Fractional Order State Equations for the Control of Viscoelastically Damped Structures

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Fractional order state equations are developed to predict the effects of feedback intended to reduce motion in damped structures. The mechanical properties of damping materials are modeled using fractional order time derivatives of stress and strain. These models accurately describe the broadband effects of material damping in the structure's equations of motion. The resulting structural equations of motion are used to derive the fractional order state equations. Substantial differences between the structural and state equations are seen to exist. The mathematical form of the state equations suggests the feedback of fractional order time derivatives of structural displacements to improve control system performance. Several other advantages of the fractional order state formulation are discussed.

Nomenclature

$\underset{\sim}{A}$	= state matrix
$-a^\beta$	= system eigenvalue
$-\underset{\sim}{a}^\beta$	= diagonal matrix of eigenvalues
$\underset{\sim}{B}$	= state control matrix
b	= viscoelastic model parameter
D^β	= β-order fractional derivative
$\hat{D}^\beta$	= modified β-order fractional derivative
E_0, E_1	= viscoelastic model parameters
$E_\beta(x)$	= β-order Mittag-Leffler function
$\tilde{F}(t)$	= applied loads prior to initial time, $t = 0$
$\hat{F}(t)$	= applied loads after initial time, $t = 0$
$\tilde{f}(t)$	= modal loads prior to initial time, $t = 0$
$\hat{f}(t)$	= modal loads after initial time, $t = 0$
$f^*(t)$	= stress operator acting on loads
$\tilde{G}(t)$	= structural pseudoloads
$g(t)$	= modal load
$\tilde{g}(t)$	= modal pseudoload
$-\underset{\sim}{G}$	= feedback gain matrix
$I^{1-\alpha}$	= one minus α-order fractional integral
$\underset{\sim}{k}_0$	= structural stiffness matrix
$\underset{\sim}{k}_1$	= structural viscostiffness matrices
N	= number of physical degrees of freedom
$\underset{\sim}{M}$	= structural mass matrix
$\tilde{t}$	= time starting at the onset of motion
t	= time starting at the initial time
t_0	= time interval between $\tilde{t} = 0$ and $t = 0$
$w(t)$	= structural displacements
x_i	= spatial coordinates
x_r	= reduced state vector
$x(t)$	= state vector
x_0	= initial state vector
$y(t)$	= modal response
$\tilde{y}(t)$	= modal response for loading prior to $t = 0$
$\hat{y}(t)$	= modal response for loading after $t = 0$
z	= impulsive load coefficients vector
β	= basis fraction $(1/n)$ for the system
Γ	= gamma function
$\varepsilon(t)$	= strain history
ϕ	= system orthonormal transformation
$\sigma(t)$	= stress history
$(E_0 + E_1 D^\alpha)$	= strain operator
$(1 + bD^\alpha)$	= stress operator

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Introduction

In the modeling of the linear elastic behavior of large space structures, damping has typically either been ignored or modeled as being linearly dependent on velocity. This damping model is adequate for very lightly damped structures and also allows a linear state space model to be defined for the structure's motion. The linear damping formulation is also well suited to the design of active control systems using state space techniques.

However, for heavily damped structures ignoring the damping is imprudent, and modeling it as being linearly dependent on velocity is inadequate. Velocity-dependent damping models, although mathematically straightforward, fail to describe the broadband mechanical behavior of damping materials. Historically, the need for more refined models has pushed the development of viscoelasticity as a discipline within engineering mechanics. Applicable viscoelastic models relate time-dependent stress and strain fields with series of ordinary time derivatives. These models yield acceptable broadband Bode plots of material properties, but they have drawbacks. Typically these models contain many terms, making them mathematically cumbersome and increasing the order of the differential equations describing the system.

As an alternative we will present accurate broadband viscoelastic damping models having only four parameters[1] and posed in terms of noninteger order time derivatives. The real strength of this approach is that these noninteger or fractional order derivatives describe inertial effects, damping effects, elastic effects, and control effects with equal precision. Substantial accompanying benefits are that the order of differentiation is kept low, reducing the effects of noise in the state, and that a potentially infinite number of additional feedback states arise at a finite number of locations to improve system performance. As a consequence, state feedback is generalized to include fractional order time derivatives of structural displacement histories. This generalization is an implementation of fractional derivative feedback, first suggested by Skaar et al.,[2] posed in a state format.

