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Rendezvous of Controlled Systems with Time-Delay

Zeungnam Bien*
Korea Advanced Institute of Science, Seoul, Korea
and
Dong H. Chyung†
The University of Iowa, Iowa City, Iowa

The control problem for rendezvous involving linear time-delay systems is investigated with a requirement that the systems involved be in a rendezvous state for a given nonzero duration of time. The problem in the paper is formulated so as to include the conventional chaser-target approach for rendezvous as a special case. For this general class of rendezvous problems, a mathematical programming technique in function space is applied and a necessary condition is derived in a form of the Pontryagin's maximum principle. It is further shown that, with a slight modification, the necessary condition is also a sufficient condition. To illustrate some use of the theory, an example problem is solved.

Nomenclature

A_i	= $(n \times n)$ system dynamic matrix
$A_0(u_1,u_2)$	= a convex cone in a function space
B_i	= $(n \times n)$ system dynamic matrix associated with delayed state
C_i	= $(n \times m)$ control matrix
$C^j[t_0,t_1]$	= space of all j-vector continuous functions on $[t_0,t_1]$
h	= time-delay
$J(\cdot)$	= cost functional
$L^j_\infty[t_0,t_1]$	= space of all j-vector essentially bounded functions on $[t_0,t_1]$
$q_i(\cdot)$	= convex function defining control restraint set
R^j	= j-dimensional Euclidean space
t	= time
t_0	= initial time point for control
t_1	= starting time point for rendezvous
t_2	= terminal time point for control
u_i	= $(m \times 1)$-vector control variable
δu_i	= $(m \times 1)$-vector perturbation variable
x_i	= $(n \times 1)$-vector phase variable
η_i	= $(1 \times n)$-vector adjoint variable
$\mu_i(t)$	= $(1 \times r)$-vector multiplier function associated with inequality constraint
$\nu(t)$	= $(1 \times n)$-vector multiplier function associated with equality constraint
$\phi_i(t)$	= $(n \times 1)$-vector initial function
Ω_i	= control restraint set

Superscripts

j	= jth component of a vector, $j = 0,1,2,\ldots$
*	= pertaining to optimality

Subscripts

i	= pertaining to the ith system involved for rendezvous
u_i	= partial derivative with respect to u_i
x_i	= partial derivative with respect to x_i

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*Associate Professor, Department of Electrical Science.
†Professor of Electrical Engineering.

Introduction

WE shall study the rendezvous problem of two controlled systems whose dynamics are of the form

$$\dot{x}_i(t) = A_i(t)x_i(t) + B_i(t)x_i(t-h) + C_i(t)u_i(t) \qquad (i=1,2) \tag{1}$$

where, for each $i = 1,2$, x_i denotes an n-vector phase variable, u_i is an m-vector control variable, and $h > 0$ denotes a constant time-delay. By rendezvous, we shall mean the situation that the phase variable x_1 and x_2 of the two dynamic systems are coincident (i.e., $x_1(t) = x_2(t)$) for some time.

The time-delay may be caused by inherent passive time-delay elements of the controlled systems or is incorporated to reflect the fact that some components of the current state information $x_i(t)$ are not immediately available to the controller. In the analytical investigation of piloted vehicles, for example, the pilot in a closed-loop control system is usually modelled as a linear dynamic element with a time-delay.[1] If control functions for a spaceship are determined based on the instructions from a distant Earth control center, a nonnegligible time-delay may exist due to long-distance transmission and processing of data.[2] Sometimes the time-delay action is intentionally introduced in the controller for an improved system implementation.[3]

The ability of two systems to rednezvous has been considered to be an important technique especially in the fields of astrodynamics and aerodynamics, and various results on rendezvous methods are available in the literature (e.g., see the references of Ref. 4). In conventional rendezvous problems, it is usual that one of the two systems involved in the rendezvous is assumed nonmaneuverable, and thus a technique of maneuvering the other system for rendezvous is sought in terms of equations of relative motion.[5] This assumption of course corresponds to the special case of either $C_1(t) \equiv 0$ or $C_2(t) \equiv 0$ in Eq. (1). In many practical cases, however, each of the systems involved in the rendezvous may be capable of exercising its own control action and thus the restriction of nonmaneuverability imposed on the target system seems not always desirable. In the case of the rendezvous between an aircraft and a tanker for midair refueling, for example, a more efficient rendezvous may result if both of the objects are actively maneuvered for rendezvous. In fact, a class of rendezvous problems without such a non-maneuverability restriction is investigated in Ref. 6 and Ref. 7 for ordinary differential systems.

In formulating a rendezvous problem, it is typically required[4] that the systems involved for rendezvous meet each other at a certain point in time $t=t_1$, or more specifically, $x_1(t_1)=x_2(t_1)$. This type of rendezvous condition may not be adequate in carrying out certain rendezvous missions. Rendezvous missions may include personnel or material transfer between two aircrafts, assembly of stations in space, or docking between two spaceships,[8] and in these cases it is necessary that the two systems maintain the rendezvous for a nonzero period of time.

