The Epoch State Navigation Filter

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The formulation of a recursive maximum-likelihood navigation system employing reference position and velocity vectors as state variables is presented. Convenient forms of the required variational equations of motion are developed, together with an explicit form of the associated state transition matrix needed to refer measurement data from the measurement time to the epoch time. Computational advantages accrue from this design in that the usual forward extrapolation of the covariance matrix of estimation errors can be avoided without incurring unacceptable system errors. Simulation data for Earth-orbiting satellites are provided to substantiate this assertion.

Introduction

INHERENT in the formulation of a navigation system for a spacecraft is the selection of an appropriate set of state variables. A common choice is the current position and velocity vectors $\boldsymbol{r}$ and $\boldsymbol{v}$, as these are the quantities that one normally wishes to estimate. The navigation technique described in this paper uses a related set of state variables, i.e., the position and velocity vectors $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ of the vehicle at some epoch time t_0. Although basically equivalent to the conventional recursive Kalman filter, this approach has some significant computational advantages.

In its simplest form, the recursive maximum-likelihood estimator[1] or Kalman filter does not include state or process noise. The covariance matrix $\boldsymbol{P}$ of estimation errors is six-dimensional and is propagated with time, usually by integrating 36 simultaneous differential equations along with six more such equations for the state vector. Then, as measurement data are obtained and incorporated to improve the state vector estimate, the covariance matrix is updated via some vector-matrix operations.

By electing $\boldsymbol{r}_0$, $\boldsymbol{v}_0$ to be the state vectors, with $\boldsymbol{P}_0$ as the associated covariance matrix, it is clear that these quantities are time invariant for undisturbed two-body motion and are altered only as the result of measurement incorporation. Under these ideal circumstances, the vectors $\boldsymbol{r}$ and $\boldsymbol{v}$ at time t are obtainable from standard closed-form two-body equations. The effect of a measurement can be extrapolated backward in time to the epoch t_0 to accomplish the update of the state vector and associated covariance matrix $\boldsymbol{P}_0$. In this manner, we can avoid the forward extrapolation of the covariance matrix which is both time-consuming and subject to problems of numerical accuracy.

In the practical case of disturbed two-body motion, each of the quantities $\boldsymbol{r}_0$, $\boldsymbol{v}_0$, and $\boldsymbol{P}_0$ changes with time but only slightly if the disturbing forces are small. The essence of the navigation scheme proposed here is that the changes in $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are tracked by solving the variational equations of motion, but the covariance matrix $\boldsymbol{P}_0$ is assumed to remain time invariant. The effect of this assumption on the total navigation accuracy can be assessed by numerical simulation. Indeed, as is shown in the last section, the numerical studies for 100-mile-alt Earth-orbiting satellites demonstrate that the errors produced under these assumptions are comparable to those obtained by conventional methods.[2]

A significant portion of this paper is devoted to developing variational equations appropriate to the proposed navigation scheme. They first appeared in Ref. 3 and were used later by Born et al.[4] in exploring the numerical characteristics of various formulations of the variational method as compared with Encke's method. Because of the particular elegance and general applicability of these variational equations, some details concerning their derivations have been included. For other formulations of the variational problem, see the references in Ref. 4.

Another section of the paper includes the development of an explicit form of the state transition matrix which is required to extrapolate the measurement geometry vectors from the time of the measurement back to the epoch time. Part of these results appeared in Ref. 3, and part are reported here for the first time. The close relationship between the partitions of the transition matrix and the variational equations is exploited to minimize the total computational requirements.

In the following section, a summary of the required navigation equations for the epoch state filter is presented. Derivations of any of the less obvious results appear in the later sections.

Navigation Equations

The technique of employing osculating reference position and velocity vectors in describing the motion of a spacecraft is well known. For the navigation scheme proposed in this paper, we select as state variables the initial position and velocity vectors $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ at the epoch time t_0. The equations of motion describe the variation with time of the epoch state due to the action of a vector disturbing acceleration $\boldsymbol{a}_d$, which represents the difference between all accelerations affecting the motion and the two-body central field. These are integrated using any suitable numerical process.

A particularly convenient form of the variational equations

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for these state variables is

$$\frac{dr_0}{dt} = R_0^* a_d; \qquad \frac{dv_0}{dt} = V_0^* a_d \tag{1}$$

where the matrices R_0^* and V_0^* are defined as

$$R_0^* = \frac{\partial r_0}{\partial v}; \qquad V_0^* = \frac{\partial v_0}{\partial v} \tag{2}$$

and evaluated at the reference time t_0. The actual position and velocity vectors r, v of the vehicle at time t are computed in the usual manner from the standard two-body formulas:

$$r = Fr_0 + Gv_0; \qquad v = F_t r_0 + G_t v_0 \tag{3}$$

where $F, \dots G_t$ are the classical Lagrangian coefficients.

