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Minimum-Variance Design of Constant-Gain Filters Subject to Stability Constraints

Spiro P. Karatsinides* and William F. Powers†
The University of Michigan, Ann Arbor, Mich.

A computer-oriented systematic approach to constant-gain filter design for time-invariant linear systems is developed. The trace of the error covariance matrix is minimized subject to constraints on the closed-loop filter eigenvalues. For continuous system representations, the eigenvalues are restricted to a region in the s plane defined by damping ratio and decrement factor inequalities, while discrete representations involve a decrement factor inequality in the z plane. Properties of inner determinants and Lyapunov equations are employed to allow a unified approach to both continuous- and discrete-time system representations. The approach is applied to three examples which illustrate filter settling time considerations, transient/steady-state gain tradeoffs, and combined control-estimation possibilities.

I. Introduction

THE response characteristics of the steady-state (constant-gain) Kalman filter are determined by the dynamical properties of the linear system representation. Stable filter operation is necessary for all practical applications and is achieved when the system is detectable and stabilizable.[1,2] However, even in cases of stable filter operation, the amount of relative stability present is a factor of major importance for certain classes of problems.

References 3 and 4 discuss the case of absence of strong filter response characteristics. This is a common situation in aerospace-type applications and is generated by neutrally stable modes unaffected by the process noise, or almost neutrally stable modes affected very lightly by the process noise. In Refs. 3 and 4 these modes are termed as undisturbable or almost undisturbable, respectively. If a transformation to modal coordinates is considered, the steady-state Kalman gains corresponding to these modes are zero or very close to zero. This results in an undesirable situation since estimation of these modes becomes open-loop or almost open-loop. The problem is not so obvious in the original system representation since none of the steady-state Kalman gains may be zero.

Theoretically this problem can be eliminated by periodically restarting a time-varying Kalman filter that avoids the steady-state gains. However, constant-gain filters are more desirable (if possible for the application) because of their implementation simplicity, ease of preflight analysis, and relatively small on-board memory requirements. Thus, it is of interest to consider possible modifications of the steady-state Kalman filter that will correct the problem.

A definite way to achieve rapid transient convergence is to design a filter with eigenvalues well inside the left half complex s plane for continuous-time systems, or well inside the unit circle of the complex z plane for discrete-time systems. Reference 5 presents a method based on inner determinants for satisfying a constant decrement factor constraint. Reference 4 gives a summary of other existing methods together with a new proposed design. The first of the methods involves the introduction of additional process noise into the system model, and this has the effect of moving the filter eigenvalues inside the left half plane. The amount of the additional noise is determined by the desired location of the filter eigenvalues. The second method considers the repositioning of the unsatisfactory eigenvalues to new, specified locations leaving the ones with satisfactory performance unchanged. Finally, the third method deals with the destabilization of the modes undisturbed by the process noise. The reason for this is that the steady-state Kalman filter eigenvalues corresponding to unstable modes that are not disturbed by the process noise are stable, and the amount of the imposed destabilization depends on the desired relocation of the filter eigenvalues.

The three methods mentioned in Ref. 4 may be unacceptable for the following reasons. First, all of them consider directly or indirectly modifications of the original dynamical system. Thus, the resultant filter may not estimate the state of the system effectively since it is not designed for the original system. Second, relocation of the filter eigenvalues well inside the left half plane (or unit circle) can usually be achieved by increasing the filter gains. This, however, causes the filter to be very sensitive to the measurement noise, which is always present. Hence, it becomes evident that from an operational

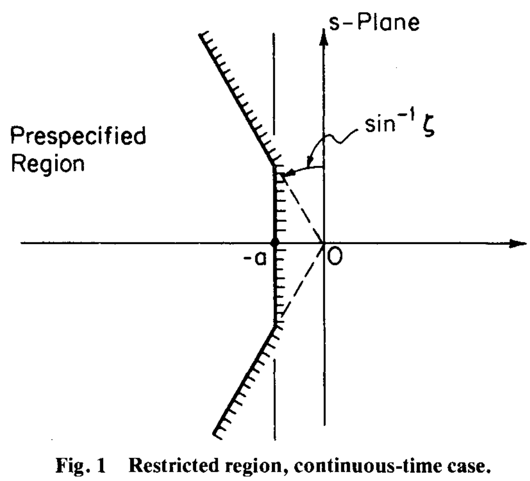

Fig. 1 Restricted region, continuous-time case.

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*Graduate Student; presently with Lear-Siegler, Grand Rapids, Mich.

†Professor, Department of Aerospace Engineering; currently Manager, Control Systems Department, Ford Motor Company Engineering and Research Staff, Dearborn, Mich.

point of view strong response characteristics of the constant gain filter alone are not adequate for acceptable performance requirements. The quality of the resulting estimates should somehow influence the design so that the resulting filter will give fast, as well as accurate, state estimates. This suggests the minimization of the trace of the error covariance matrix among all filters with eigenvalues satisfying certain prescribed stability criteria. One such set of stability criteria is damping ratio and constant decrement factor in the continuous case systems, and decrement factor alone in the discrete-time systems. Both define certain regions of the complex s and z planes, respectively, that can be viewed as restricted regions where minimization of the trace of the error covariance matrix should be attempted.

In this paper an optimization based approach to the design of constant gain filters for both continuous- and discrete-time linear time-invariant systems is developed. In Sec. II the equations for such a unified approach are developed, and in Sec. III three applications of the various capabilities of the approach are presented.