In general, it is not obvious how the continued rendezvous state can be maintained if the systems have been maneuvered only for an instant rendezvous. This difficulty becomes more serious if the systems contain time-delays. As an illustration, consider two identical scalar systems

$$\dot{x}_i=-x_i(t-1)+u_i(t) \qquad (i=1,2)$$

with constraints $|u_i(t)|\leq 1$, $i=1,2$. Suppose initially $x_1(t)=2$ and $x_2(t)=-2$ on $[-1,0]$, respectively. If the rendezvous requirement is specified as $x_1(t_1)=x_2(t_1)$ for some $t_1>0$, one can easily obtain time-optimal controls $u_i(t)=(-1)^i$ on $[0,\frac{2}{3}]$. Therefore at $t=\frac{2}{3}$, $x_1(\frac{2}{3})=x_2(\frac{2}{3})=0$. But this rendezvous state cannot be maintained thereafter, for on the interval $[\frac{2}{3},1]$ the systems are described by $\dot{x}_i(t)=(-1)^i\,2+u_i(t)$ with $\dot{x}_i(\frac{2}{3})=0$ and $|u_i|\leq 1$, which implies that, on $[\frac{2}{3},1]$, $\dot{x}_1(t)<0$ and $\dot{x}_2>0$ for all admissible controls. Thus a more realistic approach for continued rendezvous is to require that $x_1(t)=x_2(t)$ on some interval $[t_1,t_2]$. For example, the control pair $(u_1(t),\,u_2(t))$ shown in Fig. 1 drives the system to rendezvous state $x_1(t)=x_2(t)$ on $[2,3]$, and the systems can be kept together thereafter with $u_i(t)=0$ for $t>3$.

In this paper, the optimal control problem for nonzero duration rendezvous of two linear time-delay systems is investigated. A necessary condition is presented and it is shown that this necessary condition is also sufficient for optimality for normal systems. To show the use of these results, an example problem is solved.

Notational conventions are as follows. For a given matrix B, its transpose and inverse are denoted by B^T and B^{-1}, respectively. $C^j[t_0,t_1]$ denotes the Banach space of all j-vector continuous functions with sup norm topology and $L^j_\infty[t_0,t_1]$ denotes the Banach space of all j-vector essentially bounded functions with essential-sup norm topology. A vector-valued function $x(t)$ defined on an interval $[t_0,t_1]$ may be denoted by $x(\cdot)$, or simply by x, when it is considered as an element of a function space. Unless otherwise stated, a vector is a column vector. For an m-vector valued function $g(x)$, where x is an n-vector, the partial derivative $\partial g/\partial x$ is an $m\times n$ matrix and it may sometimes be denoted by subscripts g_x. Finally, for a given vector u in a Banach space X, the notation δu (with or without a subscript) denotes a perturbation vector in X with respect to u.

Optimal Rendezvous Problem

Let $h>0$ be a fixed number, and let $[t_0,t_2]$ and $[t_1,t_2]$ be given compact intervals with $t_0<t_1\leq t_2$. Consider two controlled linear systems (S_1) and (S_2) modelled by

$$\dot{x}_1(t)=A_1(t)x_1(t)+B_1(t)x_1(t-h)+C_1(t)u_1(t) \qquad (t>t_0) \tag{2}$$

$$x_1(t)=\phi_1(t) \qquad t\in[t_0-h,t_0] \tag{3}$$

and

$$\dot{x}_2(t)=A_2(t)x_2(t)+B_2(t)x_2(t-h)+C_2(t)u_2(t) \qquad (t>t_0) \tag{4}$$

$$x_2(t)=\phi_2(t) \qquad t\in[t_0-h,t_0] \tag{5}$$

respectively, where $x_i\in R^n$ is the phase variable, $u_i\in R^m$ is the control variable, and $\phi_i(t)$ is a given n-vector continuous initial function on $[t_0-h,t_0]$. $A_i(t)$, $B_i(t)$, and $C_i(t)$ are continuous matrices with compatible dimensions. Let the control restraint sets, Ω_1 and Ω_2, be given by

$$\Omega_i=\{v\in R^m\,|q^j_i(v)\leq 0\} \qquad (i=1,2;\ \ j=1,2,\ldots,r) \tag{6}$$

where $q^j_i(v)$ is a given scalar-valued smooth convex function. A control function $u_i(t)$ on $[t_0,t_2]$ is called admissible if it is measurable, essentially bounded and $u_i(t)\in\Omega_i$ a.e. (almost everywhere) on $[t_0,t_2]$. In the sequel, if $u_i(t)$ with corresponding response $x_i(t)$ is an admissible control for the system (S_i) $(i=1,2)$, the pair $(u_1(\cdot),\,u_2(\cdot))$ will be called as an admissible control pair, and $(x_1(\cdot),\,x_2(\cdot))$ will be called the corresponding response pair. The cost functional J is given by

$$J(u_1(\cdot),u_2(\cdot))=\int_{t_0}^{t_2}[s^0(x_1(t),x_2(t),t)+c^0(u_1(t),u_2(t),t)]\mathrm{d}t \tag{7}$$

where $s^0(x_1,x_2,t)$ is a scalar convex function in x_1 and x_2 for each t, and $c^0(u_1,u_2,t)$ is a scalar convex function in u_1 and u_2 for each t. The functions $s^0(x_1,x_2,t)$, $c^0(u_1,u_2,t)$, $s^0_{x_i}(x_1,x_2,t)$ $(i=1,2)$ and $c^0_{u_i}(u_1,u_2,t)$ $(i=1,2)$ are continuous in all of their arguments.

The problem is to find an optimal pair of admissible controls $(u_1(t),\,u_2(t))$ on $[t_0,t_2]$, which steer their corresponding responses $(x_1(t),\,x_2(t))$ from the given initial function states $x_i(t)=\phi_i(t)$ on $[t_0-h,t_0]$, i=1,2, to a rendezvous state on $[t_1,t_2]$, i.e.,

$$x_1(t)=x_2(t) \quad \text{on} \quad [t_1,t_2] \tag{8}$$

while minimizing the cost functional $J(u_1(\cdot),\,u_2(\cdot))$.