To avoid secular terms in the equations of motion, the reference time t_0 is permitted to vary. Rather than computing the variation of t_0 as a function of time, we choose instead to determine the true anomaly difference θ between r_0 and r as the solution of the differential equation

$$\frac{d\theta}{dt} = \frac{\sqrt{\mu p}}{r^2} + \frac{r}{\sqrt{\mu p}} \left(\frac{1}{r} r^T - \frac{1}{r_0} r_0{}^T \right) a_d \tag{4}$$

where μ is the gravitational constant and p is the instantaneous parameter of the orbit. We may calculate p from

$$p = 2r_0 - \alpha r_0{}^2 - \sigma_0{}^2 \tag{5}$$

where α, the reciprocal of the osculating semimajor axis, and σ_0 are defined by

$$\alpha = \frac{2}{r_0} - \frac{v_0{}^2}{\mu}; \qquad \sigma_0 = \frac{1}{\sqrt{\mu}} r_0 \cdot v_0 \tag{6}$$

In terms of these quantities, the Lagrangian coefficients may be calculated from

$$F = 1 - (r/p)(1 - \cos\theta); \quad \sqrt{\mu} G = (r r_0 / \sqrt{p}) \sin\theta \tag{7a}$$

$$F_t = (\sqrt{\mu}/r r_0{}^2)[r_0 \sigma_0 (1-F) - \sqrt{\mu} G]; \; G_t = 1 - (r_0/r)(1-F) \tag{7b}$$

and the length of the position vector r from

$$r = \frac{p r_0}{r_0 + (p - r_0)\cos\theta - \sqrt{p}\sigma_0 \sin\theta} \tag{8}$$

Finally, the matrices R_0^* and V_0^* are obtained as

$$R_0^* = (r_0/\mu)(1-F)[v r^T - (r - r_0) v^T] - GI \tag{9a}$$

$$V_0^* = (r/\mu)(v - v_0)(v - v_0)^T + FI \tag{9b}$$

where I is the three-dimensional identity matrix.

An important ingredient of any navigation system is frequent measurements used to improve the estimate of the vehicle state: in this case, the epoch quantities r_0 and v_0. For our present purposes, we consider only the simplest form of the recursive maximum-likelihood estimator or Kalman filter, as it is often called. Thus, we associate with each measurement (restricted to scalar quantities for simplicity) a six-dimensional vector b representing, to a first order of approximation, the variation in the measured quantity Q which would result from variations in the components of r and v at the measurement time t. Specifically,

$$\delta Q = (\delta r^T \quad \delta v^T) b \tag{10}$$

The effect of the measurement on the epoch state can be obtained by referring the b vector to the epoch time t_0 which can be accomplished by means of the state transition matrix

$$\Phi(t_0, t) = \begin{bmatrix} \dfrac{\partial r_0}{\partial r} & \dfrac{\partial r_0}{\partial v} \\ \dfrac{\partial v_0}{\partial r} & \dfrac{\partial v_0}{\partial v} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial r_0}{\partial r} & R_0^* \\ \dfrac{\partial v_0}{\partial r} & V_0^* \end{bmatrix} \tag{11}$$

The effective measurement vector b_0 is simply

$$b_0 = \Phi(t_0, t)^{T-1} b$$

and, since Φ is a symplectic matrix, we have

$$b_0 = \begin{bmatrix} V_0^* & -\dfrac{\partial v_0}{\partial r} \\ -R_0^* & \dfrac{\partial r_0}{\partial r} \end{bmatrix} b \tag{12}$$

It is important to observe that, if only the first three components of the vector b are nonzero, then just the matrices R_0^* and V_0^* are involved in computing b_0. This is the case for angle and range measurements, typical of that class of observations which can be made conveniently on board a spacecraft.

The weighting vector w appropriate for the Kalman filter is determined as

$$w = a^{-1} P_0 b_0 \tag{13}$$

where P_0 is the covariance matrix of estimation errors at the epoch, and

$$a = b_0^T P_0 b_0 + \overline{\nu^2} \tag{14}$$

with $\overline{\nu^2}$ denoting the mean-squared a priori measurement error. Then, if δQ represents the difference between the quantity actually measured and its expected value based on the current best estimate of the state, the change in the epoch state vector is simply

$$(\delta r_0^T \quad \delta v_0^T) = w^T \delta Q \tag{15}$$

As a result of the measurement, the statistics embodied in the covariance matrix P_0 must be updated. Again, the Kalman theory dictates that the new value for P_0, denoted by P_0^*, is obtained from

$$P_0^* = P_0 (I - a^{-1} b_0 b_0^T P_0) \tag{16}$$

Finally, it is necessary also to update the angle θ as a result of the measurement. This is accomplished readily by determining the change δr as

$$\delta r = V_0^{*T} \delta r_0 - R_0^{*T} \delta v_0 \tag{17}$$

and then calculating the angle θ between the new values of r and r_0.

