \varphi \\ \psi \\ \beta_g \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.853 \end{bmatrix} \mu \tag{28}$$

where $\beta_g = v_g / V$, and the units are radians and seconds. If we use only a roll-rate gyro (measurement $z_p$) and a magnetic compass (measurement $z_\psi$), the measurement model is

$$\begin{bmatrix} z_p \\ z_\psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \phi \\ \psi \\ \beta_g \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_p \\ \nu_\psi \end{bmatrix} \tag{29}$$

The compass measurement really contains a bias error, but it is unobservable, so $\psi$ in this case is heading from indicated magnetic north. The rate gyro measurement really contains a bias error (the gyro drift rate) which is observable, but, for simplicity, we ignore it here. The lateral wind $v_g$ really contains a constant component (the mean wind), but this is not observable with $z_p$ and $z_\psi$; hence $\beta$ is sideslip angle with respect to the "mean" air mass, i.e., an inertial coordinate system that moves with the mean wind velocity with respect to the ground. Thus, $\beta+\beta_g$ is the sideslip angle with respect to the "local" air mass. It is very instructive to transform Eqs. (28) and (29) into modal coordinates $\xi$; this was done using the "QR" algorithm" contained in the computer program OPT-SYS (see Ref. 10). The result is

$$\begin{bmatrix}\dot\xi_{D1}\\ \dot\xi_{D2}\\ \dot\xi_R\\ \dot\xi_S\\ \dot\xi_H\\ \dot\xi_{v_g}\end{bmatrix}=\begin{bmatrix}-0.119 & 1.490 & 0 & 0 & 0 & 0\\ -1.490 & -0.119 & 0 & 0 & 0 & 0\\ 0 & 0 & -1.254 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.0041 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -0.853\end{bmatrix}\begin{bmatrix}\xi_{D1}\\ \xi_{D2}\\ \xi_R\\ \xi_S\\ \xi_H\\ \xi_{v_g}\end{bmatrix}+\begin{bmatrix}-1.702\\ -0.610\\ -5.106\\ -0.004\\ 0\\ 3.788\end{bmatrix}[\mu] \tag{30}$$

and

$$\begin{bmatrix}z_p\\ z_\psi\end{bmatrix}=\begin{bmatrix}1 & 0 & -0.782 & 0 & 0 & -0.605\\ 0.276 & 0.295 & -0.008 & 0.995 & 1 & 0.162\end{bmatrix}\begin{bmatrix}\xi_{D1}\\ \xi_{D2}\\ \xi_R\\ \xi_S\\ \xi_H\\ \xi_{v_g}\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}v_p\\ v_\psi\end{bmatrix} \tag{31}$$

where

$$\begin{bmatrix}\beta\\ r\\ p\\ \phi\\ \psi\\ \beta_g\end{bmatrix}=\begin{bmatrix}-0.277 & -0.309 & -0.012 & 0 & 0 & -0.191\\ -0.472 & 0.376 & 0.010 & -0.004 & 0 & -0.138\\ 1 & 0 & -0.782 & 0 & 0 & -0.605\\ -0.053 & -0.667 & 0.623 & -0.104 & 0 & 0.709\\ 0.276 & 0.295 & -0.008 & 0.995 & 1 & 0.162\\ 0 & 0 & 0 & 0 & 0 & 0.225\end{bmatrix}\begin{bmatrix}\xi_{D1}\\ \xi_{D2}\\ \xi_r\\ \xi_S\\ \xi_H\\ \xi_{v_g}\end{bmatrix} \tag{32}$$

and the inverse transformation is

$$\begin{bmatrix}\xi_{D1}\\ \xi_{D2}\\ \xi_R\\ \xi_S\\ \xi_H\\ \xi_{v_g}\end{bmatrix}=\begin{bmatrix}-1.498 & -1.142 & 0.049 & 0.050 & 0 & -1.995\\ -1.818 & 1.100 & 0.015 & -0.035 & 0 & -0.715\\ -1.916 & -1.469 & -1.220 & 0.060 & 0 & -5.986\\ 0.939 & -15.27 & -7.43 & -9.06 & 0 & -0.004\\ 0 & 15.17 & 7.36 & 9.00 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 4.441\end{bmatrix}\begin{bmatrix}\beta\\ r\\ p\\ \phi\\ \psi\\ \beta_g\end{bmatrix} \tag{33}$$

