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"Direct" Wiener-Hopf Solution of Filter/Observer and Optimal Coupler Problems

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Most approaches to optimal stationary linear stochastic control depend upon time-domain techniques for both theoretical foundations and computational algorithms. In contrast, this paper discusses a "direct" Wiener-Hopf approach which avoids several difficulties occurring in time-domain methods (e.g., those which arise in connection with singular regulator and filter problems). Emphasis is placed upon solution properties in distinction to derivations. The following points of the direct Wiener-Hopf approach are emphasized: 1) linear regulator, filter/observer problems can be solved using linear algebraic equations (in distinction to solving nonlinear Riccati equations) once solution eigenvalues are obtained; 2) the control weighting matrix need not be positive definite nor is it necessary that its inverse exist; 3) the state/output weighting matrix need not be positive semidefinite; and 4) singular regulator and filter/observer problems (e.g., the cheap control problem) can be handled with a minimum of special consideration. In addition, a new class of problem is solved using the Wiener-Hopf approach. The solution specifies *both* the optimal (discrete) digital control law *and* the optimal (continuous) data hold for the case of a continuous cost function, a continuous plant, and sampled measurements.

Nomenclature

$(\bar{\cdot})$	= replace s by $-s$
$(\cdot)'$	= transpose of $(\cdot)$
$(\cdot)_*$	= replace s by $-s$ and transpose $(\cdot)$
$(\cdot)^T$	= impulse sampling, at $1/T$ Hz, of $(\cdot)$
$\varphi_{\bar{X}Y'}$	= cross-spectral density between vectors X and Y with components of X and Y continuous
$\varphi_{\bar{X}(Y^T)}{}'$	= cross-spectral density between vector X with continuous components and vector Y with discrete components

I. Introduction

THE objective herein is to develop, via the Wiener-Hopf (W-H) approach, an optimal closed-loop solution for direct digital control of continuous plants using a continuous cost function. The importance of the problem class is twofold:

1) Solution consists of two parts—optimal discrete control law and optimal continuous hold.

2) Use of a continuous cost function assures accountability for the intersample behavior of the continuous plant response.

This is worthy of further elaboration. The use of a continuous cost function in conjunction with a continuous plant model yields a control law that is optimal at all instants in time. However, the only constraint placed on the closed-loop system response is at the sampling instants when a discrete cost function is employed. The "intersample" behavior of the continuous plant response can be very unsatisfactory, especially if the data rate is low and the open-loop plant contains lightly damped modes. Also, when a discrete cost function is minimized, the data hold (the coupler between the computer and the control actuators) is specified arbitrarily by the designer (more often than not a zero-order data hold is utilized). By means of clever transformation, a continuous cost function may be used if choice of the data hold is made a priori.[1] However, the optimal discrete control law applies only for the chosen data holds, which are unlikely to have been an optimal choice. On the other hand, if a continuous cost function is minimized via the Wiener-Hopf approach, the optimal solution specifies *both* the control law and the optimal coupler.

The advantages of using modern matrix Wiener-Hopf minimization procedures are not widely appreciated in today's control community. The primary reason for this is that many experienced scientists and engineers have been alienated by the difficulty in applying the spectral factorization solution method. The factorization technique, invented by Wiener,[2] was brought to a high level of maturity by a host of other researchers.[3-8] In general, these efforts were concerned with artifacts of the factorization approach—e.g., how to circumvent the requirement for a stable open-loop plant.[7] Very few optimal control practitioners were cognizant that Wiener-Hopf equations could be solved by a direct-solution technique which made spectral factorization unnecessary[9-11] and uses mathematics no more difficult than partial fraction expansions. Moreover, the method handled multicontroller cases as well as unstable, nonminimum phase plants. We believe that increased awareness of the basic and simple methods available for solving Wiener-Hopf equations can increase modern control engineering productivity and encourage a more integrated use of time-domain and frequency-domain techniques for problem solution.

The technical development proceeds in the following sections. Section II reviews the basic solution principles for the regulator as contained in Ref. 10.

Section III deals with the optimal linear continuous stochastic control problem via the Wiener-Hopf formulation. First, the time-domain form of the linear quadratic Gaussian (LQG) optimal stochastic control problem is treated using frequency-domain methods. It is shown that the steady-state gains of the Kalman filter can be found using the W-H approach and linear solution methods (i.e., it is not necessary to solve a nonlinear Riccati equation). Next, an alternative formulation is given which does not use the separability principle and leads to a W-H equation containing two spectral matrices. One is recognizable as being the regulator spectral matrix, while the other is recognized as the spectral matrix associated with the filter/observer problem. It is shown that this W-H formulation survives, without singularity, the limiting condition wherein the measurement noise vector is set identically equal to zero and thus yields the optimal observer

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solution without the necessity of resorting to limiting forms or special partitioning (e.g., Refs. 12 and 13).

Section IV treats the optimal coupler problem using an extension of the formulation which does not use the separability principle. Again, there will be two spectral matrices which appear in the W-H equation; the "regulator" spectral matrix remains a function of the complex frequency variable s, but the "filter/observer" spectral matrix now depends on the delay operator $z = e^{sT}$.

It must be remarked that questions pertaining to the selection of the q's and r's of the regulator weighting matrices (so as to produce designs which are not only "optimal" but "satisfactory" as well) will persist for the optimal digital controller/coupler problem. In this regard, it is hoped that the numerous examples presented in Ref. 11 (which emphasize the use of nondiagonal Q and R, $R<0$, etc.) will encourage optimal control practitioners to relax self-imposed restrictions placed on the regulator weighting matrices.