Variational Equations

For the two-body problem, the position and velocity vectors r, v at time t are related to their values r_0, v_0 at some epoch time t_0 according to

$$r = Fr_0 + Gv_0; \qquad r_0 = G_t r - Gv \tag{18a}$$

$$v = F_t r_0 + G_t v_0; \qquad v_0 = -F_t r + Fv \tag{18b}$$

The Lagrangian coefficients may be expressed in terms of certain universal functions $U_n(\chi;\alpha)$. These special transcendental functions are defined, for $n=0, 1,\ldots$, by

$$U_n(\chi;\alpha)=\chi^n\left[\frac{1}{n!}-\frac{\alpha\chi^2}{(n+2)!}+\frac{(\alpha\chi^2)^2}{(n+4)!}-\cdots\right] \tag{19}$$

Their introduction in the two-body problem in place of the usual circular and hyperbolic functions renders all of the following equations "universal" in the sense that they are valid without change for all conic orbits and are void of singularities.

Specifically, we have

$$F=1-U_2/r_0=(rU_0-\sigma U_1)/r_0 \tag{20a}$$

$$\sqrt{\mu}G=r_0U_1+\sigma_0U_2=rU_1-\sigma U_2=\sqrt{\mu}(t-t_0)-U_3 \tag{20b}$$

$$F_t=-\sqrt{\mu}U_1/rr_0 \tag{20c}$$

$$G_t=1-U_2/r=(r_0U_0+\sigma_0U_1)/r \tag{20d}$$

where

$$\sqrt{\mu}\sigma=\boldsymbol{r}^T\boldsymbol{v} \tag{21a}$$

$$\alpha=2/r_0-v_0^2/\mu=2/r-v^2/\mu \tag{21b}$$

The universal form of Kepler's equation is

$$\sqrt{\mu}(t-t_0)=r_0U_1+\sigma_0U_2+U_3=rU_1-\sigma U_2+U_3 \tag{22}$$

and the following relations for r and σ also obtain:

$$r=r_0U_0+\sigma_0U_1+U_2; \qquad r_0=rU_0-\sigma U_1+U_2 \tag{23a}$$

$$\sigma=\sigma_0U_0+(1-\alpha r_0)U_1; \quad \sigma_0=\sigma U_0-(1-\alpha r)U_1 \tag{23b}$$

To relate the preceding equations to the classical ones, we can show that

$$\chi=\begin{cases}\sqrt{a}(E-E_0) & \text{ellipse}\\ \sqrt{p}(\tan\tfrac{1}{2}f-\tan\tfrac{1}{2}f_0) & \text{parabola}\\ \sqrt{-a}(H-H_0) & \text{hyperbola}\end{cases}$$

using a standard notation of celestial mechanics.

Certain identities, which exist among the functions U_n and will be required in the sequel, are

$$\begin{gathered}U_n+\alpha U_{n+2}=\chi^n/n!\\ U_0^2+\alpha U_1^2=1 \qquad U_1^2-U_0U_2=U_2\\ U_0U_3-U_1U_2=U_3-\chi U_2 \qquad U_1U_3-U_2^2=2U_4-\chi U_3\end{gathered} \tag{24}$$

along with the differential relations

$$\frac{\partial U_0}{\partial\chi}=-\alpha U_1 \tag{25a}$$

$$\frac{\partial U_n}{\partial\chi}=U_{n-1} \quad (n=1,2,\ldots,) \tag{25b}$$

$$\frac{\partial U_n}{\partial\alpha}=\tfrac{1}{2}(nU_{n+2}-\chi U_{n+1}) \tag{25c}$$

The derivation of the variational equations is a straightforward but tedious process. We report certain intermediate milestones as an aid to the serious reader.

First, we establish

$$\mathrm{d}U_0=-\alpha U_1\mathrm{d}\zeta-\tfrac{1}{2}U_1^2\mathrm{d}\alpha \tag{26a}$$

$$\mathrm{d}U_1=U_0\mathrm{d}\zeta-\tfrac{1}{2}U_1U_2\mathrm{d}\alpha \tag{26b}$$

$$\mathrm{d}U_2=U_1\mathrm{d}\zeta-\tfrac{1}{2}U_2^2\mathrm{d}\alpha \tag{26c}$$

$$\mathrm{d}U_3=U_2\mathrm{d}\zeta-\tfrac{1}{2}(U_2U_3-3U_5+\chi U_4)\mathrm{d}\alpha \tag{26d}$$