From Eq. (30), it can be seen that a) the heading mode $\xi_H$ is neutrally stable (eigenvalue, $\lambda_H=0$) and is undisturbed by the process noise (zero element in the matrix multiplying $\mu$); b) the spiral mode $\xi_S$ is very slightly damped (eigenvalue $\lambda_S=-0.0040\ \mathrm{s}^{-1}$) and is only slightly disturbed by the process noise ($-0.0038$ element in the matrix multiplying $\mu$). From Eq. (31), it can be seen that the spiral and heading modes are unobservable with the measurement $z_p$ (zero element in fourth and fifth colums of top row in matrix multiplying $\xi$). Thus, the only slight eigenvalue separation between the spiral and heading modes allows us to observe them both using the $z_\psi$ measurement. The SKF, used as an observer, would be useless because it would *not* estimate the UNS heading mode, and it would estimate the spiral mode very slowly.

The spiral and heading modes, with the measurement $z_\psi$, form a problem exactly like example 2 if one uses the transformation mentioned in the footnote to convert it into a measurement bias problem. An MDS observer was designed for this problem (see Appendix C) with the eigenvalue constraint of Eq. (10) with $\sigma=0.029\ \mathrm{s}^{-1}$ (time constant 34 s), $Q=1.63\times10^{-4}\ \mathrm{s}^{-1}$ (7 ft s$^{-1}$ rms gust with a 930-ft correlation distance), $R_p=1.5\times10^{-5}$ s, and $R_\psi=1.5\times10^{-5}\ \mathrm{s}^{-1}$. The results are shown in Table 1 and Fig. 5 and are compared with the predictions of the SKF. It is apparent that only a small penalty in rms estimate error is paid for using the MDS observer, which will not diverge (as the SKF would in this case).

**Table 1 Gains, eigenvalues, and rms estimates errors for DC-8 lateral motion estimator**

| | SKF | | MDS | | |
|---|---|---|---|---|---|
| Observer gains $[K_p K_\psi]$ | 0.051 | −0.967 | −0.073 | −0.870 | $\beta$ |
| | −1.536 | 0.411 | −1.559 | 0.429 | $r$ |
| | 2.693 | −0.004 | 2.731 | −0.033 | $p$ |
| | 0.385 | −0.789 | 0.334 | −0.752 | $\phi$ |
| | −0.004 | 0.906 | −0.034 | 0.929 | $\psi$ |
| | −1.711 | 0.655 | −1.617 | 0.583 | $\beta_g$ |
| Observer eigen-values, $s^{-1}$ | −2.349 | | −2.350 | | } $(p,\beta_g,r)$ |
| | $\pm 2.593j$ | | $\pm 2.589j$ | | |
| | −0.624 | | −0.625 | | } $(\phi,\beta,\psi)$ |
| | $\pm 0.492j$ | | $\pm 0.492j$ | | |
| | −0.00125 | | −0.0295 | | } $(\beta,\phi,\beta_g)$ |
| | 0 | | $\pm 0.0047j$ | | |
| RMS estimate errors | 3.26 | | 3.66 | | $V\tilde{\beta}$, ft $s^{-1}$ |
| | 0.219 | | 0.222 | | $\tilde{\phi}$, deg |
| | 0.211 | | 0.213 | | $\tilde{\psi}$, deg |
| | 5.21 | | 5.35 | | $V\tilde{\beta}_g$, ft $s^{-1}$ |
| Modal gains $[K_p^{(m)} K_\psi^{(m)}]$ | 5.243 | −0.367 | 5.276 | −0.392 | $\xi_{DI}$ } $DR$ |
| | −0.533 | 1.769 | −0.392 | 1.662 | $\xi_{D2}$ |
| | 9.138 | −2.711 | 8.816 | −2.465 | $\xi_R$ } $R$ |
| | 0.007 | −0.006 | 0.422 | −0.323 | $\xi_S$ } $S$ |
| | 0 | 0 | −0.560 | 0.429 | $\xi_H$ } $H$ |
| | −7.599 | 2.907 | −7.188 | 2.592 | $\xi_{v_g}$ } $v_g$ |