II. Review of the Linear Regulator Problem

The open-loop plant model used throughout the paper is

$$\dot{x} = Fx + Gu$$

$$X(s) = [Is-F]^{-1}GU + [Is-F]^{-1}x(0) = AU + Bx(0) \tag{1}$$

Assuming a control law of the form

$$U = -KX \tag{2}$$

results in the closed-loop configuration shown in Fig. 1. (K may be a function of s; it is not restricted to being a gain matrix.) The continuous cost function

$$J = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} (X_* QX + U_* RU)\,ds \tag{3}$$

where

$$X_* = X'(-s) \qquad U_* = U'(-s)$$

is minimized by taking a variation on U such that

$$U = U_0 + \lambda U_1 \tag{4}$$

where U_0 is the optimal control and U_1 is any physically realizable (exists for $t \geq 0$) but otherwise arbitrary variation. The first and second variations of J give *sufficient* conditions for an optimum (refer to Ref. 10 for the details) as

$$[R + A_* QA]\,U_0 + A_* QBx(0) = \psi \tag{5}$$

where ψ must represent a time function which is zero for $t \geq 0$.

To insure $\psi(s)$ exists for $\sigma < \sigma_2$ (e.g., in some left-half plane), pick U_0 in such a manner that any poles of Eq. (5) which exist in some right-half plane (i.e., represent functions of time which are zero for $t \leq 0$) cancel identically into each and every numerator of Eq. (5). U_0 has the form

$$U_0 = W(s)x(0) \tag{6}$$

The required compensation can be computed as

$$K = -W[B + AW]^{-1} \tag{7}$$

For the special case where K is a gain matrix, it suffices to use the initial value theorem

$$K = \lim_{|s|\to\infty} [-sW(s)] \tag{8}$$

to obtain K given $W(s)$. Finally, it is not necessary to compute the closed-loop transfer functions using

$$X(s) = [Is - F + GK]^{-1}x(0) \tag{9}$$

since

$$X(s) = [B + AW]x(0) \tag{10}$$

will work just as well. In fact, Eq. (10) remains valid even for singular cases where the K matrix has infinite entries.

What are the candidate poles for U_0? These are picked from[9,10]

$$\det[R + A_* QA] \equiv (D\bar{D})^{P-1}\Delta\bar{\Delta} \tag{11}$$

where P is the number of independent control variables, D represents the open-loop plant poles, and Δ the closed-loop poles. D can have roots anywhere in the complex s plane. We usually desire Δ to represent stable modes (i.e., having left-half plane poles); however, this need not always be the case. Consider the objective of designing a controller for a variable stability aircraft which is to represent a second aircraft having unstable phugoid or spiral modes. This is an example wherein some roots of Δ are desired to represent unstable modes.

What are the candidate zeros of U_0? These are unknown; therefore, one simply specifies polynomials with *unknown coefficients*. The number of unknown coefficients is equal to the number of "positive time" poles which must be cancelled (more precise details are available in Refs. 10 and 11).

Simplified W-H Conditions

It is not necessary to work with the complete W-H description as given in Eq. (5) since it can be reduced to two computationally simpler requirements.

Since

$$\det[R + A_* QA] \equiv (D\bar{D})^{P-1}\Delta\bar{\Delta} \tag{12}$$

let

$$U_0 = \xi(s)/\Delta = W(s)x(0) \tag{13}$$

where $\xi(s)$ is vector of polynomials having unknown coefficients. Further, describe the plant matrices A and B in terms of the open-loop poles and their adjoint (numerator) matrices. That is,

$$A = A^a/D \qquad B = B^a/D \qquad D = \text{open-loop poles} \tag{14}$$

The W-H equation becomes

$$\frac{[RD\bar{D} + A^a_* QA]\,\xi_{(s)} + \Delta A^a_* QBx(0)}{D\bar{D}\Delta} = \psi \tag{15}$$

From the previous discussion, recall that each and every numerator of ψ must contain D and Δ as an exact factor. Therefore, each numerator of ψ must be zero for those values of s such that $D = \Delta = 0$. When $D = 0$, Eq. (15) reduces to

$$A^a_* Q[A^a \xi(s) + \Delta B^a x(0)] = 0 \tag{16}$$

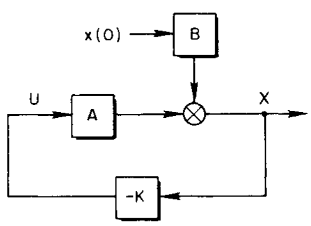


Fig. 1 Closed-loop regulator.

Since $A^a_* Q$ is already analytic in some left-half plane, it need not be considered. Thus, we have a first W-H condition:

$$A^a\xi(s)+\Delta B^a x(0)=0 \quad \text{for all} \quad s=s_j \quad \text{such that} \quad D=0 \tag{17}$$

Next, let $\Delta=0$ in Eq. (15) and obtain the second W-H condition:

$$[RD\bar{D}+A^a_* Q A^a]\xi(s)=0 \quad \text{for all} \quad s=s_j \quad \text{such that} \quad \Delta=0 \tag{18}$$

Notice that the first W-H condition is unchanged regardless of Q and R. Furthermore, the second W-H condition exists even in the limiting case for $R\to 0$ or $Q\to 0$. Finally, even though Eqs. (17) and (18) are *matrix* equations, it usually suffices to use only *one* equation from each set since Eq. (12) assures linear dependence of the other equations in each set when $D=\Delta=0$. For example, using the first component of Eq. (17) will produce the same linear set of equations, in terms of the unknown entries of $\xi(s)$, as will the second (or third) component of Eq. (17). Finally, R^{-1} does not appear anywhere in the W-H conditions, a fact which makes the direct-solution method ideal for the evaluation of singular cases.