II. Design Algorithms

A. Continuous-Time Systems

Consider the linear time-invariant continuous-time system

$$\dot{x}=Fx+Gw \tag{1}$$

$$z=Hx+v \tag{2}$$

where x is the state vector, z is the measurement vector, w, v are uncorrelated white noise processes with zero means and given covariance kernels Q and R, and F,G,H are given constant matrices. A Kalman structure state estimator for this system is given by

$$\dot{\hat{x}}=F\hat{x}+K(z-H\hat{x}) \tag{3}$$

The steady-state covariance matrix P in the error

$$\tilde{x}=\hat{x}-x \tag{4}$$

corresponding to the constant gain matrix K is given by the solution of the following continuous type algebraic Lyapunov equation[6]:

$$(F-KH)P+P(F-KH)^T+GQG^T+KRK^T=0 \tag{5}$$

To guarantee satisfaction of relative stability requirements the eigenvalues of the filter stability matrix $(F-KH)$ should be restricted to the region of the complex s plane defined by two inequality constraints (see Fig. 1). One of these constraints is the damping ratio constraint, and the other is the constant decrement factor constraint.

Satisfaction of the constraints is determined either directly through the relative position of the filter eigenvalues in the complex plane, or indirectly through inner determinants indicative of constraint violations. The latter are developed in Ref. 7. The results necessary for the present development and how they apply to the problem are as follows. Let

$$F(s)=(a'_{n+1}+ja''_{n+1})s^n+(a'_n+ja''_n)s^{n-1} +\ldots+(a'_2+ja''_2)s+(a'_1+ja''_1) \tag{6}$$

be the characteristic polynomial for an nth order system. Define

$$F(js)=(b_{n+1}+jc_{n+1})s^n+(b_n+jc_n)s^{n-1} +\ldots+(b_2+jc_2)s+(b_1+jc_1) \qquad (c_{n+1}\neq 0) \tag{7}$$

and form the matrix

$$\Delta_{2n}=\begin{bmatrix}
c_{n+1} & c_n & c_{n-1} & & \cdots 0 \cdots & & & 0\\
0 & & & & & & & \\
\vdots & & & & & & & \\
0 & c_{n+1} & c_n & c_{n-1} & c_{n-2} & & & \\
\vdots & & & & & & & \\
\vdots & 0 & c_{n+1} & c_n & c_{n-1} & \cdots & c_2 & c_1\\
0 & & & & & & & \\
\vdots & 0 & b_{n+1} & b_n & b_{n-1} & \cdots & b_2 & b_1\\
0 & b_{n+1} & b_n & b_{n-1}\ \ \Delta_2 & b_{n-2} & & & \\
\vdots & & & & & \Delta_4 & & \\
0 & & & & & & & \\
b_{n+1} & b_n & b_{n-1} & & \cdots 0 \cdots & & & 0
\end{bmatrix} \tag{8}$$

The necessary and sufficient condition for the roots of Eq. (6) to be in the open left-half plane, i.e., for the nth order system to be stable, is that Δ_{2n} in Eq. (8) is positive innerwise. (Positive innerwise here means that all the inner determinants as indicated in Eq. (8) are positive.) Let

$$F(s)=a_{n+1}s^n+a_ns^{n-1}+\ldots+a_3s^2+a_2s+a_1 \qquad (a_{n+1}>0) \tag{9}$$

be the characteristic polynomial of an nth order system. The necessary and sufficient conditions for the roots of Eq. (9) to lie in the open left-half plane can be given as follows: 1) the a_i's (or half of them) be positive and 2) the following Δ_{n-1} matrix for n even be positive innerwise:

$$\Delta^e_{n-1}=\begin{bmatrix}
a_{n+1} & a_{n-1} & \cdots & a_3 & a_1 & 0\cdots 0\\
0 & a_{n+1} & a_{n-1} & a_{n-3} & & \cdots a_1\\
\vdots & & & & & \\
\vdots & 0 & a_n & a_{n-2} & & \cdots a_2\\
\vdots & & \Delta_1 & & & \\
\vdots & a_n & a_{n-2} & a_{n-4} & & \cdots 0\\
0 & & & \Delta_3 & & \\
a_n & a_{n-2} & \cdots & a_2 & \cdots & 0
\end{bmatrix} \tag{10}$$

To determine satisfaction of the imposed constraints via the inner determinants consider the characteristic polynomial corresponding to the filter stability matrix $(F-KH)$ in the s-plane:

$$F(s)=a_{n+1}s^n+a_ns^{n-1}+\ldots+a_2s+a_1 \tag{11}$$

The complex transformation

$$w = s + \alpha \tag{12}$$

transforms the area of the s plane to the left of the constant decrement factor line at $-\alpha$ to the left-half w plane. The chracteristic polynomial Eq. (11) under this transformation takes the form

$$F_1(w) = \sum_{k=1}^{n+1} \sum_{p=1}^{n-k+2} \left[(-1)^{n-(k+p-2)} \times \binom{n-(p-1)}{n-(k+p-2)} \alpha^{n-(k+p-2)} a_{n+1-(p-1)} \right] w^{k-1} \tag{13}$$

where $\binom{k}{l}$ indicates the usual combinatorial notation. Thus, for n odd, $F_1(jw)$ of the form shown in Eq. (7) is developed, and then employed to define the innerwise matrix Δ_{2n} indicated by Eq. (8). Positivity of Δ_{2n}, in the innerwise sense, guarantees that the constant decrement factor constraint is satisfied. For n even, the innerwise matrix Δ^e_{n-1} indicated by Eq. (10) is formulated from the coefficients of Eq. (13). Positivity of Δ^e_{n-1} in the innerwise sense, in addition to having all its coefficients positive, guarantees again that the constant decrement factor constraint is satisfied.

Next, the complex transformation

$$w = se^{-j\theta} \tag{14}$$

where θ is defined by

$$\theta = \arcsin \zeta \tag{15}$$

transforms the area of the s plane to the left of the damping ratio line constraint ζ to the left-half w plane. The characteristic polynomial Eq. (11) under this transformation takes the form

$$F_1(w) = \sum_{k=1}^{n+1} \{ a_k [\cos(k-1)\theta + j\sin(k-1)\theta] \} w^{k-1} \tag{16}$$

For either n even or n odd, $F_1(jw)$ of the form shown in Eq. (7) is developed, and then employed to define the innerwise matrix Δ_{2n} indicated by Eq. (8). Positivity of Δ_{2n}, in the innerwise sense, guarantees that the damping ratio constraint is satisfied.