## Example 4: Estimating the Longitudinal Motions of an Airplane

The longitudinal motions of an airplane are modeled quite well by a fifth-order system where the states are $u$; velocity along the body $x$ axis $u$; $w$, velocity along the body $z$ axis; $q$, pitch rate; $\theta$, pitch angle, and $h$, altitude. The main disturbance inputs are the two wind velocities $u_g$ and $w_g$. The fluctuating parts of $u_g$ and $w_g$ are modeled reasonably well (at high altitudes) by first-order shaping filters with white noise inputs[8]. From Ref. 9, the longitudinal motions of a DC-8 (same flight condition as example 3) are described by

$$\begin{bmatrix} \dot u \\ \dot w \\ \dot q \\ \dot\theta \\ \dot h \\ \dot u_g \\ \dot w_g \end{bmatrix} = \begin{bmatrix} -0.015 & 0.004 & 0 & -0.0322 & 0 & -0.015 & 0.004 \\ -0.074 & -0.806 & 0.824 & 0 & 0 & -0.074 & -0.806 \\ -0.749 & -10.7 & -1.344 & 0 & 0 & -0.749 & -10.7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0.0824 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.413 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.853 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \\ u_g \\ w_g \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.413 & 0 \\ 0 & 0.853 \end{bmatrix} \begin{bmatrix} \mu_u \\ \mu_w \end{bmatrix} \tag{34}$$

where $u$, $w$, $u_g$, $w_g$ are in units of 10 ft $s^{-1}$, $q$ in units of 0.01 rad $s^{-1}$, $\theta$ in 0.01 rad, and $h$ in units of 100 ft.

If we use only a pitch-rate gyro (measurement $z_q$) and a barometric altimeter (measurement $z_h$), the measurement model is

$$\begin{bmatrix} z_q \\ z_h \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \\ u_g \\ w_g \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_q \\ v_h \end{bmatrix} \tag{35}$$

The rate gyro measurement really contains a bias error (the gyro drift rate) which is observable, but, for simplicity, we ignore it here. The barometric altimeter measurement really contains a bias error, but it is unobservable; it also has a time lag which, for simplicity, we ignore here. The longitudinal wind $u_g$ really contains a constant component (the mean wind), which is unobservable

with $z_q$ and $z_h$ (see discussion of lateral wind in example 3). In modal coordinates, $\xi$, the system in Eqs. (34) and (35) becomes:

$$\begin{bmatrix} \dot{\xi}_{S1} \\ \dot{\xi}_{S2} \\ \dot{\xi}_{P1} \\ \dot{\xi}_{P2} \\ \dot{\xi}_{E} \\ \dot{\xi}_{u_g} \\ \dot{\xi}_{w_g} \end{bmatrix} = \begin{bmatrix} -1.076 & 2.958 & 0 & 0 & 0 & 0 & 0 \\ -2.958 & -1.076 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.006 & 0.024 & 0 & 0 & 0 \\ 0 & 0 & -0.024 & -0.006 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.413 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.853 \end{bmatrix} \begin{bmatrix} \xi_{S1} \\ \xi_{S2} \\ \xi_{P1} \\ \xi_{P2} \\ \xi_{E} \\ \xi_{u_g} \\ \xi_{w_g} \end{bmatrix} + \begin{bmatrix} -0.023 & -0.882 \\ -0.110 & -3.152 \\ -0.003 & -0.089 \\ 0.065 & -0.025 \\ 0 & 0.100 \\ 0.419 & 0 \\ 0 & 1.835 \end{bmatrix} \begin{bmatrix} \mu_u \\ \mu_w \end{bmatrix} \tag{36}$$

and

$$\begin{bmatrix} z_q \\ z_h \end{bmatrix} = \begin{bmatrix} 1 & 0 & -0.007 & -0.003 & 0 & 0.054 & 0.480 \\ 0.002 & 0.002 & 1 & 0 & 1 & 0.008 & -0.003 \end{bmatrix} \begin{bmatrix} \xi_{S1} \\ \xi_{S2} \\ \xi_{P1} \\ \xi_{P2} \\ \xi_{E} \\ \xi_{u_g} \\ \xi_{w_g} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \nu_q \\ \nu_h \end{bmatrix} \tag{37}$$