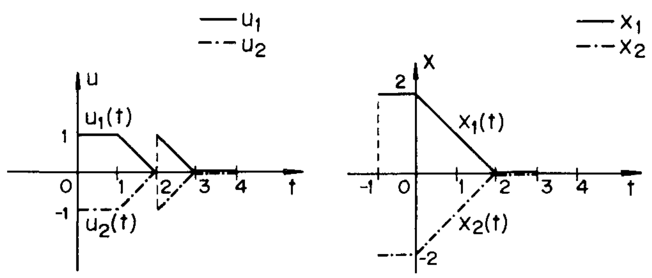

Fig. 1 Control and state function for nonzero-duration rendezvous.

Necessary Condition and Sufficient Condition

Necessary Condition

The rendezvous problem stated above can be reformulated as a mathematical programming problem in function space of the type studied in Ref. 9 by Makowski and Neustadt. To show this, let

$$x=(x^0,x_1,x_2)=(x^0,x^1_1,\ldots,x^n_1,x^1_2,\ldots,x^n_2)\in R^{2n+1}$$

$$y=(y^0,y_1,y_2)=(y^0,y^1_1,\ldots,y^n_1,y^1_2,\ldots,y^n_2)\in R^{2n+1}$$

$$u=(u_1,u_2)\in R^{2m}$$

$$f(x,y,u,t)=\begin{bmatrix} s^0(x_1,x_2,t)+c^0(u_1,u_2,t) \\ A_1x_1+B_1y_1+C_1u_1 \\ A_2x_2+B_2y_2+C_2u_2 \end{bmatrix}$$

$$\chi^j(x) = x_1^j - x_2^j \qquad (j=1,\ldots,n)$$

$$\chi^0(x) = x^0$$

$$p(x,y,u,t) = A_1(t)x_1 - A_2(t)x_2 + B_1(t)y_1 - B_2(t)y_2 + C_1(t)u_1 - C_2(t)u_2$$

and

$$q^j(u) = \begin{cases} q_1^j(u_1) & (j=1,\ldots,n) \\ q_2^{j-r}(u_2) & (j=r+1,\ldots,2r) \end{cases}$$

Also let $\phi(t)$ on $[t_0-h,t_0]$ be an $(2n+1)$-vector continuous function defined by $\phi(t)=(0,\ \phi_1(t),\ \phi_2(t))$. Then the rendezvous problem is equivalent to the following problem: Find a pair (x,u) in $C^{2n+1}[t_0-h,t_2]\times L_\infty^{2m}[t_0,t_2]$ such that $x(t)$ is absolutely continuous on $[t_0,t_2]$,

$$\dot{x}(t) = f(x(t),x(t-h),u(t),t) \qquad \text{a.e. on } [t_0,t_2] \tag{9}$$

$$x(t) = \phi(t) \qquad \text{a.e. on } [t_0-h,t_0] \tag{10}$$

$$\chi^j(x(t_2)) = 0 \qquad (j=1,\ldots,n) \tag{11}$$

$$p(x(t),x(t-h),u(t),t) = 0 \qquad \text{a.e. on } [t_1,t_2] \tag{12}$$

$$q^j(u(t)) \le 0 \qquad \text{a.e. on } [t_0,t_2] \quad (j=1,\ldots,2r) \tag{13}$$

and $\chi^0(x(t_2))$ is minimum.

The above reformulated version indicates that conventional techniques of deriving a necessary condition may not directly apply for the rendezvous problem but a certain assumption may be needed in order to handle the infinite dimensional equality constraint $p(x(t),\ x(t-h),\ u(t),\ t)=0$ on $[t_1,t_2]$. In fact, the necessary condition obtained in this paper is derived under a regularity assumption, which, as a compatibility condition of the constraints $p=0$ and $q^j\le 0$ $(j=1,\ldots,2r)$, ensures that an optimal trajectory be enclosed in "a sufficiently rich family of nearby trajectories,"[10] each of which itself satisfies the constraints. Specifically, the regularity condition is defined as follows.

For an admissible control pair (u_1,u_2), let $A_0(u_1,u_2)$ denote the set in $L_\infty^{2m}[t_0,t_2]$ defined by

$$A_0(u_1,u_2) = \Big\{\alpha(\delta u_1,\delta u_2)\ |\ \alpha>0,\ \delta u_i\in L_\infty^m[t_0,t_2],\ \delta u_i(t)=0 \text{ a.e. on } [t_0,t_1],$$
$$q_i^j(u_i(t)) + \frac{\partial q_i^j}{\partial u_i}(u_i(t))\delta u_i(t) < 0 \text{ a.e. on } [t_1,t_2], \quad (i=1,2;\ j=1,\ldots,r)\Big\} \tag{14}$$

Note that $A_0(u_1,u_2)$ is a convex cone in $L_\infty^{2m}[t_0,t_2]$. An admissible control pair $u=(u_1,u_2)$ *is said to be regular* if, for any given $\xi\in L_\infty^n[t_1,t_2]$, there exists a pair $\delta u_\xi=(\delta u_{1,\xi},\delta u_{2,\xi})$ in $A_0(u_1,u_2)$ such that

$$C_1(t)\delta u_{1,\xi}(t) - C_2(t)\delta u_{2,\xi}(t) = \xi(t) \qquad \text{a.e. on } [t_1,t_2] \tag{15}$$

Now, the necessary condition is given below. The proof of the following Theorem 1 is not provided as the result is a direct consequence of the necessary condition Theorem 3.1 in Ref. 11 for a general class of mathematical programming problem in Banach space involving delay-differential systems under an infinite dimensional equality constraint, which in turn is an extended version of the Theorem 12.1 in Ref. 9 obtained for ordinary differential systems only.