Hence, the problem posed can now be stated mathematically as follows.

Minimize: $\operatorname{tr}[P]$

Any scalar function can be employed in the procedures to be developed, e.g., a weighted sum of particular covariance matrix elements. $\operatorname{tr}[P]$ will be employed herein for notational convenience.

Subject to: Eigenvalues of $(F-KH)$ inside the restricted region shown in Fig. 1.

The approach chosen for the determination of the suboptimal gain matrix K employs an optimization algorithm based upon Powell's method and developed by Brent.[8] It is a conjugate direction method which is stable, has a good rate of convergence, and does not require an explicit calculation of the gradient. (The latter property is especially important for this class of problem because of the human effort required to develop an analytical gradient.)

The computer-based approach involves two phases. The first phase moves the eigenvalues of the filter stability matrix $(F-KH)$ inside the restricted region. The form of the performance index for this phase is

$$J_1[K] = W_1 \sum_{i=1}^{n} (1.5\alpha + g_i)^2 u_i + W_2 \sum_{i=1}^{n} [\, |g_i|\zeta - |h_i| \cos(\arcsin\zeta) \,]^2 v_i \tag{17}$$

where W_1, W_2 are simple weighting coefficients; u_i, v_i are unit step functions; and $s_i = g_i + jh_i$ $(i=1,2,\ldots,n)$ are the eigenvalues of $(F-KH)$.

At the time of the first call in this phase only the first half of $J_1[K]$ is employed, with the following values for the parameters:

$$W_1 = 1.0 \qquad W_2 = 0.0 \tag{18}$$

and

$$\begin{aligned} u_i &= 1 \quad \text{if} \quad -1.5\alpha < g_i \\ &= 0 \quad \text{if} \quad -1.5\alpha \geq g_i \end{aligned} \qquad (i=1,2,\ldots,n) \tag{19}$$

This causes all the eigenvalues to move well into the left half plane, and more specifically, to the left of the line parallel to the imaginary axis and displaced from it by -1.5α units.

At the time of the second call, the complete expression for $J_1[K]$ is employed, with the following values for the parameters:

$$W_1 = 10^6 \qquad W_2 = 1.0 \tag{20}$$

the u_i the same as in Eq. (19), but with $-\alpha$ instead of -1.5α and

$$\begin{aligned} v_i &= 1 \quad \text{if} \quad h_i \neq 0 \quad \text{and} \quad (g_i/\sqrt{g_i^2+h_i^2}) < \zeta \\ &\qquad\qquad (i=1,2,\ldots,n) \\ &= 0 \quad \text{if} \quad h_i = 0 \quad \text{or} \quad (g_i/\sqrt{g_i^2+h_i^2}) \geq \zeta \end{aligned} \tag{21}$$

This time (with the eigenvalues initially to the left of the 1.5α constant decrement factor line), the first half of the expression for $J_1[K]$ is nonzero only when some of the eigenvalues of $(F-KH)$ happen to violate the constant decrement factor constraint. For all the problems tested with the algorithm, the weighting $W_1 = 10^6$ appears to be a satisfactory value for preventing the eigenvalues from violating the constant decrement factor constraint. The second part of $J_1[K]$ causes the eigenvalues to move to the left of the damping ratio constraint line. Thus, when the optimization algorithm finishes its operation at the end of the second call, all the eigenvalues of $(F-KH)$ are inside the restricted region.

The second phase accomplishes the desired design by minimizing a performance index of the form

$$\begin{aligned} J_2[K] &= \operatorname{tr}[P_\infty] + \sum_{i=1}^{2} \sum_{j=2,4,\ldots}^{2n} [\, W|\operatorname{tr}[P_\infty]| \bullet |DD_i(j)| \,] U(DD_i(j)) \qquad (n \text{ odd}) \\ &= \operatorname{tr}[P_\infty] + \sum_{j=2,4,\ldots}^{2n} W[\, |\operatorname{tr}[P_\infty]| \bullet |DD_1(j)| \,] U(DD_1(j)) + \sum_{j=1,3,\ldots}^{(n-1)} W[\, |\operatorname{tr}[P_\infty]| \bullet |DD_2(j)| \,] U(DD_2(j)) \\ &\quad + \sum_{i=1}^{(n+1)} W[\, |\operatorname{tr}[P_\infty]| \bullet |\alpha_i| \,] U(\alpha_i) \qquad (n \text{ even}) \end{aligned} \tag{22}$$

where the following conditions exist:

1) $P_\infty[K]$ is the steady-state error covariance matrix determined by solving Eq. (5).

2) $DD_1(j)$ and $DD_2(j)$ are the inner determinants of the characteristic equation of $(F-KH)$ corresponding to the damping ratio and constant decrement factor constraints, respectively.

3) α_i, $i=1,2,...,n+1$ are the coefficients of the transformed characteristic equation in the w plane resulting from the constant decrement factor constraint.

4) $U(x)$ is a unit step function defined by

$$\begin{aligned} U(x) &= 1 \quad \text{if} \quad x<0 \\ &= 0 \quad \text{if} \quad x\geq 0 \end{aligned} \tag{23}$$

5) W (const) is an automatically determined adaptive weighting coefficient calculated by the algorithm at the iterate of the optimization process where it first encounters negative inners. Its numerical value is 10^{weight}, where weight is calculated as follows.