where

$$\begin{bmatrix} u \\ w \\ q \\ \theta \\ h \\ u_g \\ w_g \end{bmatrix} = \begin{bmatrix} 0.002 & -0.002 & -0.344 & -0.187 & 0 & 0.026 & -0.022 \\ 0.025 & -0.276 & 0.025 & 0.013 & 0 & -0.076 & -0.485 \\ 1 & 0 & -0.007 & -0.003 & 0 & 0.054 & 0.480 \\ -0.109 & -0.299 & -0.044 & 0.303 & 0 & -0.131 & -0.563 \\ 0.002 & 0.002 & 1 & 0 & 1 & 0.008 & -0.003 \\ 0 & 0 & 0 & 0 & 0 & 0.987 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.465 \end{bmatrix} \begin{bmatrix} \xi_{S1} \\ \xi_{S2} \\ \xi_{P1} \\ \xi_{P2} \\ \xi_{E} \\ \xi_{u_g} \\ \xi_{w_g} \end{bmatrix} \tag{38}$$

and the inverse transformation is

$$\begin{bmatrix} \xi_{S1} \\ \xi_{S2} \\ \xi_{P1} \\ \xi_{P2} \\ \xi_{E} \\ \xi_{u_g} \\ \xi_{w_g} \end{bmatrix} = \begin{bmatrix} -0.020 & 0.003 & 1 & -0.003 & 0 & -0.055 & -1.034 \\ -0.260 & -3.616 & 0.091 & -0.001 & 0 & -0.265 & -3.696 \\ -2.559 & 1.813 & -0.129 & -1.662 & 0 & -0.007 & -0.104 \\ -0.636 & -3.303 & 0.251 & 3.061 & 0 & -0.157 & -0.030 \\ 2.559 & -1.808 & 0.128 & 1.662 & 1 & 0 & 0.117 \\ 0 & 0 & 0 & 0 & 0 & 1.014 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.151 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \\ h \\ u_g \\ w_g \end{bmatrix} \tag{39}$$

From Eq. (36), it can be seen that the NS energy mode $\xi_E$ is undisturbed by $u_g$, but is disturbed by $w_g$. The almost NS phugoid mode, $\xi_{P1}$ and $\xi_{P2}$, is less disturbed by $u_g$ and $w_g$ than the short period mode, $\xi_{S1}$ and $\xi_{S2}$.

From Eq. (37), it can be seen that a) the energy mode is unobservable, and the phugoid mode is almost unobservable, with $z_q$ and b) the short period mode and the wind components are almost unobservable with $z_h$. Thus, only the slight eigenvalue separation between the phugoid and energy modes allows us to observe them both with the $z_h$ measurement.

A SKF was designed for this problem (see Appendix C) with $Q_u = Q_w = 1.105$ (10 ft s$^{-1}$)$^2$ s$^{-1}$ (which corresponds to 7 ft s$^{-1}$ rms gusts with an 1860-ft correlation distance) and $R_q = 0.15$ (0.01 rad s$^{-1}$)$^2$ s, $R_h = 0.05$ (100 ft)$^2$ s. The results are shown in Table 2 and Fig. 6.

## Interpretation of the Estimators in Examples 3 and 4 as a Strapdown IMU without Accelerometers

The MDS observer of Table 1 and the SKF observer of Table 2 estimate roll angle and pitch angle with an rms accuracy of 0.22 deg and 0.32 deg, respectively, and heading deviation from compass-indicated north with an rms accuracy of 0.21 deg, all in the