Theorem 1 (*Necessary Condition*): Let $(u_1^*(\cdot),\ u_2^*(\cdot))$ with corresponding response pair $(x_1^*(\cdot),\ x_2^*(\cdot))$ be an optimal pair of admissible controls that steers the responses of the two systems, S_1 and S_2, to a rendezvous state $x_1(t)=x_2(t)$ on $[t_1,t_2]$. Suppose (u_1^*,u_2^*) is regular. Then there exist an $(n+1)$ row-vector $\hat{b}=(\eta^0,b)=(\eta^0,b_1,\ldots,b^n)$, measurable row-vector functions $\nu=(\nu^1,\ldots,\nu^n)\in L_\infty^n[t_0,t_2]$, $\mu_i=(\mu_i^1,\ldots,\mu_i^r)\in L_\infty^r[t_0,t_2]$, $i=1,2$, and n row-vector absolutely continuous functions $\eta_i(t)$ on $[t_0,t_2]$, $i=1,2$, such that:

$$|\hat{b}|>0 \quad \text{and} \quad \eta^0\le 0,$$

2) $$-\dot{\eta}_i(t) = \eta^0 s^0_{x_i}(x_1^*(t),x_2^*(t),t) + (\eta_i(t)+(-1)^{i+1}\nu(t))A_i(t) + (\eta_i(t+h)+(-1)^{i+1}\nu(t+h))B_i(t+h) \qquad (\text{a.e. on } [t_0,t_2-h];\ i=1,2)$$

$$-\dot{\eta}_i(t) = \eta^0 s^0_{x_i}(x_1^*(t),x_2^*(t),t) + (\eta_i(t) + (-1)^{i+1}\nu(t))A_i(t) \qquad (\text{a.e. on } [t_2-h,t_2];\ i=1,2)$$

3) $\eta_1(t_1) = -\eta_2(t_1) = b$

4) $$\eta^0c^0(u_1^*(t),u_2^*(t),t) + \sum_{i=1}^{2}\eta_i(t)C_i(t)u_i^*(t) = \max\Big\{\eta^0c^0(u_1,u_2,t) + \sum_{i=1}^{2}\eta_i(t)C_i(t)u_i\ |\ (u_1,u_2)\in\Omega_1\times\Omega_2\Big\}$$

5) $$\eta^0c^0_{u_i}(u_1^*(t),u_2^*(t),t) + \eta_i(t)\frac{\partial q_i}{\partial u_i}(u_1^*(t)) + (\eta_i(t)+(-1)^{i+1}\nu(t))C_i(t) = 0 \qquad (\text{a.e. on } [t_1,t_2];\ i=1,2)$$

6) $\mu_i^j(t)\le 0$ and $\mu_i^j(t)q_i^j(u_i^*(t))=0$ $\qquad$ (a.e. on $[t_1,t_2]$; $j=1,\ldots,r$)

7) $|\nu(t)| = |\mu_i(t)| = 0$ $\qquad$ (a.e. on $[t_0,t_1]$; $i=1,2$)

Remark 3.1: It is not very difficult to show that the condition Eq. (15) in the definition of regularity is satisfied *only if* the matrix function $N(t)$ defined by

$$N(t) \triangleq C_1(t)C_1(t)^T + C_2(t)C_2(t)^T \tag{16}$$

is nonsingular for almost all $t\in[t_1,t_2]$ and its inverse $N(t)^{-1}$ is essentially bounded on $[t_1,t_2]$. This requirement limits the scope of the applicability of the necessary condition. For example, when the dimension m of each control vector u_i is less than a half of the dimension n of the state vector x_i, the result is not applicable. However, this restriction seems not due to a shortcoming of the method employed in the paper, but originates from the rendezvous condition [Eq. (8)] itself as an infinite dimensional constraint [Eq. (12)]. When an optimization problem includes an equality constraint such as Eq. (12), a typical approach is to make a regularity assumption such as ours as can be seen in Refs. 9, 10, and 11. Since

$$P_u(x(t),x(t-h),u(t),t)\underset{\xi}{\delta u}(t) = C_1(t)\delta u_{1,\xi}(t) - C_2(t)\delta u_{2,\xi}(t) \qquad \text{a.e. on } [t_1,t_2] \tag{17}$$

the regularity condition Eq. (15) may be said to correspond to the typical condition found in Ref. 12 of linear independence

of vectors P_u and $q_u^j(u)$, $j=1,\dots,2r$ for which $q^j(u)=0$.

It is remarked that the nonsingularity $N(t)$ a.e. on $[t_1,t_2]$ and the essential boundedness of its inverse $N^{-1}(t)$ on $[t_1,t_2]$ guarantee that the two systems can always be put to a rendezvous state on $[t_1,t_2]$ by an appropriate pair of admissible controls. That is, for given initial functions $(\phi_1(t), \phi_2(t))$ on $[t_0-h,t_0]$, one can find as in the proof of Theorem 3.1 in Ref. 13 an admissible control pair (u_1,u_2) which first drives the systems to the state $x_1(t_1)=x_2(t_1)$ and then makes $\dot{x}_1(t)=\dot{x}_2(t)$ a.e. on $[t_1,t_2]$ by

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} C_1(t)^T & 0 \\ 0 & C_2(t)^T \end{bmatrix} \begin{bmatrix} I_n \\ -I_n \end{bmatrix} N(t)^{-1} \left\{ [-I_n \quad I_n] \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right.$$

$$\left. -[I_n \quad -I_n] \begin{bmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{bmatrix} \begin{bmatrix} x_1(t-h) \\ x_2(t-h) \end{bmatrix} \right\} \qquad \text{a.e. on } [t_1,t_2]$$

Here, I_n denotes the $n\times n$ identity matrix. From the viewpoint of function-space controllability only, however, the nonsingularity of $N(t)$ a.e. on $[t_1,t_2]$ seems not necessary. That is, when the rendezvous problem is reformulated properly, one can see that the rendezvous condition is closely related to a function-scale *null* controllability. Hence, as shown in Theorem 3.2 of Ref. 13, a weaker condition involving matrices $A_i(t)$, $B_i(t)$, and $C_i(t)$ than the invertibility requirement of $N(t)$ may exist as a necessary and sufficient condition for rendezvous. But since it is not directly relevant, this condition is not studied in this paper.