Actual solution consists of solving the set of equations resulting from Eqs. (17) and (18) for the unknown polynomial coefficients of $\xi(s)$. This set of equations is linear in those polynomial coefficients.

The reader interested in more details may refer to Ref. 11 where a three-state, two-control-point example is used to highlight the mathematical fine points.

III. W-H Formulation—Optimal Linear Stochastic Control

The open-loop plant equation is modified by the addition of a process noise vector n and an output equation which includes a measurement noise vector v:

$$\dot{x}=Fx+Gu+n \qquad x(0)=x_0 \qquad y=Hx+v \tag{19}$$

The transform of Eq. (19) is:

$$X(s)=[Is-F]^{-1}GU(s)+[Is-F]^{-1}[N+x_0]$$

$$=A(s)U(s)+B(s)[N(s)+x_0] \tag{20}$$

The block diagram of the open-loop plant is shown in Fig. 2.

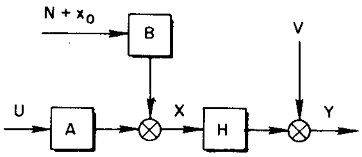


Fig. 2 Block diagram for the plant and measurements.

The time-domain formulation of the linear optimal stochastic control problem is given in Fig. 3a (Ref. 14) and the equivalent frequency-domain formulation is given in Fig. 3b.

A separability principle formulation is first used; that is, the K_1 matrix represents the regulator gains obtained independently of the output equation and process noise. The filter solution is obtained independently of the regulator solution. The task is to find the Kalman filter gains K_2. This is done by minimizing the mean square error between the state vector X and its estimate $\hat{X}$. From Fig. 3b,

$$\hat{X}=B[I+K_2HB]^{-1}K_2Y+B[I+K_2HB]^{-1}[GU+\hat{x}_0] \tag{21}$$

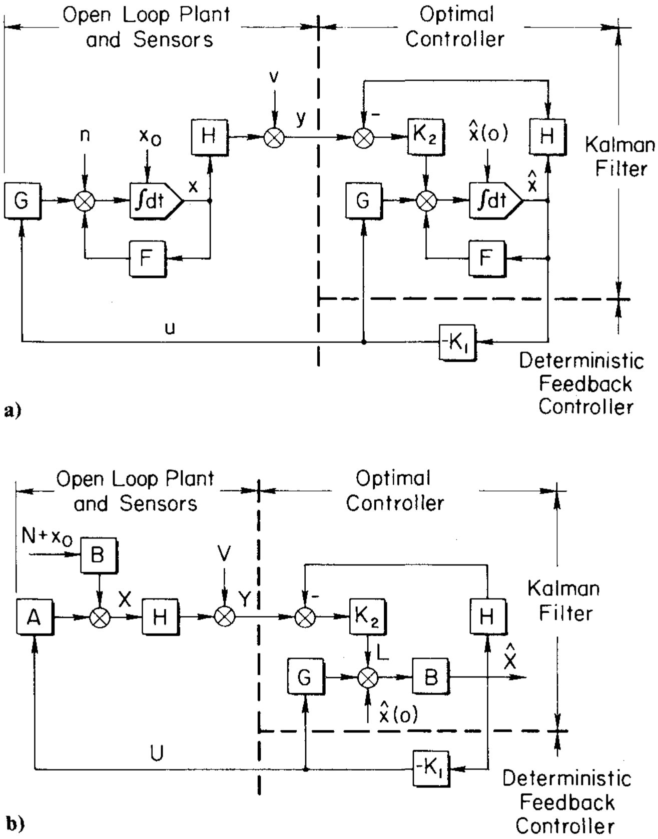
Let

$$W_a=B[I+K_2HB]^{-1}K_2 \tag{22}$$

and write the error as

$$E=\hat{X}-X=W_a\{V+HB[N+x(0)]\}$$

$$-B[N+x(0)]+B[I+K_2HB]^{-1}\hat{x}_0 \tag{23}$$



a)

b)

Fig. 3 Formulation of the linear optimal stochastic control problem: a) time domain, b) frequency domain.

Forming the expected value of E_*E and taking the gradient with respect to W_{a^*} gives the W-H equation[11]:

$$W_a[\varphi_{\bar{V}V'}+HB\varphi_{\bar{N}N'}B_*H_*]-B\varphi_{\bar{N}N'}B_*H_*=\psi \tag{24}$$

where $\varphi_{\bar{V}V'}$ is the autospectra of the measurement noise and $\varphi_{\bar{N}N'}$ is the autospectra of the process noise. The assumption of uncorrelated N and V was used in arriving at Eq. (24), although the more general case is easily treated.