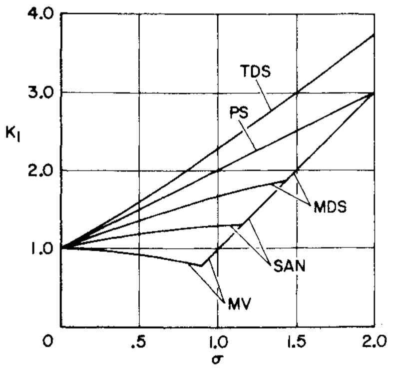


**Fig. 3 Observer gain $K_I$ as function of $\sigma$ for example 2 ($Q/R=3$).**

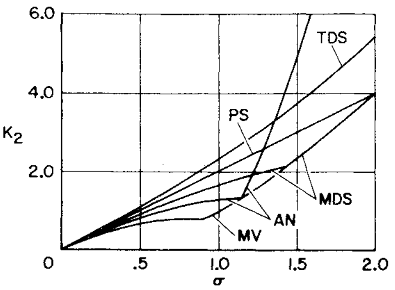


**Fig. 4 Observer gain $K_2$ as function of $\sigma$ for example 2 ($Q/R=3$).**

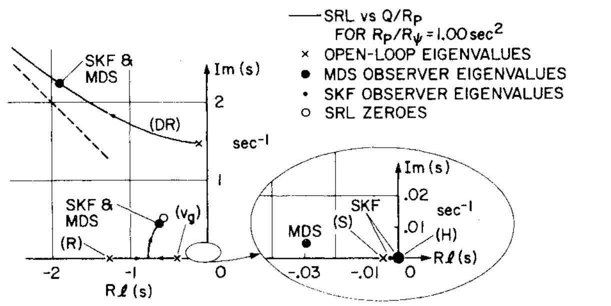


**Fig. 5 Location of eigenvalues for lateral motion observers—example 3 (DC-8 in cruise).**

presence of fairly severe turbulence. If transformed to locally level north-pointing axes, the horizontal velocity could be estimated with an rms accuracy of approximately $\sqrt{(3.66)^2+(2.44)^2}=4.44$ ft s$^{-1}$ which approaches the accuracy of current stable-platform inertial measurement units (IMU's) (about 1.5 to 3.0 ft s$^{-1}$). The two observers together, in fact, constitute a "strapdown IMU" that does not use accelerometers, gimbaled gyros, or servos; it only uses a magnetic compass, a barometric altimeter, a pitch-rate gyro, and a roll-rate gyro. Accelerations are estimated, in effect, through the coupling of accelerations to rotational motions described in the equations of motion [Eqs. (28) and (34)]. The sensitivity of rms errors to inaccuracies in the stability derivatives should, of course, be investigated. "Identifying" the stability derivatives from input-output data would help to eliminate these inaccuracies.

Note that computation of horizontal velocity and position would be a simple addition to the observers. Using VOR/DME data, this "strapdown IMU" could be updated periodically or continuously to provide quite accurate dead-reckoning. The observers obviously provide excellent state estimates for use in autopilot design.

## Divergent Linear-Quadratic Regulators

The same kind of divergence problem exists in designing optimal regulators using linear-quadratic synthesis as in designing steady-state Kalman filters.‡ It is quite possible to

‡This was pointed out to the author by T.L. Trankle.

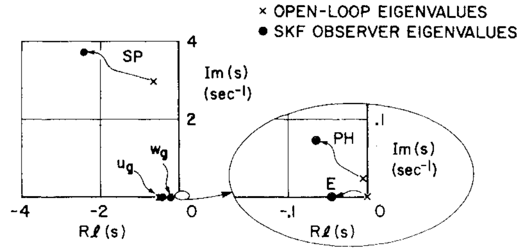


**Fig. 6 Location of eigenvalues for longitudinal motion observer—example 4 (DC-8 in cruise).**

choose outputs, $y(t)=Mx(t)$, such that one or more modes of the system, $\dot{x}=Fx$, are not observable with these outputs. The regulator gains obtained by quadratic-synthesis with performance index

$$J=\frac{1}{2}\int_0^\infty (y^TAy+u^TBu)\,dt$$

where $A$ and $B$ are positive definite matrices, and $\dot{x}=Fx+Gu$, will have zero gains on such modes (if they are stable or neutrally stable), yielding a "negligent" regulator, i.e., the eigenvalues of these modes will not be changed. This may be unacceptable if the mode is inadequately damped. Neglected (or controlled) modes can be detected by testing the system for complete observability with the outputs, i.e., $(F,M)$ must be a completely observable pair. Alternatively, if there are pole-zero cancellations (or near cancellations) in the transfer functions from the control vector $u(t)$ to the outputs $y(t)$, then the mode corresponding to the cancelled pole may be unobservable with $y$ *or* it may be uncontrollable with $u$. If a quadratic terminal penalty is imposed on the unobservable mode, then time-varying gains near the terminal time *will* control the mode, which is the dual of the TKF estimating undisturbed modes near the initial time.