First the maximum M and minimum m values of the following sets are determined:

$$\{\,|\mathrm{tr}[P_\infty]|,\ [\,|DD_i(j)|\,(j=2,4,...,2n;i=1,2), DD_i(j)<0]\,\} \qquad \text{(for } n \text{ odd)}$$

$$\begin{aligned} \{\,|\mathrm{tr}[P_\infty]|,\ &[\,|DD_1(j)|\,(j=2,4,...,2n),\ \ DD_1(j)<0],\\ &[\,|DD_2(j)|\,(j=1,3,...,(n-1)),\ \ DD_2(j)<0],\\ &[\,|\alpha_i|\,(i=1,2,...,(n+1)),\ \ \alpha_i<0]\,\} \end{aligned} \qquad \text{(for } n \text{ even)}$$

Then, defining int to be the function determining the integer part of a decimal number (e.g., int(2.35) = 2, int(−73.879) = −73), the following cases are considered:

Case 1—if $M\geq 1.0$ and $m\geq 1.0$, then

$$\text{weight} = \text{int}(\log_{10}M)+4$$

Case 2—if $M\geq 1.0$ and $m<1.0$, then

$$\text{weight} = \max\{\text{int}(\log_{10}M),\ |\text{int}(\log_{10}m)|\}+4$$

Case 3—if $M<1.0$ and $m<1.0$, then

$$\text{weight} = |\text{int}(\log_{10}m)|+4$$

Thus, $W[\,|\mathrm{tr}[P_\infty]|\bullet|DD|\,]U(DD)$ guarantees in a continuous way that the cost increases dramatically when a constraint is violated, and serves the purpose of an interior penalty function.

It should be noted that the two phase optimization approach serves a number of operational and *theoretical* purposes. 1) In Ref. 9 it is shown that a unique solution of the Lyapunov Eq. (5) exists if and only if the eigenvalues of $F-KH$ lie in the open left-half plane. Thus, the algorithm should not consider solving the Lyapunov equation until the eigenvalues are at least in the left-half plane (which may not be the case with an arbitrary $K^{(0)}$ estimate). 2) The detectability and stabilizability properties of the system guarantee only that a stable filter exists; they do not guarantee that the system can satisfy the inequality constraints of Fig. 1. (Of course, observability and controllability guarantee arbitrary eigenvalue positioning because of the pole placement theorem.) Thus, the first phase also serves the purpose of determining if the specified constraints are feasible. 3) Simulations to date indicate that convergence is more rapid, instructive, and inexpensive if the two phase approach is employed (as opposed to attacking the performance index and constraints in a single-phase manner). These comments are also applicable to the discrete-time case of the next section.

B. Discrete-Time Systems

Consider the linear time-invariant discrete-time system

$$x_{k+1} = \Phi x_k + \Gamma w_k \tag{24}$$

$$z_k = Hx_k + v_k \tag{25}$$

where the vectors and matrices involved are compatible with those of the continuous-time case systems, and w_k and v_k are white noise sequences with covariance matrices Q and R. A Kalman structure state estimator for this system is given by

$$\hat{x}_k(+) = \hat{x}_k(-) + K[z_k - H\hat{x}_k(-)] \tag{26}$$

$$\hat{x}_k(-) = \Phi\hat{x}_{k-1}(+) \tag{27}$$

or substituting Eq. (27) into Eq. (26), by

$$\hat{x}_k(+) = [(I-KH)\Phi]\hat{x}_{k-1}(+) + Kz_k \tag{28}$$

where $y_k(+)$ is the a posteriori value of the indicated variable at the sampling time t_k, and $y_k(-)$ is the a priori value at t_k. The error covariance matrices $P_k(+)$ and $P_k(-)$ corresponding to the constant gain matrix K are given by[6]

$$P_k(+) = (I-KH)P_k(-)(I-KH)^T + KRK^T \tag{29}$$

$$P_k(-) = \Phi P_{k-1}(+)\Phi^T + \Gamma Q \Gamma^T \tag{30}$$

In the steady state $(k\to\infty)$, the following equalities hold:

$$P_{k-1}(+) = P_k(+) \qquad P_{k-1}(-) = P_k(-)$$

Hence, the subscript k can be dropped, and the corresponding quantities can be designated as $P(+)$ and $P(-)$, respectively. Combining the steady-state form of Eqs. (29) and (30) into one equation by eliminating $P(-)$, the following relation for $P(+)$ is obtained:

$$\begin{aligned} &[(I-KH)\Phi]P(+)[(I-KH)\Phi]^T - P(+) = \\ &-[(I-KH)\Gamma Q\Gamma^T(I-KH)^T + KRK^T] \end{aligned} \tag{31}$$

This is a discrete-type Lyapunov equation resulting in the a posteriori steady-state error covariance matrix $P(+)$ associated with the filter of Eq. (28).

A relative stability requirement in the z plane is the restriction of the eigenvalues of the filter stability matrix

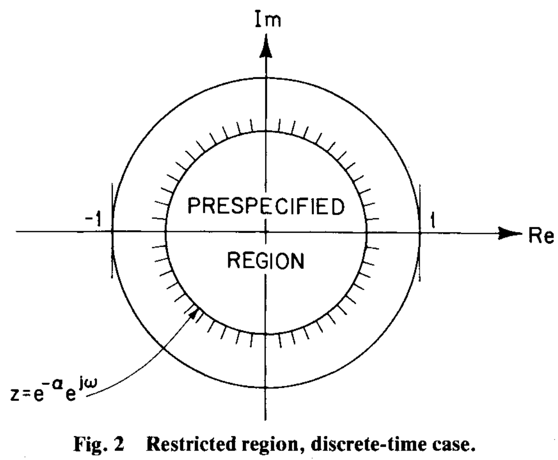

Fig. 2 Restricted region, discrete-time case.