The unknown of the W-H equation, W_a, is postmultiplied by a spectral matrix, which we will term the filter/observer spectral matrix. Equation (24) can be solved using the direct-solution method in exactly the same manner as the regulator problem. Moreover, once W_a is found, the K_2 feed-forward matrix can be computed using

$$K_2=B^{-1}W_a[I-HW_a]^{-1} \tag{25}$$

A simpler result is obtainable, if one considers the properties of W_a as a solution to a W-H equation. Rewrite Eq. (25) as

$$K_2[I-HW_a]=[Is-F]W_a \tag{26}$$

and observe that, when W_a is a proper rational function and K_2 is a gain matrix, one can find K_2 directly by letting $|s|\to\infty$

$$K_2=\lim_{|s|\to\infty}[sW_a(s)] \tag{27}$$

One cannot use Eq. (27) in the singular cases which occur when some or all of the measurement noise components are zero. In this event, $(I-HW_a)$ is singular.

Once either K_2 or W_a is known, the controller transfer functions can be computed using (refer to Fig. 3b):

$$U=-K_1[I+(I-W_aH)AK_1]^{-1}W_aY \tag{28}$$

or

$$U=-K_1[Is-F+GK_1+K_2H]^{-1}K_2Y \tag{29}$$

Clearly, Eq. (28) persists even when the feed-forward gain matrix K_2 is singular. Equation (28) gives an interesting limiting form. Suppose $\varphi_{\bar{V}V'}\equiv 0$ (no measurement noise) and H is invertible. Then,

$$W_a=H^{-1}\Rightarrow U=-K_1H^{-1}Y \tag{30}$$

The following illustrative example clarifies the mathematical details:

$$\varphi_{\bar{V}V'}\neq 0$$

Suppose the open-loop plant is described by

$$\dot{x}=\begin{bmatrix} 0 & 1 & 0 \\ -5 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix} x+\begin{bmatrix} 0 & 1 \\ -4 & 2 \\ 1 & 0 \end{bmatrix} U+\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{31}$$

and $Y=HX+V$, giving

$$X=AU+B[N+x(0)]=\frac{\begin{bmatrix} -4s+1 & s^2+2s+1 \\ s(-4s+1) & s(2s-3) \\ s^2-s+1 & s+1 \end{bmatrix}}{s(s^2+4)}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$
$$+\frac{\begin{bmatrix} s^2-1 & s+1 & 5 \\ -5s & s(s+1) & 5s \\ s-1 & 1 & s^2-s+5 \end{bmatrix}}{s(s^2+4)}[N+x(0)] \tag{32}$$

First, solve for the regulator gains; call this matrix K_1. If

$$R=\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \qquad Q=\begin{bmatrix} 2 & -1 & 5 \\ -1 & 0 & 2 \\ 5 & 2 & 18 \end{bmatrix} \tag{33}$$

we would find, using the algebraic methods of Ref. 10 (see Ref. 11 for details),

$$U_0=-\frac{\begin{bmatrix} s^2+4s+9 & 0 & 2s^2-2 \\ 2s^2+3s+9 & s^2+5s+6 & 6s^2+7s+32 \end{bmatrix}}{(s+1)(s+2)(s+3)}x(0) \tag{34}$$

Application of the initial value theorem gives the feedback gains as

$$K_1=\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 6 \end{bmatrix} \tag{35}$$

Further, the matrix of "regulator" closed-loop transfer function is computed as

$$X=[B+AW]x(0)$$
$$\equiv\frac{\begin{bmatrix} (s+1)(s+3) & 0 & -6(s+1) \\ -5(s+3) & (s+2)(s+3) & s+32 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}}{(s+1)(s+2)(s+3)}x(0) \tag{36}$$

where

$$\Delta_1=(s+1)(s+2)(s+3) \tag{37}$$

The second step is to find the Wiener filter W_a, which is tantamount to specifying the Kalman gains K_2. The filter portion of the problem has the W-H equation

$$W_a[\varphi_{\bar{V}V'}+HB\varphi_{\bar{N}N'}B_*H_*]-B\varphi_{\bar{N}N'}B_*H_*=\psi \tag{38}$$

For demonstration purposes, assume only one output measurement. Let

$$Y=HX+V \qquad H=[1,\ \ 0,\ \ 0] \tag{39}$$

and also let the noise sources be described by the intensities

$$\varphi_{\bar{V}V'}=1 \qquad \varphi_{\bar{N}N'}=\begin{bmatrix} 85 & 0 & 0 \\ 0 & 1690 & 0 \\ 0 & 0 & 505 \end{bmatrix} \tag{40}$$

A straightforward computation gives

$$\varphi_{\bar{V}V'}+HB\varphi_{\bar{N}N'}B_*H_*=\Delta_2\bar{\Delta}_2/D\bar{D} \tag{41}$$

where

$$\Delta_2=(s+4)(s+5)(s+6) \qquad \bar{\Delta}_2=-s^3+15s^2-74s+120 \tag{42}$$

Also, one finds

$$B\varphi_{\bar{N}N'}B_*H_*=\frac{\begin{bmatrix} 85s^4-1860s^2+14{,}400 \\ -2115s^3+14{,}740s \\ 85s^3+2440s^2-4300s+14{,}400 \end{bmatrix}}{D\bar{D}} \tag{43}$$

The W-H equation now takes the form

$$\frac{\begin{bmatrix} a_0s^2+a_1s+a_2 \\ b_0s^2+b_1s+b_2 \\ c_0s^2+c_1s+c_2 \end{bmatrix}}{\cancel{\Delta_2}}\ \frac{\cancel{\Delta_2}\bar{\Delta}_2}{D\bar{D}}$$
$$-\frac{\begin{bmatrix} 85s^4-1860s^2+14{,}400 \\ -2115s^3+14{,}740s \\ 85s^3+2440s^2-4300s+14{,}400 \end{bmatrix}}{D\bar{D}}=\psi \tag{44}$$