## Conclusions

Steady-state Kalman filters (SKF) used as observers will diverge if there are undisturbed, neutrally stable (UNS) modes

**Table 2 Gains, eigenvalues, and rms estimate errors for DC-8 longitudinal motion estimator (SKF)**

| | | | |
|---|---|---|---|
| SKF gains $[K_qK_h]$ | 0.060 | 0.039 | $u$ |
| | 0.263 | −0.159 | $w$ |
| | 3.519 | 0.040 | $q$ |
| | 0.001 | −0.052 | $\theta$ |
| | 0.013 | 0.151 | $h$ |
| | −0.024 | 0.067 | $u_g$ |
| | −1.286 | 0.126 | $w_g$ |
| SKF eigenvalues sec$^{-1}$ | $-3.102\pm4.111j$ | | $(h,q)$ |
| | −0.416 | | $(\theta,w,w_g)$ |
| | −0.314 | | $(u_g)$ |
| | $-0.066\pm0.065j$ | | $(\theta,u)$ |
| | −0.033 | | $(u,\theta)$ |
| rms estimate errors | 2.44 | | $\tilde{u}$, ft s$^{-1}$ |
| | 5.12 | | $\tilde{w}$, ft s$^{-1}$ |
| | .318 | | $\tilde{\theta}$, deg |
| | 8.68 | | $\tilde{h}$, ft |
| | 6.75 | | $\tilde{u}_g$, ft s$^{-1}$ |
| | 5.70 | | $\tilde{w}_g$, ft s$^{-1}$ |
| Modal gains $[K_q^{(m)}K_h^{(m)}]$ | 4.850 | −0.095 | $\xi_{S1}$ } $SP$ |
| | 3.474 | 0.080 | $\xi_{S2}$ |
| | 0.001 | −0.319 | $\xi_{P1}$ } $PH$ |
| | 0.011 | 0.358 | $\xi_{P2}$ |
| | −0.008 | 0.470 | $\xi_E$ } $E$ |
| | −0.024 | 0.068 | $\xi_{u_g}$ } $u_g$ |
| | −2.767 | 0.271 | $\xi_{w_g}$ } $w_g$ |

in the system model. Modal destabilization (MDS) is a simple method for modifying the SKF gains to avoid divergence, with only a slight decrease in estimation accuracy. Eigenvalue-constrained minimum variance (MV) observers give the smallest decrease in estimation accuracy for a given attenuation time of estimation errors.

Estimator design in modal coordinates is similar to design using transfer functions (TF). However, it is superior to TF design in determining observability and disturbability (or controllability), particularly for multi-input, multi-output systems. The Euler angles and velocities of an airplane can be estimated with an accuracy approaching that of current inertial measurement units (IMU's) using only rate gyros, a magnetic compass, a barometric altimeter, and modified SKF's.

## Appendix A: Necessary Conditions for Gains in Constrained Minimum Variance (MV) Observer

The problem is stated in the text; choose the gains $K$ to minimize Eq. (9), subject to Eqs. (8) and (13). Adjoin Eqs. (8) and (13) to Eq. (9) with Lagrange multiplers $\Lambda$ (a matrix) and $\mu$ (a vector):

$$J = tr\{2MP + \Lambda[(F-KH)P + P(F-KH)^T + \Gamma Q \Gamma^T + KRK^T] + \mu^T C(K)\} \tag{A1}$$

where

$$\mu_i \begin{cases} = 0 \text{ if } C_i(K) > 0 \\ \leq 0 \text{ if } C_i(K) = 0 \end{cases} \tag{A2}$$