Remark 3.2: One may view the rendezvous condition $x_1(t)=x_2(t)$ as $2n$ sets of conflicting bounded state-space constraints

$$x_1^j(t) \ge x_2^j(t), \qquad x_1^j(t) \le x_2^j(t) \qquad (j=1,\dots,n; \quad t\in[t_1,t_2])$$

and can derive a necessary condition as in Ref. 14. However, the result obtained in this manner would be somewhat different from ours in that, for example, the necessary condition does not require the regularity assumption but the nontriviality condition such as condition 1 of Theorem 1 would not be obtained by this approach.

Remark 3.3: For each $(x_1,x_2,y_1,y_2)\in R^n\times R^n\times R^n\times R^n$ and $t\in[t_1,t_2]$, let $W(x_1,x_2,y_1,y_2,t)$ denote the set of all pairs $(u_1,u_2)\in\Omega_1\times\Omega_2$ such that

$$A_1(t)x_1 - A_2(t)x_2 + B_1(t)y_1 - B_2(t)y_2 + C_1(t)u_1 - C_2(t)u_2 = 0$$

and

$$\left\{ C_1(t)v_1 - C_2(t)v_2 \,|\, v_i\in R^m, \quad \frac{\partial q_i^j}{\partial u_i}(u_i)v_i < 0 \quad \text{if } q_i^j(u_i)=0, \quad (j=1,\dots,r; \quad i=1,2) \right\} = R^n$$

Then it can be shown that the conditions 5 and 6 imply the following maximum condition:

8) $$\eta^0 c^0(u_1^*(t),u_2(t),t) + \sum_{i=1}^{2} \eta_i(t)C_i(t)u_i^*(t) = \max\left\{ \eta^0 c^0(u_1,u_2,t) + \sum_{i=1}^{n} \eta_i(t)C_i(t)u_i \,\middle|\, (u_1,u_2)\in V^*(t) \right\} \quad \text{a.e. on } [t_1,t_2]$$

where

$$V^*(t) = W(x_1^*(t),x_2^*(t),x_1^*(t-h),x_2^*(t-h),t)$$

To show this, let $(u_1(t), u_2(t))$ on $[t_0,t_2]$ be an arbitrary regular control pair satisfying the relation

$$C_1(t)u_1(t) - C_2(t)u_2(t) = -(A_1(t)x_1^*(t) - A_2(t)x_2^*(t) + B_1(t)x_1^*(t-h) - B_2(t)x_2^*(t-h)) \tag{18}$$

Let

$$M(u) = \int_{t_1}^{t_2} \left\{ (\eta^0 c^0(u_1(t),u_2(t),t) + \sum_{i=1}^{2} \eta_i(t)C_i(t)u_i(t)) - (\eta^0 c^0(u_1^*(t),u_2^*(t),t) + \sum_{i=1}^{2} \eta_i(t)C_i(t)u_i^*(t)) \right\} dt$$

Using condition 5 of the necessary condition, one can obtain

$$M(u) = \eta^0 \int_{t_1}^{t_2} \Big[c^0(u_1(t),u_2(t),t) - c^0(u_1^*(t),u_2^*(t),t) - c^0_{u_1}(u_1^*(t),u_2^*(t),t)(u_1(t)-u_1^*(t)) - c^0_{u_2}(u_1^*(t),u_2^*(t),t)(u_2(t)-u_2^*(t)) \Big] dt$$
$$- \int_{t_1}^{t_2} \left[\mu_1(t)\frac{\partial q_1}{\partial u_1}(u_1^*(t))(u_1(t)-u_1^*(t)) + \mu_2(t)\frac{\partial q_2}{\partial u_2}(u_2^*(t))(u_2(t)-u_2^*(t)) \right] dt$$
$$- \int_{t_1}^{t_2} \nu(t)\,[(C(t)u(t) - C(t)u(t)) - (C_1(t)u_1^*(t) - C_2(t)u_2^*(t))]\, dt$$

Recall that the function $c^0(u_1,u_2,t)$ is convex in (u_1,u_2) and $q_i(u)$ is convex in u_i, $i=1,2$. Further, the last term is identically zero. Hence $M(u)\le 0$. Arguing in the same manner as in Ref. 9, one can now show that $M(u)\le 0$ for all regular controls $u=(u_1,u_2)$ satisfying Eq. (18) implies condition 8.

Condition 8, together with condition 4, states that a regular optimal control maximizes the Hamiltonian of the system among the regular controls satisfying the relation Eq. (18). That is, the necessary condition of Theorem 1 is essentially a maximum condition of Pontrygin type in Chap. 6 of Ref. 10.

Sufficient Condition

In the following, it is shown that the necessary condition obtained earlier is also a sufficient condition if $\eta^0<0$, which together with Theorem 1 provides a means of solving optimal rendezvous problems as illustrated in the next section. The proof for this sufficiency result will be given in the Appendix.