where $\mathrm{d}\zeta$ is defined as

$$\mathrm{d}\zeta=\mathrm{d}\chi+\tfrac{1}{2}U_3\mathrm{d}\alpha \tag{27}$$

Using these, we calculate the differentials for r_0 and $\sqrt{\mu}(t-t_0)$ as

$$\mathrm{d}r_0=-\sigma_0\mathrm{d}\zeta-\tfrac{1}{2}(r_0+r)U_2\mathrm{d}\alpha-U_1\mathrm{d}\sigma+U_0\mathrm{d}r \tag{28a}$$

$$\sqrt{\mu}\mathrm{d}(t-t_0)=r_0\mathrm{d}\zeta+\tfrac{1}{2}\sqrt{\mu}c\mathrm{d}\alpha-U_2\mathrm{d}\sigma+U_1\mathrm{d}r \tag{28b}$$

with the symbol c, introduced for notational convenience and defined by

$$\sqrt{\mu}c=3U_5-\chi U_4-\sqrt{\mu}(t-t_0)U_2 \tag{29}$$

The differentials of the epoch quantities $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are obtained after some extensive manipulation and can be expressed in the form

$$\begin{aligned}\mathrm{d}\boldsymbol{r}_0=&-\frac{r_0}{\sqrt{\mu}}\boldsymbol{v}_0\mathrm{d}\zeta+\tfrac{1}{2}U_2(\boldsymbol{r}-\boldsymbol{r}_0)\mathrm{d}\alpha+\frac{U_2}{\sqrt{\mu}}\boldsymbol{v}\mathrm{d}\sigma\\ &+\left(\frac{U_2}{r^2}\boldsymbol{r}-\frac{U_1}{\sqrt{\mu}}\boldsymbol{v}\right)\mathrm{d}r+G_t\mathrm{d}\boldsymbol{r}-G\mathrm{d}\boldsymbol{v}\end{aligned} \tag{30}$$

$$\begin{aligned}\mathrm{d}\boldsymbol{v}_0=&\frac{\sqrt{\mu}}{r_0^2}\boldsymbol{r}_0\mathrm{d}\zeta-\frac{1}{r_0}(\boldsymbol{v}-\boldsymbol{v}_0)(\tfrac{1}{2}rU_2\mathrm{d}\alpha+U_1\mathrm{d}\sigma)\\ &+\frac{1}{r}[U_0(\boldsymbol{v}-\boldsymbol{v}_0)+F_t\boldsymbol{r}]\mathrm{d}r-F_t\mathrm{d}\boldsymbol{r}+F\mathrm{d}\boldsymbol{v}\end{aligned} \tag{31}$$

The matrix elements of the variational equations for $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are determined by first observing that

$$\frac{\partial\alpha}{\partial\boldsymbol{v}}=-\frac{2}{\mu}\boldsymbol{v}^T \tag{32a}$$

$$\frac{\partial\sigma}{\partial\boldsymbol{v}}=\frac{1}{\sqrt{\mu}}\boldsymbol{r}^T \tag{32b}$$

$$\frac{\partial\zeta}{\partial\boldsymbol{v}}=\frac{1}{\sqrt{\mu}r_0}\left(c\boldsymbol{v}^T+U_2\boldsymbol{r}^T-\mu\frac{\partial t_0}{\partial\boldsymbol{v}}\right) \tag{32c}$$

Then we may write

$$\begin{aligned}\boldsymbol{R}_0^*=&\frac{r_0}{\mu}(1-F)[\boldsymbol{v}\boldsymbol{r}^T-(\boldsymbol{r}-\boldsymbol{r}_0)\boldsymbol{v}^T]-G\boldsymbol{I}\\ &+\boldsymbol{v}_0\left[\frac{\partial t_0}{\partial\boldsymbol{v}}-\frac{r_0}{\mu}(1-F)\boldsymbol{r}^T-\frac{c}{\mu}\boldsymbol{v}^T\right]\end{aligned} \tag{33}$$

$$\begin{aligned}\boldsymbol{V}_0^*=&\frac{r}{\mu}(\boldsymbol{v}-\boldsymbol{v}_0)(\boldsymbol{v}-\boldsymbol{v}_0)^T+F\boldsymbol{I}\\ &-\frac{\boldsymbol{r}_0}{r_0^3}\left[\mu\frac{\partial t_0}{\partial\boldsymbol{v}}-r_0(1-F)\boldsymbol{r}^T-c\boldsymbol{v}^T\right]\end{aligned} \tag{34}$$

It is desirable[4] from the point of view of numerical integration to avoid secular or mixed terms in the variational equations of motion. These arise in Eqs. (33) and (34) due to the presence of the quantity c defined in Eq. (29). Therefore, the temptation of arbitrarily choosing the epoch time t_0 to be a constant will be avoided. Instead, we elect to have t_0 vary in such a way that

$$\mu\frac{\partial t_0}{\partial \boldsymbol{v}}=r_0(1-F)\boldsymbol{r}^T+c\boldsymbol{v}^T \tag{35}$$

so that $\boldsymbol{R}_0^*$ and $\boldsymbol{V}_0^*$ reduce to the expressions given in Eq. (9).