Consider differential changes in $J$ due to differential changes in $K$, holding the constraints satisfied:

$$dJ = 2tr\{[M + \Lambda(F-KH)]dP + [(RK^T - HP)\Lambda + \mu^T \frac{\partial C}{\partial K}]dK\} \tag{A3}$$

Choose $\Lambda$ to be

$$\Lambda = -M(F-KH)^{-1} \tag{A4}$$

Then a necessary condition for a stationary value of $J$ is that the coefficient of $dK$ in Eq. (A3) vanish:

$$(HP - RK^T)M(F-KH)^{-1} + \mu^T \frac{\partial C}{\partial K} = 0 \tag{A5}$$

Suppose there are $q$ gains, $K_i$, $[n(n+1)]/2$ elements in $P$, and $p$ constraints in Eq. (13). Then, Eqs. (A5, 8, and 13) are $q$, $[n(n+1)]/2$, and $p$ equations, respectively, for determining $K$, $P$, and $\mu$. This is a typical nonlinear programming problem; solution algorithms are given, for example, in Ref. 11. As an initial guess, one should use the gains from one of the approximate solutions like the MDS or SAN observers.

## Appendix B: Observers for Estimating a Constant Disturbance for a System with One Real Eigenvalue (Example 2)

From Eqs. (8) and (23), the steady-state covariance matrix is given by

$$0 = \begin{bmatrix} -1-K_1, 1 \\ -K_2, 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -1-K_1, -K_2 \\ 1, 0 \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} + R \begin{bmatrix} K_1^2, K_1K_2 \\ K_1K_2, K_2^2 \end{bmatrix} \tag{B1}$$

The solution to Eq. (B1) is

$$P_{11} = Q + \frac{R(K_1^2 + K_2)}{2(1+K_1)} \tag{B2}$$

$$P_{12} = \tfrac{1}{2} RK_2 \tag{B3}$$

$$P_{22} = K_2[P_{11} + (R/2)(1-K_1)] \tag{B4}$$

The augmented-stability characteristic equation [see Eq. (11)] is

$$s^2 + (1 + K_1 - 2\sigma)s + K_2 + \sigma^2 - (1+K_1)\sigma = 0 \tag{B5}$$

The Routh stability conditions for Eq. (B5), and hence the explicit gain constraints [see Eq. (13)], are

$$1 + K_1 - 2\sigma \geq 0 \tag{B6}$$

$$K_2 + \sigma^2 - (1+K_1)\sigma \geq 0 \tag{B7}$$

**MV Observer**

Suppose we wish to minimize $P_{11}$. In Eq. (9), this corresponds to the choice

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{B8}$$

Minimizing $P_{11}$ in Eq. (B2) with respect to $K_1$ and $K_2$ with the constraints in Eq. (B6) and (B7) is a straightforward nonlinear programming problem. The solution may be found in closed form as

$$\left. \begin{aligned} K_1 &= \sqrt{1 + \frac{Q}{R} - \sigma^2} - 1 \\ K_2 &= \sigma\sqrt{1 + \frac{Q}{R} - \sigma^2} - \sigma^2 \end{aligned} \right\} \text{ for } \sigma \leq \frac{1}{\sqrt{5}}\sqrt{1 + \frac{Q}{R}} \tag{B9a}$$

and

$$\left. \begin{aligned} K_1 &= 2\sigma - 1 \\ K_2 &= \sigma^2 \end{aligned} \right\} \text{ for } \sigma \geq \frac{1}{\sqrt{5}}\sqrt{1 + \frac{Q}{R}} \tag{B9b}$$

The minimum value of the variance $P_{11}$ is

$$P_{11} = \begin{cases} R\left(\sqrt{1 + \frac{Q}{R} - \sigma^2} + \frac{\sigma}{2} - 1\right), & \sigma \leq \frac{1}{\sqrt{5}}\sqrt{1 + \frac{Q}{R}} \\ R\left(\frac{1 + \frac{Q}{R}}{4\sigma} + \frac{5}{4}\sigma - 1\right), & \sigma \geq \frac{1}{\sqrt{5}}\sqrt{1 + \frac{Q}{R}} \end{cases} \tag{B10}$$