Theorem 2 (Sufficient Condition): Let (u_1^*, u_2^*) with the corresponding response pair (x_1^*, x_2^*) be an admissible control pair which steers the responses of the systems (S_1) and (S_2) from the initial function state $(x_1(t), x_2(t)) = (\phi_1(t), \phi_2(t))$ on $[t_0 - h, t_0]$ to a rendezvous state $x_1^*(t) = x_2^*(t)$ on $[t_1, t_2]$. Suppose there exist an $(n+1)$-vector $\hat{b} = (\eta^0, b)$ with $\eta^0 < 0$, measurable functions $\nu \in L_\infty^n[t_0, t_2]$, $\mu_i \in L_\infty^r[t_0, t_2]$, $i = 1,2$ and absolutely continuous functions $\eta_i(t)$ on $[t_0, t_2]$, $i = 1,2$ such that conditions 2-7 in Theorem 1 hold. Then (u_1^*, u_2^*) is an optimal control pair.

Example

The problem of rendezvous considered so far is formulated in view of practical aspects. The rendezvous approach suggested in this paper is more general than the practicing conventional one, and hence it can be a more effective and useful rendezvous method. As for time-delay systems, examples of physical systems with time delays are numerous.[15] In particular, the remote control system or the teleoperator control system for satellites and deep-space vehicles is reported to exhibit a serious time-delay problem.[2] In this case, the time-delay occurs due to signal transmission, coding, data processing, passivity of the process and human reaction factors, etc. Also the time-delay may be intentionally incorporated in the design of certain controllers to enhance the system performance.[16] The effectiveness of the theory developed in the paper and difficulties involved as well may be well illustrated by solving a realistic problem, which unfortunately is not available to the authors. Also, it seems that the dynamics involved for practical systems are usually of high-order as discussed in Ref. 17. To avoid complications due to complex computations, a simple problem involving fictitious first-order systems is solved in the following.

An Example Problem and Solution

In order to apply Theorem 1 to obtain candidates of optimal control pair for a given rendezvous problem, it is natural to first investigate whether or not the regularity condition is satisfied. If $\Omega_i = R^m$, $i = 1,2$, then the regularity can be determined algebraically in terms of $C_i(t)$, $i = 1,2$. That is, the invertibility of $N(t)$ in Eqs. (3-8) a.e. on $[t_1, t_2]$ and its essential boundedness on $[t_1, t_2]$ implies that the regularity assumption is satisfied. If Ω_i is a strictly proper subset of R^m, then the set $A_0(u_1^*, u_2^*)$ is not fully known a priori in general, and hence the regularity assumption may become difficult to check before solving. In many cases, however, the apparent difficulty can be circumvented by utilizing the fact that the necessary condition with $\eta^0 < 0$ is also a sufficient condition as shown previously. That is, one *may* solve a rendezvous problem first by simply assuming the regularity, then applying Theorem 1 to obtain a candidate and finally applying Theorem 2 to see if the candidate is optimal. This is a typical procedure of solving a problem using a necessary and sufficient pair as illustrated in the following example. Of course, it is possible that certain problems would not be solved by this approach.

Consider two scalar linear systems with time-delay

$$\dot{x}_1(t) = -x(t) - 0.5x_1(t-1) + u_1(t) \qquad (t>0)$$

$$x_1(t) = 2 \qquad (-1 \le t \le 0)$$

$$\dot{x}_2(t) = -2x_2(t) + 0.2x_2(t-1) + u_2(t) \qquad (t>0)$$

$$x_2(t) = 0 \qquad (-1 \le t \le 0)$$

The problem is to find a pair of control functions $(u_1(t), u_2(t))$ on the interval $[0,2]$ with $|u_1(t)| \le 1$ and $|u_2(t)| \le 1$, respectively, such that the corresponding response pair $(x_1(t), x_2(t))$ satisfies the condition

$$x_1(t) = x_2(t) \quad \text{on} \quad [1,2]$$

while minimizing the cost functional

$$J(u_1(\cdot), u_2(\cdot)) = \int_0^2 (x_2(t) - x_1(t))\mathrm{d}t$$

Let $(u_1^*(t), u_2^*(t))$ with corresponding response pair $(x_1^*(t), x_2^*(t))$ be an optimal control pair. Note that $N(t) = 2$ for each $t \in [1,2]$ and so $N(t)^{-1} = ½$. To apply Theorem 1, suppose $(u_1^*(t), u_2^*(t))$ is regular. Then there exist numbers η^0 and b, real-valued essentially-bounded measurable functions $\mu_i(t)$, $i = 1,2$, $\nu(t)$ on $[0,2]$ and real-valued absolutely continuous functions $\eta_i(t)$, $i = 1,2$, on $[0,2]$ such that

1) $|\eta^0| + |b| > 0 \qquad (\eta^0 \le 0)$

2)
$$-\dot{\eta}_1(t) = \begin{cases} \eta^0 - \eta_1(t) - 0.5(\eta_1(t+1) + \nu(t+1)) & \text{(a.e. on } [0,1]) \\ -\eta^0 - (\eta_1(t) + \nu(t)) & \text{(a.e. on } [1,2]) \end{cases}$$

$$-\dot{\eta}_2(t) = \begin{cases} \eta^0 - 2\eta_2(t) + 0.2(\eta_2(t+1) - \nu(t+1)) & \text{(a.e. on } [0,1]) \\ \eta^0 - 2(\eta_2(t) - \nu(t)) & \text{(a.e. on } [1,2]) \end{cases}$$

3) $\eta_1(2) = b = -\eta_2(2)$

4) $\eta_1(t)u_1^*(t) + \eta_2(t)u_2^*(t)u_2^*(t)$

$$= \max\{\eta_1(t)u_1 + \eta_2(t)u_2 \mid |u_1| \le 1, |u_2| \le 1\} \qquad \text{(a.e. on } [0,1])$$

5) $|\eta_1(t) + \nu(t)| + 2\mu_1(t)u_1^*(t) = 0 \qquad$ (a.e. on $[1,2]$)