Since each component of ψ must contain D, the open-loop roots, nine unknown coefficients are needed to accomplish the cancellation. Of course, each ψ component can be treated separately, so the basic problem is solving three equations for three unknowns, rather than nine equations for nine unknowns. Letting $s=0,2j$ gives these equations; for example,

$$(a_0s^2+a_1s+a_2)\Big|_{s=0,2j}=\frac{85s^4-1860s^2+14{,}400}{-s^3+15s^2-74s+120}\Bigg|_{s=0,2j}$$

The final result is

$$W_a=\frac{\begin{bmatrix}15s^2+70s+120\\ 70s^2+60s\\ 13s^2+2s+120\end{bmatrix}}{(s+4)(s+5)(s+6)} \tag{45}$$

Finding the Kalman gains is now a simple task, since

$$K_2=\lim_{|s|\to\infty}[sWa(s)]=\begin{bmatrix}15\\ 70\\ 13\end{bmatrix} \tag{46}$$

Once *either* W_a or K_2 is known, the input/output transfer functions can be computed using Eq. (28) or (29).

$$U=-\frac{\begin{bmatrix}41s^2+60s+19\\ 178s^2+486s+701\end{bmatrix}}{s^3+21s^2-7s-27}Y$$

$$=-\begin{bmatrix}\dfrac{41s+19}{s^2+20s-27}\\ \hdashline \dfrac{178s^2+486s+701}{(s+1)(s^2+20s-27)}\end{bmatrix}Y \tag{47}$$

The transfer functions describing the controller in this example are unstable. It is well-known that this may happen in solving the stochastic control problem.

Singular cases can also be treated using the direct method since Eqs. (24) and (28) can always be solved.

Solution Without Separability

The linear stochastic optimal control problem can be solved without using the separability principle. This can be very useful when: 1) the regulator part of the solution is singular ($K_1\to\infty$), and 2) it is desirable to reduce the computational burden (e.g., if one has a very high-order system but only a few measurable outputs and controllers, it is more reasonable to compute a few transfer functions than attempt to estimate a large number of states).

We may proceed according to Fig. 4 where, for conciseness, initial conditions on the state are zero. For the system of Fig. 4, minimize the performance index

$$J=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}(\tilde X_*\tilde Q\tilde X+\tilde U_*\tilde R\tilde U)\,ds \tag{48}$$

where $(\tilde{\cdot})$ denotes the expected value of $(\cdot)$.

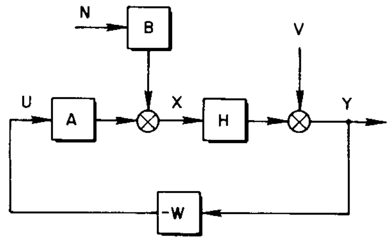


Fig. 4 Direct solution of the linear stochastic optimal control problem.

First, form the integrand of the performance index

$$\Phi=\tilde X_*\tilde Q\tilde X+\tilde U_*\tilde R\tilde U \tag{49}$$

using the equations (refer to Fig. 4)

$$U=-[I+WHA]^{-1}W[V+HBN] \tag{50}$$

$$X=BN+AU=BN-A[I+WHA]^{-1}W[V+HBN] \tag{51}$$

Note, from Eq. (50), the closed-loop stability is determined by

$$W_a=-[I+WHA]^{-1}W \tag{52}$$

Thus, we can minimize Eq. (48) with respect to W_a and assure closed-loop stability. Using the variational method of Ref. 11, the W-H equation becomes

$$[R+A_*QA]W_a[\varphi_{\tilde VV'}+HB\varphi_{\tilde NN'}B_*H_*]+A_*QB\varphi_{\tilde NN'}B_*H_*=\psi \tag{53}$$

for the special case of uncorrelated N and V. Therefore, solve the W-H equation for W_a and compute W using Eq. (52), i.e.,

$$W=-Wa[I+HAW_a]^{-1} \tag{54}$$

The separability principle is still very much in evidence in Eq. (53), since $[R+A_*QA]$ determines that group of closed-loop poles which correspond to the optimal regulator solution, while $[\varphi_{\tilde VV'}+HB\varphi_{\tilde NN'}(\tilde H\tilde B)']$ determines the remaining closed-loop poles which correspond to the optimal filter/observer solution. Equation (53) can also be solved using the algebraic methods previously discussed. (In addition, a spectral factorization algorithm is given in the appendix.)

Application of this approach gives the direct relationship of the controller to the output since

$$U=-WY \tag{55}$$

An effect of formulating the problem in this way can be a dramatic reduction in dimensionality of the problem solution. Application to the illustrative example given in the previous subsection yields the exact same transfer function W but with less intermediate detail.

IV. Optimal Coupler Problem

The prime objective of the research effort reported herein is to develop an optimal closed-loop solution for direct digital control of continuous plants using a continuous cost function. As noted in the introduction, the solution consists of two parts: an optimal discrete control law and the optimal (continuous) data holds. We have elected to proceed with the development by extending the second method of the previous section in a manner which accounts for sampled output signals, which are to be processed by a digital computer and outputted through a data hold (coupler) to the control actuators. In the case of the continuous stochastic control problem, two spectral matrices came into play; the matrix which premultiplied the unknown of the W-H equation was recognized as the regulator spectral matrix, whereas the

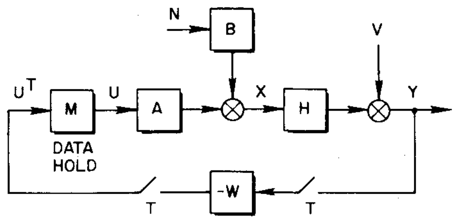


Fig. 5 Linear stochastic optimal discrete control of a continuous plant.

postmultiplier was recognized as the filter/observer spectral matrix. Both were, in the illustrative example, rational functions of s. In the W-H equation for the optimum coupler case, it will be seen that the premultiplier remains a function of s while the postmultiplier becomes a function of the shift operator $z=e^{sT}$.