A relative stability requirement in the z plane is the restriction of the eigenvalues of the filter stability matrix $(I-KH)\Phi$ to the region of the complex z plane shown in Fig. 2. This is the region inside the circle $z=e^{-\alpha}e^{j\omega}$ (ω free parameter) describing a constant decrement factor constraint α. In a manner similar to the continuous-time systems case, satisfaction of this constraint is determined either directly through the relative position of the eigenvalues in the complex z plane, or through inner determinants indicative of the constraint violations. To establish the criteria for relative stability determination through inner determinants, let

$$F(z)=a_{n+1}z^n+a_nz^{n-1}+\ldots+a_2z+a_1 \tag{32}$$

be the characteristic polynomial of an nth-order discrete system in the z plane. The necessary and sufficient condition for the roots of Eq. (32) to lie inside the unit circle is that the following Δ_{2n} matrix be positive innerwise:

$$\Delta_{2n}=\left[\begin{array}{cccccccccc}
a_{n+1} & a_n & \cdots\cdots & \cdots & a_2 & 0 & 0 & \cdots & \cdots & a_1 \\
0 & 0 & a_{n+1} & a_n & a_{n-1} & 0 & 0 & a_1 & \cdot\cdot & \\
0 & \cdots\cdots & \cdots\cdots & a_{n+1} & a_n & 0 & a_1 & a_2 & \cdot\cdot & \\
0 & \cdots\cdots & \cdots\cdots & \cdots & a_{n+1} & a_1 & a_2 & a_3 & \cdot\cdot & \\
0 & \cdots\cdots & \cdots\cdots & \cdots & \bar{a}_1 & \bar{a}_{n+1} & \bar{a}_n & \bar{a}_{n-1} & \cdot\cdot & \\
0 & \cdots\cdots & \cdots\cdots & \bar{a}_1 & \bar{a}_2 & 0\ \Delta_2 & \bar{a}_{n+1} & \bar{a}_n & \cdot\cdot & \\
0 & \cdots\cdots & \bar{a}_1 & \bar{a}_2 & \bar{a}_3 & 0 & 0\ \Delta_4 & \bar{a}_{n+1} & \cdot\cdot & \\
\bar{a}_1 & \bar{a}_2 & \bar{a}_n & 0 & 0\ \cdots & \cdots & \cdots & \cdots & \Delta_6 & \bar{a}_{n+1}
\end{array}\right] \tag{33}$$

where the bar over a quantity indicates its complex conjugate.

To determine satisfaction of the constant decrement factor constraint via the inner determinants, consider a characteristic polynomial corresponding to the filter stability matrix $(I-KH)\Phi$ of the form shown in Eq. (32). The complex transformation

$$w=e^{\alpha}z \tag{34}$$

transforms the area of the z plane inside the $z=e^{-\alpha}e^{j\omega}$ circle to the area inside the unit circle of the w plane. The characteristic polynomial of Eq. (32) under this transformation takes the form

$$F_1(w)=(a_{n+1}e^{-n\alpha})w^n+(a_ne^{-(n-1)\alpha})w^{n-1}$$
$$+\ldots+(a_2e^{-\alpha})w+a_1 \tag{35}$$

From this the innerwise matrix of Eq. (33) is formulated, and positivity in the innerwise sense guarantees that the constant decrement factor constraint is satisfied.

Hence, the problem in this case can be stated as follows.

Minimize: $\operatorname{tr}[P(+)]$

Subject to: eigenvalues of $(I-KH)\Phi$ inside the restricted region shown in Fig. 2

Note that by defining a new matrix B from $(I-KH)\Phi$ through the bilinear transformation

$$B=[\{(I-KH)\Phi\}^T-I]^{-1}[\{(I-KH)\Phi\}^T+I] \tag{36}$$

the steady-state discrete type Lyapunov equation given by Eq. (31) is transformed into a steady-state continuous type Lyapunov equation (see Ref. 10) given by

$$B^TP(+)+P(+)B$$
$$=-\tfrac{1}{2}(B^T-I)[(I-KH)Q(I-KH)^T+KRK^T](B-I) \tag{37}$$

Thus, the need for additional algorithms for the solution of Eq. (31) is eliminated and both Eqs. (5) and (31) can be solved by the same algorithm.

The approach for the determination of the suboptimal gain matrix K again proceeds in two phases. The first phase moves the eigenvalues

$$z_i=g_i+jh_i \qquad (i=1,2,\ldots,n)$$

of $(I-KH)\Phi$ inside the circle $z=e^{-\alpha}e^{j\omega}$ by minimizing a performance index (based upon the distances of these eigenvalues from the constant decrement factor constraint) of the form

$$J_1[K]=\sum_{i=1}^{n}\left(\sqrt{g_i^2+h_i^2}-e^{-\alpha}\right)^2u_i \tag{38}$$

where u_i is the unit step function defined by

$$\begin{aligned} u_i&=1 \quad \text{if} \quad g_i^2+h_i^2>e^{-\alpha} \\ &=0 \quad \text{if} \quad g_i^2+h_i^2\le e^{-\alpha} \qquad (i=1,2,\ldots,n) \end{aligned} \tag{39}$$

The second phase completes the design by minimizing a second performance index of the form

$$J_2[K]$$
$$=\operatorname{tr}[P(+)]+\sum_{j=2,4,\ldots}^{2n}W[\,|\operatorname{tr}[P(+)]|\bullet|DD(j)|\,]U(DD(j)) \tag{40}$$

where the following conditions exist.

1) $P(+)$ (a function of K) is the discrete steady-state error covariance matrix determined by solving Eq. (31).

2) $DD(j)$ are the inner determinants of the characteristic equation of $(I-KH)\Phi$, corresponding to the constant decrement factor constraint.

3) $U(x)$ is a unit step function defined by Eq. (23).

4) W(const) is an adaptive weighting coefficient computed the first time a negative inner is encountered during the iteration process. It is determined as in statement 5 of $J_2[K]$ for the continuous case with the only difference that M and m in the present case are defined for the set

$$\{\,|\operatorname{tr}[P(+)]|,\quad \{|DD(j)| \quad (j=2,4,\ldots,2n),\quad DD(j)<0\}\,\}$$

III. Applications

An algorithm was developed to solve the problem formulated in the previous sections. The algorithm incorporates in a single package the solutions to both continuous- and discrete-time problems. Due to certain similarities between the two problems, parts of the algorithm are used interchangeably; thus reducing its size and memory requirements. Among others, these similarities allow for the computation of inner determinants by the same subroutine due to the unifying property of all innerwise matrices created to be left triangular zero. Also, the simultaneous solution of both continuous and discrete type Lyapunov equations is made possible due to the availability of the bilinear transformation given by Eq. (36) that transforms the discrete type equation into a continuous one.