To reap the benefits of this approach, however, one must become comfortable with the concept of fractional order

differentiation. Although the convolution operator that produces these time derivatives at first appears alien, fractional differentiation in the Laplace transform domain is exceedingly simple. Multiplying a transform by s^α, in effect, produces the transform of the α-order derivative, when initial conditions are zero.

The development and applications of fractional order derivatives in viscoelasticity and structural dynamics are well documented.[1,3–12,14,15] The models are consistent with thermodynamic constraints[7] and have their foundation in classical molecular theories predicting the macromechanical properties of viscoelastic materials.[5]

The resulting structural equations of motion serve as the foundation for the state equations, but they are substantially different. The hereditary nature of the structural equations suppresses the existence of homogeneous solutions found in the state equations. In addition, the two sets of equations employ different operators that lead to different requirements for initial conditions. It should come as no surprise that the generalized or fractional order state equations comprise a generalization of the initial value problem. The generalization begins with the structural equations of motion.

Structural Equations of Motion

The structural equations of motion differ from classical formulations in that fractional order derivatives are used to model the viscoelastic damping phenomenon. The extended Riemann-Liouville fractional derivative is a linear operator,[18]

$$D^\alpha_{(t)}[w(t)] \equiv \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{w(\tau)}{\Gamma(1-\alpha)(t-\tau)^\alpha}\,\mathrm{d}\tau, \qquad 0 \le \alpha \le 1 \tag{1}$$

and serves as the basis of the generalized model of the viscoelastic phenomenon. The most general form of the models is

$$\sigma(t, x_i) + \sum_{p=1}^{N} b_p D^{\alpha_p}_{(t)}[\sigma(t, x_i)]$$
$$= E_0\epsilon(t, x_i) + \sum_{p=1}^{N} E_p D^{\alpha_p}_{(t)}[\epsilon(t, x_i)] \tag{2}$$

where the derivatives acting on the stress $\sigma(t, x_i)$ and strain $\epsilon(t, x_i)$ fields are of real, rational fractional order. Note that this model becomes the classical viscoelastic model[16] when the orders of differentiation are taken to be integers.

The Fourier transform of the fractional derivative of a function has a special property when the function is zero for negative time:

$$F\{D^\alpha_{(t)}[x(t)]\} = (i\omega)^\alpha F[x(t)] \tag{3}$$

where $(i\omega)^\alpha$ is the principal root, and the Fourier transform is

$$F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\,\mathrm{d}t \tag{4}$$

This property, Eq. (3), is that the transform of the α-order derivatives is the transform parameter $i\omega$ raised to the α power times the transform of the function. Note the similarity of this transform with the Fourier transform of an ordinary derivative.

The attractive feature of the fractional derivative operator is the ability to vary the degree of its frequency dependence through the choice of α. As a direct result, fractional derivative models are capable of modeling linear, frequency-dependent phenomena not easily captured by the transforms of ordinary derivatives. This leads to models that accurately predict frequency responses over several decades of frequency and yet contain very few, typically four, parameters.[7]

In the time domain, the four-parameter model for uniaxial deformation takes the form

$$(1 + bD^\alpha)\sigma(t) = (E_0 + E_1 D^\alpha)\epsilon(t) \tag{5}$$

where b, E_0, E_1, and α are the parameters. The order of differentiation α is taken to be real and rational $\alpha = q/n$. This model has been used to construct general three-dimensional constitutive equations for linear, homogeneous, isotropic viscoelastic materials.[3] When these general constitutive equations are employed, it can be shown that the form of the finite-element equations of motion is[4]

$$b\underline{M}D^{2+\alpha}w(t) + \underline{M}D^2 w(t) + \underline{k}_1 D^\alpha w(t)$$
$$+ \underline{k}_0 w(t) = bD^\alpha F(t) + F(t) \tag{6}$$

where $F(t)$ is a vector of applied forces. Note that the equations of motion are posed in terms of three real, square, symmetrical matrices. In general the viscostiffness matrix $\underline{k}_1$ will not be a linear combination of $\underline{M}$ and $\underline{k}_0$, and usually the equations of motion cannot be decoupled in their present form.

