Technical Comments

Comment on "Improved Time-Domain Stability Robustness Measures for Linear Regulators"

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IN this Comment we discuss the results in the paper by Petkovski regarding stability margins in the face of known directional perturbations.[1] In that work, the author deals with a special class of perturbations problems that can be stated as follows:

Find the maximum value $\bar{e}$ such that the system

$$\dot{x} = (A + eE)x \tag{1}$$

is stable for all $0 \le e < \bar{e}$, where E is a given perturbations matrix and A is a stable matrix describing the unperturbed system.

Our critique of the approach proposed in Ref. 1 is twofold. First, the perturbations model proposed may not be realistic from a practical point of view. The requirement that all perturbations move in a certain fixed direction E by the same relative scaling e is quite restrictive since it may very well be the case that perturbations are uncorrelated; i.e., every element of E, say E_{ij}, may undergo changes in an independent and possibly random fashion.

For this restrictive perturbations model to be judged adequate, we have to assume that the designer has perfect knowledge of the directional properties of the possible perturbations; i.e., a certain elemental perturbation will always be of a certain sign (negative or positive) and with a fixed relative scaling to all other possible element perturbations. Thus, we exclude cases where instability is caused by a certain element being perturbed by 10% while another is perturbed by 5% in the opposite direction (sign).

This assumption may be impractical since in most cases the engineer does not have an a priori (not to mention, perfect) knowledge of the directional information and the relative scaling of the different modeling errors. Such information, as will be discussed below, is important for the method in Ref. 1 to arrive at correct answers.

Second, there are difficulties associated with the proposed method of solution. In Ref. 1, an iterative procedure for evaluating $\bar{e}$ is proposed, where the parameter $\bar{e}$ is given by $\Sigma_i^\infty e_i$, with

$$e_i < \frac{1}{\sigma_{\max}(|P|E)_s}$$

P solves the Lyapunov equation

$$\bar{A}^T P + P\bar{A} + 2I = 0$$

and

$$\bar{A} = A + \sum_{p=0}^{i-1} e_p E$$

Unfortunately, no proof was given as to the convergence of this algorithm to the optimal $\bar{e}$. Furthermore, by examining the examples solved in Ref. 1, we found cases for which $(A + \bar{e}E)$ is strictly stable; i.e., the given bound is too conservative. We also detected cases for which a finite $\bar{e}$ was given as an upper bound on possible positive perturbations in a given direction E, although the system matrix cannot be destabilized with any positive perturbations in the given direction E. That is to say, the method described in Ref. 1 is inaccurate.

To qualify the preceding statements, let us examine one of the examples given in Ref. 1. In particular, let

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \tag{2}$$

and

$$E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{3}$$

In Ref. 1, it was found that the maximum possible positive perturbation $\bar{e}$ such that $e \ge \bar{e} \rightarrow (A + eE)$ is unstable if $\bar{e}$ = 1.5583. Our investigation shows that no positive e such that $(A + eE)$ is unstable exists. That is, for any $e \ge 0$, eigenvalues of $(A + eE)$ are strictly in the left half-plane. Yet, if we allow e to take negative values, $(A + eE)$ becomes unstable for values of $e \le -1$. To see that, note that

$$A + eE = \begin{bmatrix} -3 & -2 \\ 1+e & 0 \end{bmatrix} \tag{4}$$

Thus, the roots of $(A + eE)$ are

$$\lambda = \frac{-3 \pm \sqrt{1-8e}}{2} \tag{5}$$

The only way for one of these roots to have a nonnegative real part is for

$$e \le -1$$

This discussion illustrates the problems that arise if too much confidence is placed on the directional perturbations matrix E. Furthermore, it shows that the method in Ref. 1 fails under certain circumstances.

If the perturbations matrix in the preceding counterexample is replaced by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

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then $\bar{e}$ is given in Ref. 1 by 0.8981. Our investigation shows that the least conservative value of $\bar{e}$ is

$$\bar{e} = 1 \tag{7}$$

That is, the method proposed in Ref. 1 results in a conservative estimate of the maximum allowed perturbation.