The situation of interest is depicted in Fig. 5. For brevity, $N+x(0)$ is taken as N; that is, the initial conditions are taken as zero.

Let the integrand of the cost function be Φ, the expected value of the usual quadratic function:

$$\Phi=\tilde{X}_*\tilde{Q}\tilde{X}+\tilde{U}_*\tilde{R}\tilde{U} \tag{56}$$

First, develop expressions for the continuous X and U (the superscript T will be used to denote that a signal is impulsively sampled at $1/T$ samples per second; a prime will be used to denote the transpose):

$$U=-M[I+W^T(HAM)^T]^{-1}W^T[V+HBN]^T \tag{57}$$

$$X=-AM[I+W^T(HAM)^T]^{-1}W^T[V+HBN]^T+BN \tag{58}$$

Optimize with respect to the matrix of transfer functions relating $[V+HBN]$ to U^T; therefore, let

$$W_a^T=-[I+W^T(HAM)T]^{-1}W^T \tag{59}$$

and

$$\xi_1=V+HBN \qquad \xi_2=BN \tag{60}$$

so that Eqs. (57) and (58) become

$$U=MW_a^T\xi_1^T \qquad X=AMW_a^T\xi_1^T+\xi_2 \tag{61}$$

Substituting Eqs. (61) into Eq. (56) gives the integrand of the performance index

$$\Phi=[\xi_{1*}^T W_{a*}^T M_*]R[MW_a^T\xi_1^T]$$
$$+[\xi_{2*}+\xi_{1*}^T W_{a*}^T M_* A_*]Q[AMW_a^T\xi_1^T+\xi_2'] \tag{62}$$

Next, take the gradient of Eq. (62) (e.g., see Ref. 11) with respect to the unknown MW_a^T, take the expectation, and arrive at the W-H equation:

$$[R+A_*QA]MW_a^T\varphi'(\xi_1^T\xi_1^{T'})+A_*Q\varphi'(\xi_1^T\xi_2')=\psi \tag{63}$$

Assuming V and N to be independent noise processes gives (see Fig. 6):

$$\varphi_{\xi_1^T\xi_1^{T'}}=(1/T)[\varphi_{\tilde{V}V'}+HB\varphi_{\tilde{N}N'}B_*H_*]^T$$
$$=(1/T)[\varphi_{\tilde{V}V'}^T+(HB\varphi_{\tilde{N}N'}B_*H_*)^T] \tag{64}$$

The $1/T$ scale factor is in keeping with the definitions given in Chap. 10 of Ref. 15. (The use of $1/T$ in Ref. 15 appears to be a "matter of convenience." Other reference sources, for example, Ref. 16, do not use it.) In a like manner, the cross

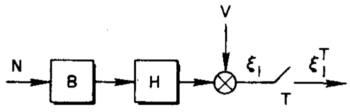


Fig. 6 Spectral noise model.

spectra between the sampled vector ξ_1^T and the continuous vector ξ_2 is simply the scaled continuous spectra[15]:

$$\varphi_{\xi_1^T\xi_2}=(1/T)B\varphi_{\tilde{N}N'}B_*H_* \tag{65}$$

The W-H equation now takes the form

$$[R+A_*QA]MW_a^T[(1/T)\varphi_{\tilde{V}V'}^T+(1/T)(HB\varphi_{\tilde{N}N'}B_*H_*)^T]$$
$$+(1/T)A_*QB\phi_{\tilde{N}N'}B_*H_*=\psi \tag{66}$$

Summary of Optimal Coupler Problem

Given the digitally controlled continuous system of Fig. 5, the W-H equation resulting from minimizing the expected value of a quadratic index is

$$[R+A_*QA](MW_a^T)\left[\frac{\varphi_{\tilde{V}V}^T}{T}+\frac{1}{T}(HB\varphi_{\tilde{N}N'}B_*H_*)^T\right]$$
$$+\frac{A_*QB\varphi_{\tilde{N}N'}B_*H_*}{T}=\psi \tag{67}$$

where W_a^T is defined as

$$W_a^T=-[I+W^T(HAM)^T]^{-1}W^T \tag{68}$$

When Eq. (67) has been solved for MW_a^T, W_a^T is chosen to be the discrete portion and M (the coupler) is taken as the continuous part. A straightforward computation then provides the digital control law

$$W^T=-W_a^T[I+(HAM)^TW_a^T]^{-1} \tag{69}$$

which, in turn, yields the equation relating the continuous control output to the sampled measurements

$$U=-MW^TY^T \tag{70}$$

The W-H equation [Eq. (67)] can be solved in two ways: 1) the direct method or 2) spectral factorization. The first approach requires only the application of the fundamental concept that MW_a^T must be constructed in a manner which forces any pole (be it in the s or z domain) which can generate a positive function of time, to cancel into each and every component of ψ. This is an algebraically simple technique. The more computationally difficult spectral factorization solution method is described in the appendix (in reality, a new approach since the spectral matrix is factorized as the product of three matrices rather than the traditional two).