To overcome this obstacle to spectral analysis and begin the derivation of the state equations, we will pose the structural equations of motion in terms of two real, square symmetrical matrices, for which an orthonormal transformation exists. To begin this process, one takes advantage of the composition property of the fractional order derivative

$$D^\alpha\{D^\gamma[w(t)]\} = D^{\alpha+\gamma}[w(t)] \tag{7}$$

and poses the structural equations of motion as

$$[b\underline{M}(D^\beta)^m + \underline{M}(D^\beta)^r + \underline{k}_1(D^\beta)^q + \underline{k}_0]w(t)$$
$$= [1 + b(D^\beta)^q]F(t) \tag{8}$$

where m, r, and q are integers, and $(D^\beta)^m$ is the β-order derivative taken m times:

$$\beta m = 2 + \alpha$$

$$\beta r = 2$$

$$\beta q = \alpha$$

$$\beta = 1/n$$

The variable β is chosen to be the largest fraction of the form $1/n$, where n is an integer common to all of the rational orders of differentiation in the structural equations of motion. As we will see in Eq. (10), this form for β is necessary to ensure that velocities (and hence initial velocities needed for the initial value problem) appear in the fractional order state vector. The most general form of these equations of motion is

$$\sum_{p=0}^{m} \underline{c}_p (D^\beta)^p w(t) = [1 + b(D^\beta)^q]F(t) = f^*(t) \tag{9}$$

where the $\underline{c}_p$ are real and constant, although many may be zero, and $f^*(t)$ is the result of the viscoelastic stress operator acting on the applied forces, $F(t)$, as shown in Eq. (8).

Equation (9) describes the structures with N degrees of freedom producing N equations of order βm that can be alternatively posed as $m \cdot N$ equations of order β. In matrix

form the $m \cdot N$ equations of β order are

$$D^{\beta}\left[\begin{array}{c|ccc} \underline{0} & & & \\ \vdots & & H & \\ 0 & & & \\ \underline{0} & & & \\ \hline \underline{c}_m & \cdots & \underline{c}_3\underline{c}_2 & \underline{c}_1 \end{array}\right]\left\{\begin{array}{c} (D^{\beta})^{m-1}w(t) \\ \vdots \\ (D^{\beta})^2 w(t) \\ (D^{\beta})^1 w(t) \\ w(t) \end{array}\right\}$$

$$+\left[\begin{array}{ccc|c} & & & \underline{0} \\ & & & \vdots \\ & -H & & \underline{0} \\ & & & \underline{0} \\ \hline \underline{0} & \cdots & \underline{0}\ \underline{0} & \underline{c}_0 \end{array}\right]\left\{\begin{array}{c} (D^{\beta})^{m-1}w(t) \\ \vdots \\ (D^{\beta})^2 w(t) \\ (D^{\beta})^1 w(t) \\ w(t) \end{array}\right\}=\left\{\begin{array}{c} 0 \\ \vdots \\ 0 \\ 0 \\ f^*(t) \end{array}\right\} \tag{10}$$

where the lowest set of partitioned equations is seen to be Eq. (9). The matrix $[H]$ is chosen so that both square matrices of order $m \cdot N$ become symmetrical and the top $(m-1) \cdot N$ equations are satisfied identically. This is accomplished by constructing H so that all matrices $\underline{c}_p$ lying on any given diagonal running from lower left to upper right in the first matrix of Eq. (10) are equal. We will refer to this form of the equations of motion as the expanded equations of motion.

For example, if α is one-half in Eq. (6), then β is one-half, making $m = 5$ in Eq. (8), and the expanded equations of motion become

$$D^{\frac{1}{2}}\begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} & b\underline{M} \\ \underline{0} & \underline{0} & \underline{0} & b\underline{M} & \underline{M} \\ \underline{0} & \underline{0} & b\underline{M} & \underline{M} & \underline{0} \\ \underline{0} & b\underline{M} & \underline{M} & \underline{0} & \underline{0} \\ b\underline{M} & \underline{M} & \underline{0} & \underline{0} & \underline{k}_1 \end{bmatrix}\left\{\begin{array}{c} (D^{\frac{1}{2}})^4 w(t) \\ (D^{\frac{1}{2}})^3 w(t) \\ (D^{\frac{1}{2}})^2 w(t) \\ (D^{\frac{1}{2}})^1 w(t) \\ w(t) \end{array}\right\}$$

$$+\begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & -b\underline{M} & \underline{0} \\ \underline{0} & \underline{0} & -b\underline{M} & -\underline{M} & \underline{0} \\ \underline{0} & -b\underline{M} & -\underline{M} & \underline{0} & \underline{0} \\ -b\underline{M} & -\underline{M} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{k}_0 \end{bmatrix}\left\{\begin{array}{c} (D^{\frac{1}{2}})^4 w(t) \\ (D^{\frac{1}{2}})^3 w(t) \\ (D^{\frac{1}{2}})^2 w(t) \\ (D^{\frac{1}{2}})^1 w(t) \\ w(t) \end{array}\right\}$$

$$=\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ (1 + bD^{1/2})F(t) \end{array}\right\} \tag{11}$$