To illustrate the various capabilities of the approach, three examples are presented. These applications are designed to emphasize features such as estimator settling time improvement, unity of the approach to handle both continuous- and discrete-time systems, and combined controller-estimator gain design.

A. Third-Order System with Bias Estimation

The present example demonstrates the feasibility of the algorithm to improve the steady-state Kalman filter if its transient performance is not satisfactory. The system contains three state variables and two measurements. The thrid state variable is a bias-type term modeled as a first-order Markov process. The complete system and measurement equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ w_2 \\ w_3 \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{42}$$

with noise covariance kernels given by

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 64.698 & 0 \\ 0 & 0 & 0.09 \end{bmatrix} \qquad R = \begin{bmatrix} 500 & 0 \\ 0 & 66.698 \end{bmatrix} \tag{43}$$

The eigenvalues of the dynamic system stability matrix are

$$s_{1,2} = 0.0 \qquad s_3 = 0.01 \tag{44}$$

The system is completely observable and completely controllable by the process noise, and thus the steady-state Kalman filter is stable. However, due to the relatively small magnitude of the variance of the w_3 component of the process noise [see Eqs. (43)], estimation of the bias state x_3 is very slow. Indeed, the steady state Kalman gain matrix is

$$K = \begin{bmatrix} 0.848 & -0.112\times 10^{-3} \\ 0.360 & -0.397\times 10^{-4} \\ -0.222\times 10^{-5} & 0.481\times 10^{-1} \end{bmatrix} \tag{45}$$

resulting in eigenvalues of the filter stability matrix $(F-KH)$ at

$$-0.424 \pm j\,0.424 \qquad \text{and} \qquad -0.038 \tag{46}$$

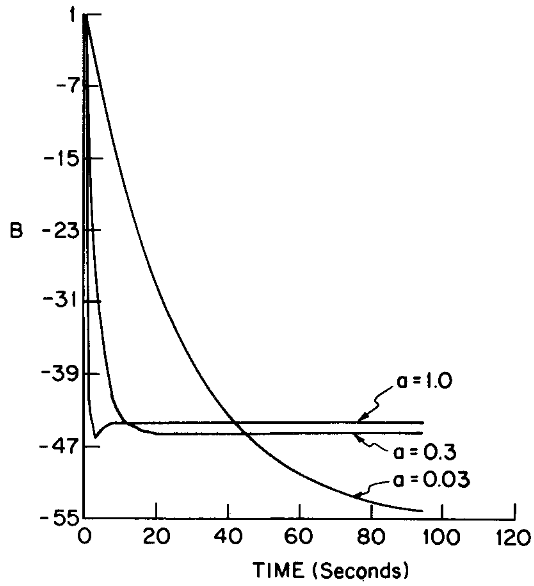

Fig. 3 Bias estimates for $\alpha = 0.03$, 0.3, and 1.0.

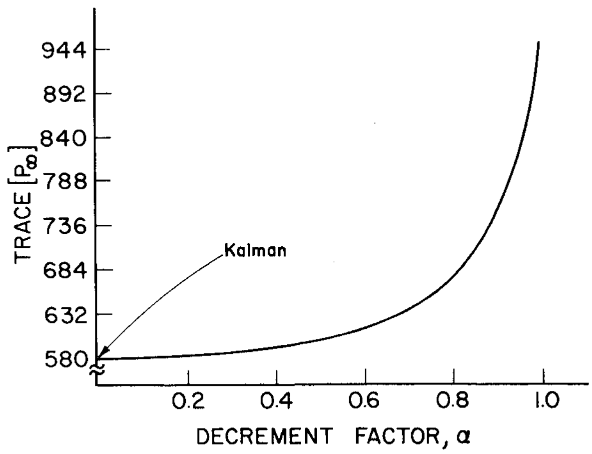

Fig. 4 Trace of error covariance matrix vs constant decrement factor constraint (all cases for $\zeta = 0.5$).

The trace of the error covariance matrix is

$$\operatorname{tr}[P] = 579.858 \tag{47}$$

The -0.038 eigenvalue in Eq. (46) causes a very slow estimation of the bias state. The relative position of the other two eigenvalues is satisfactory, and estimation of the corresponding states x_1 and x_2 reaches a steady state in less than 10 and 20 s, respectively. However, the bias state requires a time interval of over 100 s to reach steady state (see Fig. 3) and this is not acceptable from an operational point of view. Therefore, a constant gain suboptimal filter should be designed with the main goal of relocating the real eigenvalue further inside the left-half s plane without causing a large decrease in estimator accuracy.

To insure that the complex eigenvalues remain satisfactory, $\zeta = 0.5$ was imposed as the damping ratio constraint for all the design cases while the decrement factor constraint was systematically varied. Figure 4 shows the relation between $\operatorname{tr}[P]$ and the constant decrement factor constraint α for the various designs.

To demonstrate the improvement in the response of the suboptimal estimator the design resulting from $\alpha = 0.3$ and $\alpha = 1.0$ are compared to the Kalman filter. The gains resulting

from the preceding designs are

$$[K]_{\alpha=0.3} = \begin{bmatrix} 0.84329 & 0.15192\times10^{-1} \\ 0.35843 & -0.94358\times10^{-2} \\ -0.74760\times10^{-3} & 0.31006 \end{bmatrix} \quad (48)$$

and

$$[K]_{\alpha=1.0} = \begin{bmatrix} 0.20850\times10^{1} & 0.36607 \\ 0.94300 & 0.98676 \\ -0.19258 & 0.18515\times10^{1} \end{bmatrix} \quad (49)$$

with corresponding traces of the error covariance matrices and filter eigenvalues

$$\mathrm{tr}[P] = 587.522 \quad \text{and} \quad -0.423\pm j0.425, \quad -0.3000036 \quad (50)$$

$$\mathrm{tr}[P] = 943.559 \quad \text{and} \quad -1.9265, \quad -1.00000332\pm j0.00432 \quad (51)$$

respectively.