$|\eta_2(t) - \nu(t)| + 2\mu_2(t)u_2^*(t) = 0 \qquad$ (a.e. on $[1,2]$)

6) $\mu_i(t) \le 0$ and $\mu_i(t)(u_i^*(t)^2 - 1) = 0$

(a.e. on $[1,2]$; $i = 1,2$)

7) $|\nu(t)| = |\mu(t)| = 0 \qquad$ (a.e. on $[0,1]$)

As noted in Remark 3.3, conditions 5 and 6 imply

8) $\eta_1(t)u_1^*(t) + \eta_2(t)u_2^*(t)$

$$= \max\{\eta_1(t)u_1 + \eta_2(t)u_2 \mid (u_1, u_2) \in V^*(t)\} \qquad \text{(a.e. on } [1,2])$$

where $V^*(t)$ denotes the set of pairs (u_1, u_2) with $|u_1| \le 1$ and $|u_2| \le 1$ such that $(u_1, u_2) \ne (1, -1)$, or $(u_1, u_2) \ne (-1, 1)$ and

9) $$-x_1^*(t) + 2x_2^*(t)0.5x_1^*(t-1) - 0.2x_2^*(t-1) + u_1 - u_2 = 0$$

Since $(u_1^*(t), u_2^*(t)) \in V^*(t)$ on $[1,2]$, $(u_1^*(t), u_2^*(t))$ cannot be $(1, -1)$ or $(-1, 1)$ for any $t \in [1,2]$ and further must satisfy condition 9 on $[1,2]$. Hence assume that $|u_1^*(t)| < 1$ a.e. on $[1,2]$. Then from condition 5, $u_1(t) = 0$, a.e. on $[1,2]$. Hence by condition 4, $\eta_1(t) = -\nu(t)$ on $[1,2]$. Also, it can be shown from condition 1 and by the standard contraposition argument that $(u_1^*(t), u_2^*(t))$ being an optimal control pair for rendezvous implies $\eta^0 < 0$. Therefore let $\eta^0 = -1$ without loss of generality. By solving the equations

given by conditions 2 and 3,

$$\eta_1(t)=\begin{cases}1+be^{(t-1)} & \text{on } [0,1]\\ b-(t-2) & \text{on } [1,2]\end{cases}$$

$$\eta_2(t)=\begin{cases}-\tfrac{1}{2}+(-b-\tfrac{1}{2}) & \text{on } [0,1]\\ -b+(t-2) & \text{on } [1,2]\end{cases}$$

where $b=\eta_1(2)=-\eta_2(2)$. By computing in a standard manner (using the maximum conditions 4 and 8 and the boundary conditions), one obtains the following.

$$u_1^*(t)=\begin{cases}1 & [0,S]\\ -1 & (S,1)\\ 0.5x_1^*(t-1)-x_2^*(t)+0.2x_2^*(t-1) & [1,2]\end{cases}$$

$$u_2^*(t)=\begin{cases}-1 & [0,T]\\ +1 & (T,1)\\ 0 & [1,2]\end{cases}$$

$$x_1^*(t)=\begin{cases}2e^{-t} & [0,S]\\ -2+2(1+e^{-s})e^{-(t-s)} & [S,1]\\ x_2^*(t) & [1,2]\end{cases}$$

$$x_2^*(t)=\begin{cases}-\tfrac{1}{2}+\tfrac{1}{2}e^{-2t} & [0,T]\\ \tfrac{1}{2}+(-1+\tfrac{1}{2}e^{-2T}) & [T,1]\\ [x_2(1)+(2t-1)/20]e^{-2(t-1)}-1/20 & [1,1+T]\\ 1/20+[x_2(1+T)-1/20+(-2/10 & \\ \quad +e^{-2T}/10)(t-1-T)]e^{-2(t-1-T)} & [1+T,T]\end{cases}$$

where

$$x_2(1)=\tfrac{1}{2}+(-1+\tfrac{1}{2}e^{-2T})e^{-2(1-T)}$$

$$x_2(1+T)=[x_2(1)+(2T-1)/20]e^{-2T}-1/20$$

and, T and S are solutions of the equations

$$5/2+(-1+\tfrac{1}{2}e^{-2T})e^{-2(1-T)}=2(1+e^{-S})e^{-(1-S)}$$

$$e^{-2(T-1)}=2e^{-(S-1)}-1 \qquad (S\simeq 0.594,\ T\simeq 0.653)$$

One may invoke the sufficient condition (Theorem 2) to conclude that the control pair $(u_1^*(\cdot),\ u_2^*(\cdot))$ shown above is in fact an optimal solution.

Concluding Remarks

In this paper, nonzero time duration rendezvous problems involving controlled time-delay systems were investigated. For a fixed terminal-time optimal rendezvous problem, a necessary condition and a sufficient condition were provided, and the effectiveness of the results was illustrated by a simple example. These results can be easily extended to systems with multiple delays in the state and control variables.

The theory developed in this paper is applicable for the case when both of the systems involved can exercise actions for rendezvous as well as for the conventional problem of chaser-target approach for rendezvous. Some difficulties exist, however, in that the regularity assumption in the necessary condition is sometimes not easy to check a priori. As suggested in the solution of the example, the necessary and sufficient condition pair may work, but certain problems would not be solved by this approach.