Equation (35) implies that an additional variational equation must be solved numerically in addition to Eqs. (1). Again the presence of the term involving c in Eq. (35) makes it undesirable to determine t_0 by numerical integration.

Although somewhat in violation of the spirit of the variational method, the problem can be eliminated by solving instead the differential equation for the true anomaly difference θ between $\boldsymbol{r}$ and $\boldsymbol{r}_0$. An additional advantage accrues in that the necessity of repeatedly solving Kepler's equation also is avoided. The penalty paid, however, is that the equations are not valid for rectilinear motion, since θ is not defined for $p=0$.

To calculate the variation in the true anomaly difference, we first obtain the differential for σ_0 as

$$\mathrm{d}\sigma_0=-(1-\alpha r_0)\,\mathrm{d}\zeta+\tfrac{1}{2}(r_0+r)\,U_1\mathrm{d}\alpha+U_0\mathrm{d}\sigma \tag{36}$$

Then, by comparing Eqs. (7) and (20), we observe that

$$pU_1=r\sqrt{p}\sin\theta-r\sigma_0(1-\cos\theta) \tag{37a}$$

$$pU_2=rr_0(1-\cos\theta) \tag{37b}$$

Now the variation is calculated with respect to $\boldsymbol{v}$ for each of Eqs. (37) using the differentials of U_1 and U_2 as obtained from Eqs. (26). There result two expressions for $\partial\zeta/\partial\boldsymbol{v}$, the first having as its coefficient F and the second the coefficient G.

We solve these two equations for $\partial\zeta/\partial\boldsymbol{v}$ by multiplying the first by G_t, the second by F_t, subtracting, and using the identity

$$FG_t-F_tG=1$$

After considerable simplification, using Eq. (5) in particular, we obtain

$$\frac{\partial\zeta}{\partial\boldsymbol{v}}=\frac{r_0}{\sqrt{p}}\frac{\partial\theta}{\partial\boldsymbol{v}}-\frac{1}{2r}\left(\frac{1}{r_0}U_1U_2+\frac{1}{p}(r_0U_1+\sigma_0U_2)\right]\frac{\partial p}{\partial\boldsymbol{v}}$$

$$-\frac{r}{2r_0}U_1U_2\frac{\partial\alpha}{\partial\boldsymbol{v}}-\frac{U_2}{rr_0}(r_0+\sigma U_1)\frac{\partial\sigma}{\partial\boldsymbol{v}} \tag{38}$$

Finally, substituting from Eqs. (32) and using the readily derived variational derivative for p,

$$\mu\frac{\partial p}{\partial\boldsymbol{v}}=2\boldsymbol{r}^T(r\boldsymbol{v}^T-\boldsymbol{v}\boldsymbol{r}^T) \tag{39}$$

yields

$$\frac{\partial\theta}{\partial\boldsymbol{v}}=\frac{r}{\sqrt{\mu p}}\left(\frac{1}{r}\boldsymbol{r}^T-\frac{1}{r_0}\boldsymbol{r}_0{}^T\right) \tag{40}$$

and Eq. (4), the differential equation for θ, is established.

State Transition Matrix

Two of the four submatrices, which constitute the state transition matrix used in extrapolating the measurement geometry vector $\boldsymbol{b}$ from the measurement time to the epoch time, have been determined in the previous section. These are sufficient, as already noted, if the measurement capability of the navigation system includes only range and angle observations. In general, however, we need to determine also $\partial\boldsymbol{r}_0/\partial\boldsymbol{r}$ and $\partial\boldsymbol{v}_0/\partial\boldsymbol{r}$ appearing in Eq. (11) as partitions of the transition matrix.

For this purpose, we first obtain

$$\frac{\partial\alpha}{\partial\boldsymbol{r}}=-\frac{2}{r^3}\boldsymbol{r}^T,\quad\frac{\partial\sigma}{\partial\boldsymbol{r}}=\frac{1}{\sqrt{\mu}}\boldsymbol{v}^T,\quad\frac{\partial r}{\partial\boldsymbol{r}}=\frac{1}{r}\boldsymbol{r}^T \tag{41a}$$

$$\frac{\partial\zeta}{\partial\boldsymbol{r}}=\frac{1}{\sqrt{\mu}}\left[\frac{\mu c}{r_0r^3}\boldsymbol{r}^T+(\boldsymbol{v}-\boldsymbol{v}_0)^T-\frac{\mu}{r_0}\frac{\partial t_0}{\partial\boldsymbol{r}}\right] \tag{41b}$$