Optimal Coupler—Scalar Example

A scalar, single-control-point example affords the opportunity to gain familiarity with the mathematical manipulations. Let

$$A=B=1/(s+1) \tag{71}$$

$$R=1 \quad Q=8 \quad \varphi_{\tilde{V}V'}=1 \quad \varphi_{\tilde{N}N'}=15 \tag{72}$$

Therefore,

$$R+A_*QA=1+\frac{1}{-s^2+1}=\frac{-s^2+9}{-s^2+1} \tag{73}$$

Let

$$GG_* = \frac{\varphi^T_{VV'}}{T} + \frac{1}{T}(HB\varphi_{\tilde{N}N'}B_*H_*)^T = \frac{1}{T^2} + \frac{1}{T}\left[\frac{15}{-s^2+1}\right]^T$$
$$= \frac{1}{T^2} + \frac{15}{2T}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]^T \tag{74}$$

The power spectra of sampled white noise has been taken as $1/T^2$ rather than $1/T$ (see Ref. 15 for a discussion). Continuing, we find

$$GG_* = \frac{1}{T^2} + \frac{15}{2T}\left[\frac{z}{z-e^{-T}} - \frac{z}{z-e^{T}}\right]$$
$$= \frac{1}{T^2}\left[1+\frac{(15/2)Tz(e^{-T}-e^{T})}{(z-e^{-T})(z-e^{T})}\right]$$
$$= \frac{1}{T^2}\left[\frac{z^2-\{e^T+e^{-T}-(15/2)T(e^{-T}-e^T)\}z+1}{(z-e^{-T})(z-e^{T})}\right]$$
$$= \frac{1}{T^2}\frac{(z-e^{-bT})(z-e^{bT})}{(z-e^{-T})(z-e^{T})} \tag{75}$$

Thus, the optimal filter/observer pole is defined by $z-e^{-bT}$; the value of b as a function of T is given as

T	b
0	4
0.01	3.9998
0.1	3.9769
1	3.0285

The only remaining computation is

$$\frac{1}{T}A_*QB\varphi_{\tilde{N}N'}B_*H_* = \frac{120}{T(-s+1)^2(s+9)} \tag{76}$$

The W-H equation is, therefore,

$$\frac{(-s+3)(s+3)}{(-s+1)(s+1)}MW_a^T\frac{(z-e^{-bT})(z-e^{bT})}{T^2(z-e^{-T})(z-e^{T})}$$
$$+\frac{120}{T(-s+1)^2(s+1)} = \psi \tag{77}$$

The application of the direct approach requires that MW_a^T be such that the numerator of ψ cancels all those poles which can produce positive time functions; that is, cancel the terms $s+1$ and $z-e^{-T}$. Clearly, a selection of

$$MW_a^T = \frac{a_0(z-e^{-T})}{(s+3)(z-e^{-bT})} \tag{78}$$

where a_0 is an undetermined coefficient, is sufficient to achieve this goal. Substitution of Eq. (78) into Eq. (77) gives

$$\frac{(-s+3)(s+3)}{(-s+1)(s+1)}\,\frac{a_0(z-e^{-T})}{(s+3)(z-e^{-bT})}\,\frac{(z-e^{-bT})(z-e^{bT})}{(z-e^{-T})(z-e^{T})}$$
$$+\frac{120T}{(-s+1)^2(s+1)} = \psi T^2 \tag{79}$$

or

$$\frac{(-s+1)(-s+3)a_0(z-e^{bT})+120T(z-e^{T})}{(-s+1)^2(s+1)(z-e^{T})} = \psi T^2 \tag{80}$$

In Eq. (80), a_0 must be selected so that the numerator is zero when $s=-1$; therefore,

$$a_0 = -\frac{120T(e^{-T}-e^{T})}{2(4)(e^{-T}-e^{bT})} = -15T\frac{(1-e^{2T})}{[1-e^{(1+b)T}]} \tag{81}$$

For example, when $T=1$ second,

$$MW_a^T = \frac{-1.736929333(z-e^{-T})}{(s+3)(z-e^{-bT})} \qquad T=1.0 \quad b=3.0285 \tag{82}$$

Notice the general result. The data-hold poles are defined by the (continuous) regulator spectral matrix, while the "discrete" poles of W_a^T are determined by the sampled filter/observer spectral matrix. Letting

$$M = \frac{1}{s+3} \qquad W_a^T = \frac{a_0(z-e^{-T})}{(z-e^{-bT})} \tag{83}$$

one then proceeds to compute the digital control law using the equation

$$W^T = -W_a^T[I+(HAM)^TW_a^T]^{-1} \tag{84}$$

First,

$$(HAM)^T = \left[\frac{1}{(s+1)(s+3)}\right]^T = \left[\frac{½}{s+1} - \frac{½}{s+3}\right]^T$$
$$= \frac{1}{2}\left[\frac{z}{z-e^{-T}} - \frac{z}{z-e^{-3T}}\right] = \frac{1}{2}\frac{(e^{-T}-e^{-3T})z}{(z-e^{-T})(z-e^{-3T})} \tag{85}$$