Simulations of the three sets of gains indicate the following. For the first state variable x_1, the resulting estimates are so close to each other that they are practically indistinguishable. For the second state variable x_2, the estimates resulting from the Kalman gains and also the gains corresponding to the $\alpha=0.3$ design are again so close to each other that they can be considered indistinguishable. The x_2 estimate resulting from the $\alpha=1.0$ design is slightly different. Finally, the essential improvement of the suboptimal estimation becomes apparent in the estimation of the x_3 state variable. The resulting estimates are shown in Fig. 4. The suboptimal filters achieve their steady-state values at $t\approx$ 10-20 s whereas the Kalman filter has not reached steady state at $t\approx 100$ s.

B. Discrete-Time Example

The unity of the approach for both continuous and sampled-data systems is illustrated by a discrete-time suboptimal filter design subject to a constant decrement factor constraint. The system considered is a third-order model motivated by a bioengineering application and described by the following process and measurement equations

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{k+1} = \begin{bmatrix} -6 & 0 & 0 \\ 6 & 0.3 & 0 \\ 0 & 0.832 & -0.75 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + \begin{bmatrix} 1 \\ 0.3 \\ 0.8 \end{bmatrix} w_k$$

$$w_k \sim N(0,3) \quad (52)$$

$$z_k = [0\ 0\ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_k + v_k \qquad v_k \sim N(0,4) \quad (53)$$

The system is completely controllable by the process noise, and thus, the steady-state Kalman filter is stable. The steady-state Kalman gain matrix is

$$K = \begin{bmatrix} 0.631767\times10^{1} \\ -0.598004\times10^{1} \\ 0.982325\times10^{0} \end{bmatrix} \quad (54)$$

resulting in eigenvalues of the stability matrix $(I-KH)\Phi$ at

$$\lambda_1 = -0.74733 \qquad \lambda_2 = 0.18348 \qquad \lambda_3 = -0.17402 \quad (55)$$

with

$$\mathrm{tr}[P_\infty] = 0.5340133\times10^{3} \quad (56)$$

From Eqs. (55), the stability characteristics of the filter are satisfactory due to the fact that the three eigenvalues are well inside the unit circle. However, for illustration purposes two additional designs are attempted. Both employ an initial guess of the gain matrix that places the eigenvalues of the filter stability matrix at $0.8\pm j\,0.5$ and -0.5.

The first design is obtained with a constant decrement factor constraint equal to 0.001. This defines a circular restricted region with radius equal to $e^{-0.001}=0.999$, which contains all three eigenvalues λ_1, λ_2, λ_3 of the steady-state Kalman filter. The resultant simulation duplicates exactly the analytic results of the steady-state Kalman calculations.

The second design is with a constant decrement factor constraint equal to 0.8. The corresponding circular restricted region has a radius equal to $e^{-0.8}=0.44933$. This region, of course, does not include the steady-state Kalman eigenvalues, and the resulting design is different. The new gain matrix is

$$K = \begin{bmatrix} 0.641356\times10^{1} \\ -0.608211\times10^{1} \\ 0.966735\times10^{0} \end{bmatrix} \quad (57)$$

with filter eigenvalues at

$$\lambda_1 = 0.226312 \quad \lambda_2 = -0.448985 \quad \lambda_3 = 0.441959 \quad (58)$$

and an increased trace equal to

$$\mathrm{tr}[P_\infty] = 0.5492222\times10^{3} \quad (59)$$

C. Attitude Control of a Solar Probe

Because of the fact that the exact state is not readily available in the optimal quadratic control of linear stochastic systems, the Kalman estimate of the state is employed to close the feedback loop of the resulting linear optimal controller. However, due to the frequently encountered lack of strong response characteristics of the steady-state Kalman filter, the performance of the system may not be satisfactory and introduction of suboptimal estimation becomes a necessary tool.

An example where this situation occurs is in the attitude control of a solar probe, described in Ref. 5. The system is third-order with the true anomaly ν of the trajectory as the independent variable. The state consists of the two spinaxis angles $x_1(\nu)$, $x_2(\nu)$ and the uncertainty of the torque due to the solar pressure $x_3(\nu)$, that is modeled as a first-order Markov stochastic process. There is only one measurement, the second spinaxis angle $x_2(\nu)$, and two controls, u_1 and u_2. The complete state and measurement equations are

$$\dot{x} = Fx + Du + Gw \qquad w \sim N(0, 4\times10^{-4})$$

$$z = Hx + v \qquad v \sim N(0, 1.4\times10^{-7}) \quad (60)$$

where

$$F = \begin{bmatrix} 0 & 360 & 1 \\ -360 & 0 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (61)$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad H = [0\ 1\ 0] \quad (62)$$

and

$$u \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

To complete the problem, the following performance criterion is chosen for minimization by the control vector u

$$J = E\left\{\int_0^\infty [x^T A x + u^T B u]\,\mathrm{d}t\right\} \tag{63}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \tag{64}$$

This problem satisfies the conditions of the separation principle and one may apply the standard linear-quadratic Gaussian theory to determine an optimal controller and state estimator without iteration. The resultant steady-state controller is

$$u(\nu) = C\hat{x}(\nu) \qquad C = \text{const} \tag{65}$$

where $\hat{x}(\nu)$ is the steady-state Kalman estimate of the state. The steady-state optimal control and estimation gain matrices are

$$C = \begin{bmatrix} -10 & \approx 0 & -7.96667\times 10^{-4} \\ \approx 0 & -10 & -2.7755\times 10^{-2} \end{bmatrix} \tag{66}$$

and

$$K = \begin{bmatrix} -4.5026\times 10^{-5} \\ 1.8005\times 10^{-1} \\ -1.1366\times 10^{1} \end{bmatrix} \tag{67}$$

with filter eigenvalues at

$$\lambda_{1,2} = -7.423919\times 10^{-2} \pm j\,360 \qquad \lambda_3 = -0.3649 \tag{68}$$

and

$$\mathrm{tr}[P] = 5.7292\times 10^{-4} \tag{69}$$

However, since the eigenvalues λ_1, λ_2 are very close to the imaginary axis, estimation of the spinaxis angles is not satisfactory. This degrades the performance of the controller to the point where it cannot stabilize the system satisfactorily.