so that, from Eqs. (30) and (31), we have

$$\frac{\partial\boldsymbol{r}_0}{\partial\boldsymbol{r}}=\frac{r_0}{\mu}(\boldsymbol{v}-\boldsymbol{v}_0)(\boldsymbol{v}-\boldsymbol{v}_0)^T+G_t\boldsymbol{I}$$

$$+\frac{1}{r^3}[r_0(1-F)\boldsymbol{r}_0-c\boldsymbol{v}_0]\boldsymbol{r}^T+\boldsymbol{v}_0\frac{\partial t_0}{\partial\boldsymbol{r}} \tag{42}$$

$$\frac{\partial\boldsymbol{v}_0}{\partial\boldsymbol{r}}=\frac{\mu c}{r^3r_0{}^3}\boldsymbol{r}_0\boldsymbol{r}^T+\frac{1}{r^2}(\boldsymbol{v}-\boldsymbol{v}_0)\boldsymbol{r}^T+\frac{1}{r_0{}^2}\boldsymbol{r}_0(\boldsymbol{v}-\boldsymbol{v}_0)^T$$

$$-F_t\left[\boldsymbol{I}-\frac{1}{r^2}\boldsymbol{r}\boldsymbol{r}^T+\frac{1}{\mu r}(\boldsymbol{v}-\boldsymbol{v}_0)\boldsymbol{r}^T(\boldsymbol{v}\boldsymbol{r}^T-\boldsymbol{r}\boldsymbol{v}^T)\right]$$

$$-\frac{\mu}{r_0{}^3}\boldsymbol{r}_0\frac{\partial t_0}{\partial\boldsymbol{r}} \tag{43}$$

with $\partial t_0/\partial\boldsymbol{r}$ still to be determined.

In the previous section, we found it desirable to specify $\partial t_0/\partial\boldsymbol{v}$ as in Eq. (35). Therefore, we write

$$\frac{\partial t_0}{\partial\boldsymbol{r}}=\frac{\partial t_0}{\partial\boldsymbol{v}}\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{r}}$$

where the matrix $\partial\boldsymbol{v}/\partial\boldsymbol{r}$ is known[5] to be symmetric. Thus, we have

$$\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{r}}=\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{v}_0}\left(\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{v}_0}\right)^{-1}=\left(\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{v}_0}\right)^{T-1}\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{v}_0}\right)^T$$

Now, because of the symplectic nature of the state transition matrix, it is readily shown that

$$\left(\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{v}_0}\right)^T=-\frac{\partial\boldsymbol{r}_0}{\partial\boldsymbol{v}}=-\boldsymbol{R}_0^*;\quad\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{v}_0}\right)^T=\frac{\partial\boldsymbol{r}_0}{\partial\boldsymbol{r}}$$

so that, from the foregoing and Eq. (35), we obtain

$$\frac{\partial t_0}{\partial\boldsymbol{r}}=-\frac{1}{\mu}[r_0(1-F)\boldsymbol{r}^T+c\boldsymbol{v}^T]\boldsymbol{R}_0^{*-1}\frac{\partial\boldsymbol{r}_0}{\partial\boldsymbol{r}} \tag{44}$$

We conclude by substituting Eq. (44) into Eq. (42) and solving for $\partial\boldsymbol{r}_0/\partial\boldsymbol{r}$. There results

$$\frac{\partial\boldsymbol{r}_0}{\partial\boldsymbol{r}}=\boldsymbol{R}_0^*\left\{\boldsymbol{R}_0^*+\frac{\boldsymbol{v}_0}{\mu}[r_0(1-F)\boldsymbol{r}^T+c\boldsymbol{v}^T]\right\}^{-1}$$

$$\times\left\{\frac{r_0}{\mu}(\boldsymbol{v}-\boldsymbol{v}_0)(\boldsymbol{v}-\boldsymbol{v}_0)^T+\frac{1}{r^3}[r_0(1-F)\boldsymbol{r}_0-c\boldsymbol{v}_0]\boldsymbol{r}^T+G_t\boldsymbol{I}\right\} \tag{45}$$

The matrix $\partial\boldsymbol{v}_0/\partial\boldsymbol{r}$ is obtained directly from Eqs. (43-45) without further calculations. Note that, in computing the

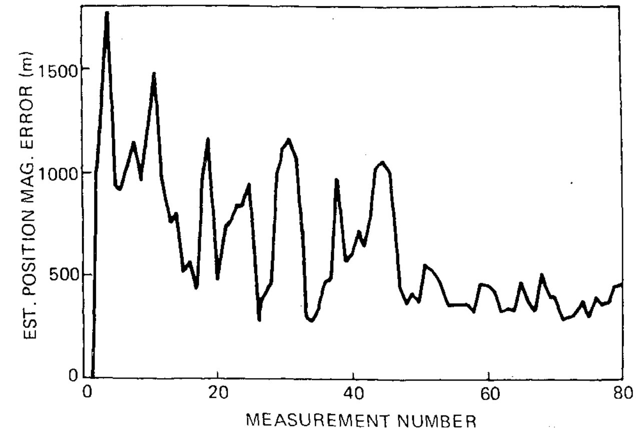

Fig. 1 Magnitude of error in the position estimate during orbit.