It is worth noting here that the stability-margin problem as stated above can be formulated as that of finding a zero of a nonlinear (also, nondifferentiable) function in one parameter. Namely, let

$$f(e) = \max[R\lambda(A + eE)] \tag{8}$$

where $\lambda(\ )$ corresponds to the set of eigenvalues of the given matrix and R stands for taking the real part of a complex number. Then, the minimum value $|\bar{e}|$ such that $f(\bar{e}) = 0$ is actually the stability margin that we are seeking. We have successfully checked the results in the above two counterexamples using a simple bisection algorithm to find the zeros of the function f in Eq. (8).

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References

[1]Petkovski, D. B., "Improved Time-Domain Stability Robustness Measures for Linear Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 4, 1989, pp. 595–598.

Reply by Author to Nasser M. Khraishi

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OUR paper[1] deals with a special type of perturbations that move in a certain fixed direction $\bar{E}$ by the same relative scaling e ($0 \le e \le \bar{e}$). As pointed out in the paper, the main reason for choosing such a structure for the perturbation matrix is the need to incorporate the directional information in the perturbation matrix, which was not the case with the perturbation characterization proposed in Refs. 2-4. In this way, we considerably improved the perturbation bounds.[2-4] However, it should be pointed out that our approach can also be applied to the more general cases of perturbation characterizations proposed in Refs. 2-4. Furthermore, although it may seem that our assumption is quite restrictive, there are at least two classes of systems widely used in practice, where the perturbation directions are well-defined:

1) Singularly perturbed systems,[5-7] where the directional perturbations in the open-loop matrix A and in the control actuating matrix B, are defined by

$$\Delta A = \frac{1}{\lambda}\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \qquad \Delta B = \frac{1}{\lambda}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

2) Weakly coupled systems[6]

$$\Delta A = \epsilon\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}, \qquad \Delta B = \epsilon\begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}$$

As is well known, the entries of these matrices are known, and are not necessarily positive numbers, and they cannot be properly characterized by the approaches proposed in Refs. 2–4, where e_{ij} are assumed to be positive numbers.

For a more general case, when every element of $\bar{E}$ may undergo changes in an independent and possibly random fashion, we proposed a new approach on stability robustness characterization, based on the concept of interval matrices.[8] However, although a wide choice of measures of the "size" of the perturbations has been suggested in the area of robustness analysis for multivariable systems, a basic need still remains for more refined tests and measures of robustness.

As in the case of most of the results published in the literature, our stability robustness criterion only provides *sufficient conditions*, and assumes that $e \ge 0$. However, it still represents an improvement over the results published in Refs. 2-4. We have not claimed that our algorithm leads to exact stability margins.

Therefore, the so-called counterexamples only pointed out that we have proposed only sufficient conditions, restricted to the case when $e \ge 0$, and that further research is needed to overcome the conservatism of the existing stability robustness criteria for multivariable control systems.

The case when e is not restricted to being a positive number can easily be included by using the following theorem:

Theorem 1. The perturbed system

$$\dot{x}(t) = (A + e\bar{E})x(t) \tag{1}$$

where A is asymptotically stable matrix and $\bar{E}$ is a perturbation matrix, is asymptotically stable if the following inequalities are satisfied:

$$\lambda_{\min}^{-1}(Y) = e_{\min} < e < e_{\max} = \lambda_{\max}^{-1}(Y) \tag{2}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the minimum and maximum eigenvalues of $(\cdot)$, respectively, and

$$Y = \tfrac{1}{2}(\bar{E}^T P + P\bar{E}) \tag{3}$$

and the matrix P is the positive definite solution of

$$A^T P + PA + 2I = 0 \tag{4}$$

Proof. The proof proceeds by using the argument of Lyapunov theory. Consider the positive definite function $V(x)$ for the perturbed system

$$V(x) = x^T P x \tag{5}$$

Since P is the positive definite matrix, it remains to examine $\dot{V}(x)$. Taking the time derivative of $V(x)$ along the solution of Eq. (1), it follows that

$$\dot{V}(x) = x^T[2I - e(\bar{E}^T P + P\bar{E})]x \tag{6}$$

making the simplification by using the Lyapunov function, Eq. (4).

Asymptotic stability follows if $\dot{V}(x)$ is negative definite, which follows if

$$2I - e(\bar{E}^T P + P\bar{E}) > 0 \tag{7}$$

To prove the conditions given by Eq. (2), recall the following lemma.

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