Both the general form, Eq. (10), and the example in Eq. (11) are now posed in terms of two real, square, symmetrical matrices for which an orthonormal transformation exists:

$$\left\{\begin{array}{c} (D^{\beta})^{m-1}w_{(t)} \\ \vdots \\ (D^{\beta})^2 w(t) \\ (D^{\beta})^1 w(t) \\ w(t) \end{array}\right\}=\begin{bmatrix} \\ \Phi \\ \\ \end{bmatrix}\left\{\begin{array}{c} y_{m \cdot N}(t) \\ \vdots \\ y_3(t) \\ y_2(t) \\ y_1(t) \end{array}\right\} \tag{12}$$

This transformation leads to a system of $m \cdot N$ uncoupled differential equations of order β, where $y(t)$ are the modal coordinates:

$$D^{\beta}\left[\diagdown I_{\diagdown}\right]\{y(t)\} + \left[\diagdown a^{\beta}_{\diagdown}\right]\{y(t)\} = [\phi]^T\{f^*(t)\} = \{f(t)\} \tag{13}$$

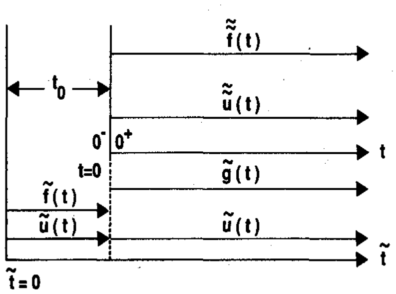

Fig. 1 Time scales for the loads and responses of the initial value problem.

Constructing the Modal State Equations

The decoupled structural equations of motion or basis equations [Eq. (13)] individually take the form

$$(D^{\beta} + a^{\beta})y(t) = f(t), \qquad \beta = 1/n \tag{14}$$

where the subscripts have been dropped to simplify notation. Green's function solutions for these equations are relatively straightforward, and the resulting expressions for the forced response of the structure can be shown to be real, continuous, and causal.[3] The solutions to Eq. (14) may be viewed as particular solutions of the structural equations of motion.

It is important to note that the only homogeneous solution to Eq. (14) is the trivial solution. This is consistent with a strict interpretation of Eq. (2), the generalized viscoelastic constitutive model. Inherent in the model is the implication that at time zero the viscoelastic material should be in its virgin, undeformed state and the structure is commencing motion from a quiescent state. An attempt to impose nontrivial initial conditions implies the existence of previous motion that is inconsistent with the hereditary viscoelastic model; hence, homogeneous solutions are not needed.

Our objective is to construct modal equations having nontrivial homogeneous solutions that serve as the basis of the initial value problem. Consider the case where the structure's motion described in Eq. (14) started from rest at time $\tilde{t} = 0$. At some later time, $\tilde{t} = t_0$, one starts a second clock measuring time t as shown in Fig. 1. Our intent is to describe the ensuing structural motion (for $t \geq 0$) in terms of its initial state $t = 0$, its displacement history prior to $t = 0$, and the loads applied at and after $t = 0$.

To begin, we pose Eq. (14) in terms of the $\tilde{t}$ time scale shown in Fig. 1:

$${}_0D^{\beta}_{\tilde{t}}\overset{*}{y}(\tilde{t}) + a^{\beta}\overset{*}{y}(\tilde{t}) = \overset{*}{f}(\tilde{t}), \quad \tilde{t} = t + t_0 \tag{15}$$

Note that the terminals of integration appear as subscripts on the fractional derivative convolution operator. The asterisk indicates a function of $\tilde{t}$. A function without an asterisk indicates the same function defined in terms of the t scale, also shown in Fig. 1:

$$\overset{*}{y}(\tilde{t}) = y(\tilde{t} - t_0) = y(t), \quad \tilde{t} - t_0 = t \tag{16}$$