Therefore,

$$(HAM)^TW_a^T = \frac{1}{2}\frac{(e^{-T}-e^{-3T})z}{(z-e^{-T})(z-e^{-3T})}\,\frac{a_0(z-e^{-T})}{(z-e^{-bT})}$$

and

$$I+(HAM)^TW_a^T$$
$$= \frac{z^2-\{e^{-3T}+e^{-bT}-(a_0/2)(e^{-T}-e^{-3T})\}z+e^{-(3+b)T}}{(z-e^{-3T})(z-e^{-bT})}$$
$$\equiv \frac{(z-e^{-\alpha T})(z-e^{-T})}{(z-e^{-3T})(z-e^{-bT})} \tag{86}$$

where $\alpha=2+b$. That is, as a consequence of the W-H process, the numerator of $I+(HAM)^TW_a^T$ contains the "open-loop root," $z-e^{-T}$, as an *exact* factor. Therefore,

$$W^T = \frac{-a_0(z-e^{-T})}{z-e^{-bT}}\,\frac{(z-e^{-3T})(z-e^{-bT})}{(z-e^{-\alpha T})(z-e^{-T})} = \frac{-a_0(z-e^{-3T})}{(z-e^{-\alpha T})} \tag{87}$$

The output equation

$$U = -MW^TY^T = -\underbrace{\left[\frac{1}{s+3}\right]}_{M}\underbrace{\left[\frac{-a_0(z-e^{-3T})}{z-e^{-\alpha T}}\right]}_{W^T}Y^T \tag{88}$$

can be written alternatively as

$$U = -\underbrace{\left[\frac{1-e^{-T(s+3)}}{s+3}\right]}_{M}\underbrace{\left[\frac{-a_0z}{z-e^{-\alpha T}}\right]}_{W^T}Y^T \tag{89}$$

That is, the data hold can be viewed in a manner quite similar to the zero-order hold (see Fig. 7). It is now apparent that the optimal coupler solution defines a data hold which forces the continuous variables of the plant to follow a path, during the intersample period, that is, in a sense, "scheduled" by the constraints placed on the solution by the (continuous) spectral matrix $R+A_*QA$.

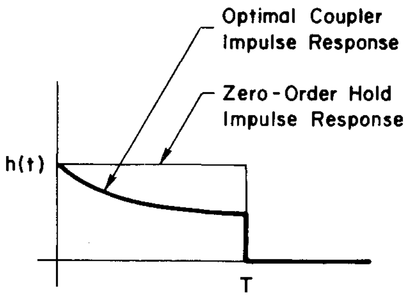


Fig. 7 Impulse response of zero-order hold and optimal coupler.

An interesting limiting case for this example, where N and V are zero (only initial condition excitation), can be found in Ref. 11.

V. Conclusion

This paper presents and demonstrates a "direct" Wiener-Hopf solution for stationary multivariable linear stochastic control problems. This direct approach avoids difficulties which tend to arise in connection with singular regulator and filter problems when other methods of solution are used.

A new class of problem is solved using the direct Wiener-Hopf approach. The solution specifies *both* the optimal (discrete) digital control law *and* the optimal (continuous) data hold for coupling the digital control law to the plant actuators for the case of a continuous cost function, a continuous plant, and sampled measurements. Although the primary focus was on the direct-solution method, a new spectral factorization algorithm was given. This algorithm factorizes the spectral matrix into the product of three matrices and does not require the control weighting matrix to be positive definite.

Appendix: Solution Using Spectral Factorization

The regulator W-H equation is

$$[R+A_*QA]U+A_*QBx(0)=\psi \tag{A1}$$

Since we may write

$$[R+A_*QA]=[I+A_*K']R[I+KA]=F_*RF \qquad \text{(A2)†}$$

one may verify, by direct substitution, that

$$U_0=-F^{-1}R^{-1}[F_*^{-1}A_*QBx(0)]_+ \tag{A3}$$

is a solution to Eq. (A1). This solution requires only that R^{-1} exist.

The Wiener-Hopf equation for the filter-observer problem is:

$$[R+A_*QA]W_a[\varphi_{\tilde{V}V'}+HB\varphi_{\tilde{N}N'}B_*H_*]$$

$$+A_*QB\varphi_{NN'}B_*H_*=\psi \tag{A4}$$

Setting

$$\varphi_{\tilde{V}V'}+HB\varphi_{NN'}B_*H_*=GG_* \qquad \text{(A5)†}$$

$$R+A_*QA=F_*RF \tag{A6}$$

$$A_*QB\varphi_{\tilde{N}N'}B_*H_*=N \tag{A7}$$

gives the following form for Eq. (A4):

$$(F_*RF)W_aGG_*+N=\psi \tag{A8}$$

By direct substitution, one may verify

$$W_a=-F^{-1}R^{-1}[F_*^{-1}NG_*]_+G^{-1} \tag{A9}$$

satisfies Eq. (A8); that is,

$$-F_*[F_*^{-1}NG_*^{-1}]_+G_*+F_*F_*^{-1}NG_*^{-1}G_*=\psi$$

$$=F_*\{-[F_*^{-1}NG_*^{-1}]_++F_*^{-1}NG_*^{-1}\}G_*$$

$$=F_*\{F_*^{-1}NG_*^{-1}\}_-G_* \tag{A11}$$

Equation (A11) satisfies the W-H requirement of forcing ψ to consist of time functions which are nonzero only for negative time. This is so as a consequence of the definitions of the three factors on the right-hand side of Eq. (A11).

†F and G in this appendix are the indicated spectral matrix factors. In the main text, F and G are the system and control distribution matrices, respectively.

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