Appendix: Proof of Theorem 2

Let $(u_1(\cdot),\ u_2(\cdot))$ with the corresponding response pair $(x_1(\cdot),\ x_2(\cdot))$ be an admissible pair which steers the responses of the systems to a rendezvous state, i.e., $x_1(t)=x_2(t)$, on $[t_1,t_2]$. Let

$$R=\eta^0[J(u_1(\cdot),u_2(\cdot))-J(u_1(\cdot),u_2(\cdot))]$$

Since $\eta^0<0$, it suffices to show that $R\le 0$ for all such control pairs $(u_1(t),\ u_2(t))$. Noting that $(\eta_1(t),\ \eta_2(t))$ is absolutely continuous on $[t_0,t_2]$, one can add and subtract

$$\int_{t_0}^{t_2}\mathrm{d}\eta_1(t)(x_1(t)-x_1^*(t)) \text{ and } \int_{t_0}^{t_2}\mathrm{d}\eta_2(t)(x_2(t)-x_2^*(t)),$$

respectively to R, and obtain

$$\begin{aligned}R=&\eta^0(J(u_1(\cdot),u_2(\cdot))-J(u_1^*(\cdot),u_2^*(\cdot))\\ &+\int_{t_0}^{t_2}\mathrm{d}\eta_1(t)(x_1(t)-x_1^*(t))+\int_{t_0}^{t_2}\mathrm{d}\eta_2(t)(x_2(t)-x_2^*(t))\\ &-\int_{t_0}^{t_2}\mathrm{d}\eta_1(t)\int_{t_0}^{t}[A_1(s)(x_1(s)-x_1^*(s))\\ &+B_1(s)(x_1(s-h)-x_1^*(s-h))+C_1(s)(u_1(s)-u_1^*(s))]\mathrm{d}s\\ &-\int_{t_0}^{t_2}\mathrm{d}\eta_2(t)\int_{t_0}^{t}[A_2(s)(x_2(s)-x_2^*(s))\\ &+B_2(s)(x_2(s-h)-x_2^*(s-h))-C_2(s)(u_2(s)-u_2^*(s))]\mathrm{d}s\\ =&\eta^0[J(u_1(\cdot),u_2(\cdot))-J(u_1(\cdot),u_2(\cdot))]\\ &+\int_{t_0}^{t_2}[(\dot\eta_1(t)+\eta_1(t)A_1(t))(x_1(t)-x_1^*(t))\\ &+\eta_1(t)B_1(t)(x_1(t-h)-x_1^*(t-h))]\mathrm{d}t\\ &+\int_{t_0}^{t_2}\eta_1(t)C_1(t)(u_1(t)-u_1^*(t))\mathrm{d}t\\ &-\eta_1(t_2)(x_1(t_2)-x_1^*(t_2))\\ &+\int_{t_0}^{t_2}[(\eta_2(t)+\eta_2(t)A_2(t))(x_2(t)-x_2^*(t))\\ &+\eta_2(t)B_2(t)(x_2(t-h)-x_2^*(t-h))]\mathrm{d}t\\ &+\int_{t_0}^{t_2}\eta_2(t)C_2(t)(u_2(t)-u_2^*(t))\mathrm{d}t\\ &-\eta_2(t_2)(x_2(t_2)-x_2^*(t_2))\end{aligned}$$

Noting that $x_1(t_2)=x_2(t_2)$ and $x_1^*(t_2)=x_2^*(t_2)$ and using conditions 2, 3, and 5, one obtains

$$\begin{aligned}R=&\eta^0\int_{t_0}^{t_2}[s^0(x_1(t)x_2(t),t)-s^0(x_1^*(t),x_2^*(t),t)\\ &-s^0_{x_1}(x_1(t),x_2^*(t),t)(x_1(t)-x_2^*(t))\\ &-s^0_{x_2}(x_1^*(t),x_2^*(t),t)(x_2(t)-x_2^*(t))]\mathrm{d}t\\ &-\int_{t_0}^{t_2-h}\nu(t+h)[B_1(t+h)(x_1(t)-x_1^*(t))\\ &-B_2(t+h)(x_2(t)-x_2^*(t))]\mathrm{d}t\end{aligned}$$

$$-\int_{t_2-h}^{t_2} \nu(t)\,[A_1(t)(x_1(t)-x_1^*(t))-A_2(x_2(t)-x_2^*(t))]\,\mathrm{d}t$$

$$+\eta^0\int_{t_0}^{t_2}[c^0(u_1(t),u_2(t),t)-c^0(u_1^*(t),u_2^*(t),t)]\,\mathrm{d}t$$

$$+\int_{t_0}^{t_2}\eta_1(t)C(t)(u(t)-u^*(t))\,\mathrm{d}t$$

$$+\int_{t_0}^{t_2}\eta_2(t)C_2(t)(u_2(t)-u_2^*(t))\,\mathrm{d}t$$

Since $\dot{x}_1(t)=\dot{x}_2(t)$ and $\dot{x}_1^*=\dot{x}_2^*(t)$ on $[t_1,t_2]$, and since $s^0(x_1,x_2,t)$ is convex in (x_1,x_2), conditions 4, 6, and 7 imply that

$$R=\eta^0\int_{t_1}^{t_2}[c^0(u_1(t),u_2(t),t)-c^0(u_1^*(t),u_2^*(t),t)$$

$$-c^0_{u_1}(u_1^*(t),u_2^*(t),t))(u_1(t)-u_1^*(t))$$

$$-c^0_{u_2}(u(t),u(t),t))(u(t)-u(t))]\,\mathrm{d}t$$

$$-\int_{t_1}^{t_2}[\mu_1(t)\frac{\partial q_1}{\partial u_1}(u_1^*(t))(u_1(t)-u_1^*(t))$$

$$+\mu_2(t)\frac{\partial q_2}{\partial u_2}(u_2^*(t))(u_2(t)-u_2^*(t))]\,\mathrm{d}t$$

Since $c^0(u_1,u_2,t)$ is convex in (u_1,u_2) and $q_i(u_i)$ is convex in u_i, it follows that $R\leq 0$.

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