The corresponding eigenvalues are

$$(s_1,s_2)=(-\sigma,\ -\sqrt{1+\frac{Q}{R}-\sigma^2}+\sigma)\ \text{for}\ \sigma\leq\frac{1}{\sqrt{5}}\sqrt{1+\frac{Q}{R}} \tag{B11a}$$

and

$$(s_1,s_2)=(-\sigma,-\sigma)\ \text{for}\ \sigma\geq\frac{1}{\sqrt{5}}\sqrt{1+\frac{Q}{R}} \tag{B11b}$$

For $\sigma<(1/\sqrt{5})\sqrt{1+(Q/R)}$, Eq. (B6) is the "tight" (equality) constraint. For $\sigma>(1/\sqrt{5})\sqrt{1+(Q/R)}$, *both* constraints in Eqs. (B6) and (B7) are tight.

**SAN Observer**

In this approach, the input $x_2$ is modeled as a "random walk" process,

$$\dot{x}_2=w_2 \tag{B12}$$

where $w_2$ is a zero-mean, white noise process with spectral density $Q_2$. The steady-state Kalman filter (SKF) is then designed, varying $Q_2/R$ until the filter eigenvalues satisfy Eq. (10). For $\sigma>1.16$, the eigenvalues become a complex conjugate pair with real part equal to $-\sigma$. At $\sigma=1.16$, the AN observer is identical to the MV observer.

**PS Observer**

Here the zero eigenvalue of the SKF is shifted to $s=-\sigma$, while the other eigenvalue (at $s=-\sqrt{1+Q/R}$) is held fixed.

**MDS Observer**

In this approach, the input $x_2$ is modeled as a first-order unstable system with eigenvalue $s=\alpha$:

$$\dot{x}_2=\alpha x_2 \tag{B13}$$

The SKF is then designed, yielding gains $K_1$ and $K_2$, which are used in Eqs. (25) and (B2) to find the observer eigenvalues and the variance $P_{11}$; $\alpha$ is varied until the eigenvalues satisfy Eq. (10). For $\sigma>1.414$, the eigenvalues become a complex conjugate pair with real part equal to $-\sigma$. At $\sigma=1.414$, the DS observer is identical to the MV observer.

**TDS Observer**

Here the total system is destabilized by an amount $\sigma/2$:

$$\dot{x}_1=\left(\frac{\sigma}{2}-1\right)x_1+x_2+w,\ \dot{x}_2=\frac{\sigma}{2}x_2 \tag{B14}$$

The SKF is then designed, yielding gains $K_1$ and $K_2$, which are again used in Eqs. (25) and (B2) to find the observer eigenvalues and the variance $P_{11}$. The zero eigenvalue will be shifted to $s=-\sigma$.

## Appendix C: Observers for Estimating the Lateral and Longitudinal Motions of a DC-8 (Examples 3 and 4)

The primary tool used here was the OPTSYS computer program,[10] modified to include an arbitrary observer gain ($K$) option [to compute $P$ in Eq. (8) and the eigenvalues of $F-KH$]. An option was also added to compute the left eigenvector matrix, $T^{-1}$[examples in Eqs. (33) and (B9)], the modal disturbance matrix, $T^{-1}\Gamma$ [examples in Eqs. (28) and (36)], the modal measurement matrix, $HT$ [examples in Eqs. (30) and (37)], and the modal filter gains, $T^{-1}K$ (examples in Tables 1 and 2). The left eigenvector matrix is simply the inverse of the right eigenvector matrix. The columns of the right eigenvector matrix $T$ are the eigenvectors (for a complex eigenvalue, the first column is the real part and the second column is the imaginary part of the eigenvector); Eqs. (31) and (38) are examples of eigenvector matrices.

Designing in modal coordinates is essentially like using transfer functions in partial fraction form instead of in pole-zero form.

It is actually more informative than transfer function (TF) methods since a pole-zero cancellation in the TF matrix from $w(s)$ to $Hx(s)$ indicates that that mode is *either* undisturbable by $w$ *or* unobservable with $Hx$. Inspection of the modal disturbance matrix and the modal measurement matrix clears up this ambiguity. If the mode is undisturbable we can shift the eigenvalue; if it is unobservable, we cannot.

## Acknowledgment

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