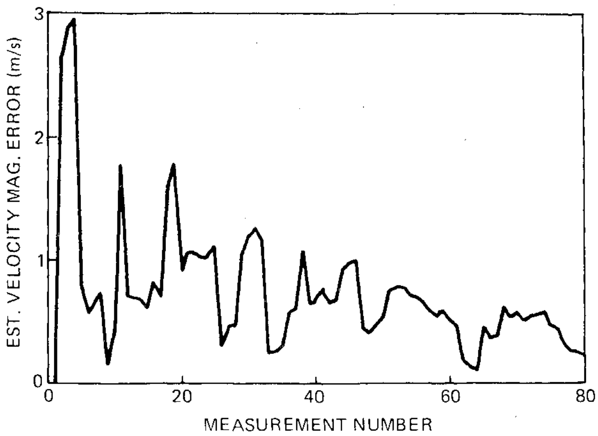

Fig. 2 Magnitude of error in the velocity estimate during orbit.

variations of r_0 and v_0 with respect to r, the inversion of one 3×3 matrix is required.

Finally, it is worth remarking that $\partial v_0 / \partial r$ can be obtained without using Eq. (43) but at the expense of also inverting the matrix R_0^*. We have

$$\frac{\partial v_0}{\partial r} = -R_0^{*T-1}\left(I - V_0^{*T}\frac{\partial r_0}{\partial r}\right) \tag{46}$$

as can be demonstrated using the basic properties of the transition matrix.

Simulation Results

To evaluate the performance of the epoch state navigation technique, as compared with the more conventional method detailed in Ref. 1, a number of simulation studies[2] were performed. A circular equatorial Earth orbit of 185.2 km alt (100 n.m.), disturbed only by the J_2 term in the Earth's gravitational potential, served as the reference trajectory for most of the study.

Measurement data were obtained and incorporated at intervals of 10-deg central angle, for a total of 80 measurements in all, i.e., for more than two complete revolutions. For simplicity, the measurement geometry vector b was chosen to be a unit vector along one of the geocentric coordinate axes. The particular choice of axis was permuted cyclically to avoid bias. The error in the measurement ν was produced by a random number generator having an rms value of 1 km. The initial covariance matrix P_0 was chosen to be diagonal with three equal position elements of $(500\text{ m})^2$ and three equal velocity elements of $(5\text{ m/s})^2$. Had larger values for these elements been chosen, they would have been reduced quickly to approximately this size after only a few measurements because of the rms value of ν.

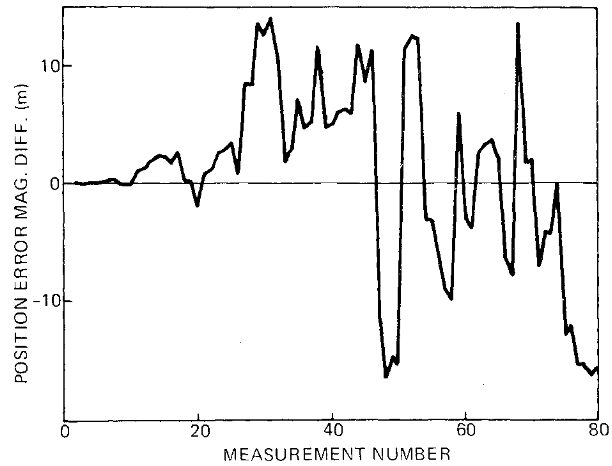

Fig. 3 Difference between position error magnitudes (epoch state minus conventional filter) during orbit.

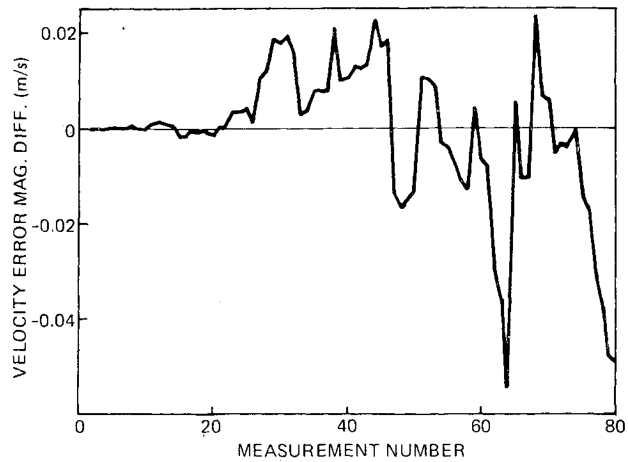

Fig. 4 Difference between velocity error magnitudes (epoch state minus conventional filter) during orbit.