To correct the problem a number of suboptimal designs were systematically obtained by employing the continuous part of the algorithm. The design produced with $\zeta = 0.4$ and $\alpha = 0.35$ is suggested as a good compromise between the requirements of accuracy and stability. Thus, the resulting suboptimal filter gain matrix is

$$K = \begin{bmatrix} 0.128827\times 10^{2} \\ 0.265927\times 10^{3} \\ -0.111634\times 10^{2} \end{bmatrix} \tag{70}$$

placing the eigenvalues of the filter at

$$\lambda_{1,2} = -142.948 \pm j\,327.534 \qquad \lambda_3 = -0.364822 \tag{71}$$

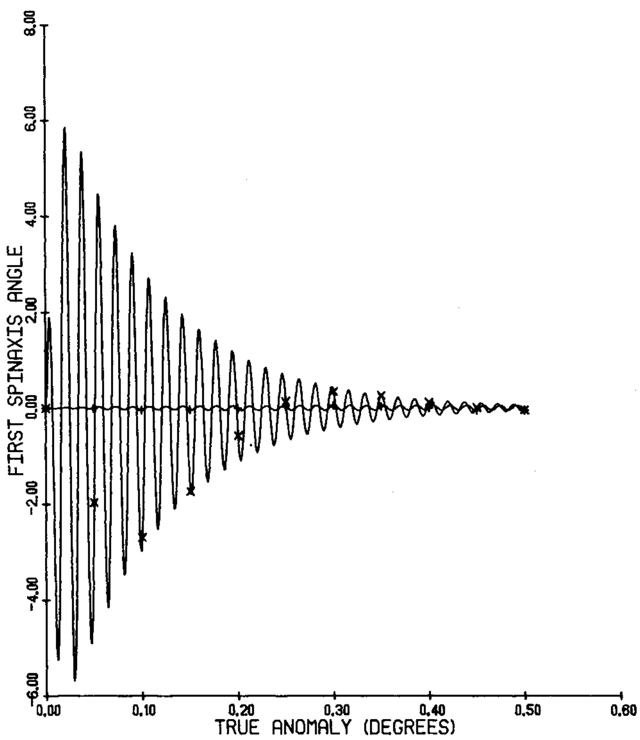

Fig. 5 Solar probe estimator for first spinaxis angle: + —Kalman filter; × —suboptimal design with $\alpha = 0.35$, $\zeta = 0.4$.

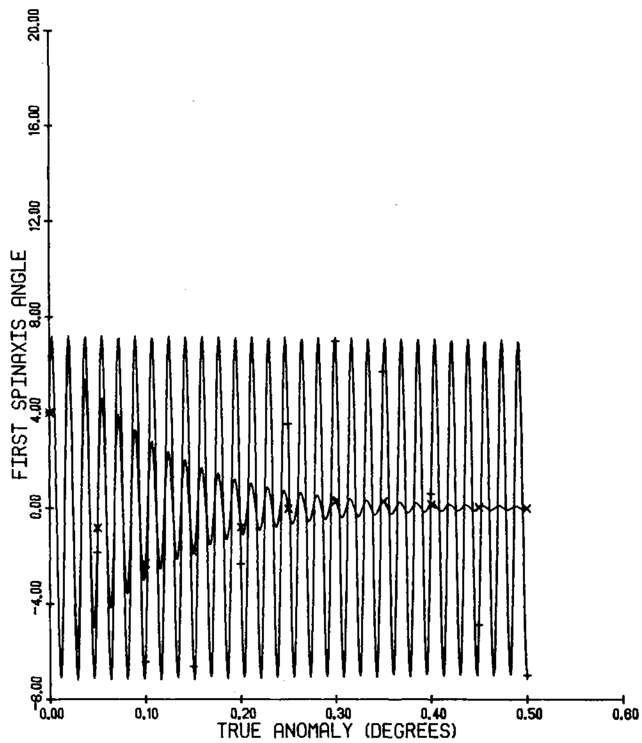

Fig. 6 Solar probe stochastic quadratic controller output: + —Kalman filter; × —suboptimal design with $\alpha = 0.35$, $\zeta = 0.4$.

and resulting in a trace of the error covariance matrix

$$\mathrm{tr}[P] = 6.097\times 10^{-4} \tag{72}$$

The superiority of the suboptimal estimation for this case becomes apparent in Figs. 5 and 6. Figure 5 shows the optimal (Kalman) and suboptimal ($\zeta = 0.4$, $\alpha = 0.35$) estimates for the first spinaxis angle corresponding to an oscillatory input

measurement. The performance of the controller for this spinaxis angle when the feedback loop is closed with the previously given estimates is shown in Fig. 6. Thus, although the two different cases displayed are in phase with respect to the frequency of their oscillations, the case involving the controller with the suboptimal estimator follows a smooth decaying oscillation that brings the spinaxis angle to the desired zero condition and keeps it there. Conversely, when the steady-state Kalman estimate is used, the effort of the controller is nearly negligible and the system continues oscillating with near-zero damping. Similar behavior is exhibited for the second spinaxis angle.

IV. Conclusions

A procedure for designing constant-gain minimum-variance filters subject to stability constraints is developed for time-invariant linear continuous- and discrete-time systems corrupted by additive white noise. The approach is especially useful for systems in which the steady-state Kalman gains result in unacceptably long settling times and/or lightly damped estimators that perform poorly in control systems applications. Computer programming effort is reduced by utilizing properties of inner determinants and Lyapunov equations that apply to both continuous- and discrete-time systems.

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