The epoch state variational equations, Eqs. (1) and (4), were integrated by a fourth-order Runge-Kutta process with a 5-deg step size, assuming a time-invariant covariance matrix. For the conventional method, the state was propagated by Encke's method (again with a fourth-order Runge-Kutta algorithm), the covariance matrix was extrapolated using the numerically integrated state transition matrix. In the latter case, a 1-deg integration step was required to obtain the same accuracy as achieved by the variational formulation.

The performances of the two navigation systems were practically indistinguishable. In Figs. 1 and 2 are plotted the magnitudes of the difference between the estimated and true values of position and velocity, respectively, for the epoch state filter, while in Figs. 3 and 4 are shown the performance differences between the two techniques. More specifically, in Fig. 3, we have plotted the difference between the magnitudes of the errors for the epoch state position and the conventionally estimated position, and in Fig. 4 are plotted the same differences for velocity. Thus, although the position estimate magnitude varies between 1800 and 500 m, the performance difference ranges between +13 and −18 m. When the sign is positive, the performance of the conventional method is better, and for negative values, the epoch state method is superior. Similarly, the magnitude of the error in the velocity estimate does not exceed 3 m/s in either case, whereas the performance difference ranges between +0.02 and −0.05 m/s.

The last two plots, Figs. 5 and 6, show the excursion history of the epoch state estimates of position and velocity magnitude differences from their true initial values. Thus, we

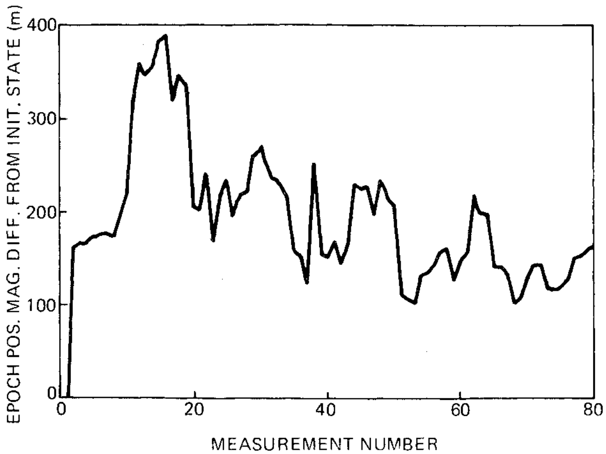

Fig. 5 Excursion history of epoch state position magnitude during orbit.

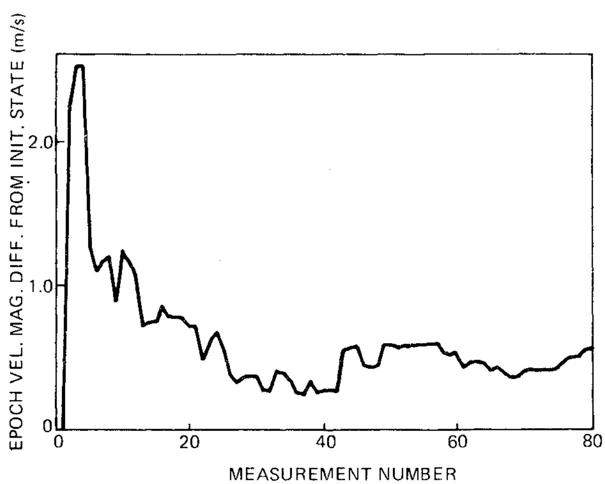

Fig. 6 Excursion history of epoch state velocity magnitude during orbit.

see that the epoch position and velocity remain within 400 m and 2.5 m/s of their starting values. These results lend some credence to the assumption of a time-invariant covariance matrix.

Further simulation results are reported in Ref. 2. For example, the disturbing acceleration was increased tenfold in magnitude, with no qualitative change in the performance difference of the two navigation methods.

Conclusions

The formulation of a recursive maximum-likelihood navigation system employing reference position and velocity vectors as state variables permits roughly an order-of-magnitude reduction in the computational time from that which would be required by more conventional methods. There is, however, some storage penalty associated with the epoch state filter implementation, since the equations for propagating the state variables are somewhat more complex. Both memory and processing time are, of course, dependent on the specific host computer.

The epoch state filter has been shown to perform almost identically to the conventional filter as far as accuracy is concerned. Although Figs. 3 and 4 indicate that sometimes one formulation has better performance than the other, the conclusion to be drawn here is that the details of their difference are insignificant from the standpoint of the absolute performance, as recorded in Figs. 1 and 2.

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