Casting Eq. (15) in terms of the functions without the asterisks yields

$${}_0D^{\beta}{}_{t_0}y(\tilde{t} - t_0) + {}_{t_0}D^{\beta}_{\tilde{t}}y(\tilde{t} - t_0) + a^{\beta}y(\tilde{t} - t_0) = \tilde{\tilde{f}}(\tilde{t} - t_0) \tag{17}$$

where $\tilde{\tilde{f}}(t - t_0)$ is the applied force for $\tilde{t} > t_0$ (see Fig. 1). With τ taken as the dummy variable in these convolutions, the transformation of variable $u = \tau - t_0$ leads to

$${}_{-t_0}D^{\beta}_{0}y(t) + {}_0D^{\beta}_{t}y(t) + a^{\beta}y(t) = \tilde{\tilde{f}}(t), \qquad t > 0$$

or

$${}_0D^{\beta}_{t}y(t) + a^{\beta}y(t) = \tilde{\tilde{f}}(t) - \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-t_0}^{0}\frac{y(u)}{(t-u)^{\beta}}\,\mathrm{d}u \tag{18}$$

Using the transformation of variables $\eta = t - u$ produces

$$ {}_0D_y^\beta y(t) + a^\beta y(t) = \tilde{\tilde{f}}(t) - \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{t+t_0}\frac{y(t-\eta)}{\eta^\beta}\,\mathrm{d}\eta \quad (19) $$

The integral term is effectively a pseudoforcing function inducing the hereditary effects of motion between $t = -t_0$ and $t = 0$.

Note that the fractional derivative now operates over an interval beginning at $t = 0$. In essence, one is posing the problem in terms of a fractional derivative not dependent on motion prior to $t = 0$, while retaining the history effects of the pseudoforcing function. Using the generalized Leibnitz rule to differentiate the two convolution integrals in Eq. (18) and noting that $y(t)$ at $t = -t_0$ is zero yields

$$ {}_0I_t^{1-\beta}\dot{y}(t) + a^\beta y(t) = \tilde{\tilde{f}}(t) - \frac{1}{\Gamma(1-\beta)}\int_t^{t+t_0}\frac{\dot{y}(t-\eta)}{(\eta)^\beta}\,\mathrm{d}\eta \quad (20) $$

where

$$ {}_0I_t^\delta x(t) \equiv \frac{1}{\Gamma(\delta)}\int_0^t \frac{x(t-\tau)}{\tau^{1-\delta}}\,\mathrm{d}\tau \quad (21) $$

is the Riemann-Liouville δ-order integral.[17] Since the modal equation [Eq. (21)] contains the $(1-\beta)$-order integral of the first derivative of $y(t)$, something very similar to β-order differentiation is taking place in this operation. This modified form of differentiation is denoted by a "hat" over the operator:

$$ {}_0D_t^\beta x(t) = \frac{x(0)t^{-\beta}}{\Gamma(1-\beta)} + {}_0I_t^{1-\beta}\dot{x}(t) \equiv \frac{x(0)t^{-\beta}}{\Gamma(1-\beta)} + {}_0\hat{D}_t^\beta x(t) \quad (22) $$

The relationship between this modified operator, $\hat{D}^\beta$, and the Riemann-Liouville fractional order differentiation and integration is given earlier.[18] Posing Eq. (21) in terms of the modified operator produces the modal state equations:

$$ \hat{D}^\beta y(t) + a^\beta y(t) = \tilde{\tilde{f}}(t) + \tilde{g}(t) = g(t) \quad (23a) $$

$$ \tilde{g}(t) = -\frac{1}{\Gamma(1-\beta)}\int_t^{t+t_0}\frac{\dot{y}(t-\eta)}{(\eta)^\beta}\,\mathrm{d}\eta \quad (23b) $$

Note the similar appearance of Eqs. (14) and (23). Recall that Eq. (14) is based on the $\tilde{t}$ time scale and has a trivial homogeneous solution. On the other hand, Eq. (23) is based on the t time scale, possesses a nontrivial homogeneous solution, and accounts for the effects of previous motion through the initial value $y(0)$ and pseudoforcing function $\tilde{g}(t)$.

Constructing the Fractional Order State Equations

The overall goal is to determine the nature of the fractional order state equations from the modal state equations. The immediate goal is to use the modal state equations [Eq. (23)] to predict the structural response, where the relaxation effects induced by previous motion are accounted for by the pseudoforcing functions $\tilde{g}(t)$. The transient structural response will be a superposition of the homogeneous solutions of the modal state equations and will be shown to satisfy the initial conditions. The forced structural response will be constructed from the particular solutions to the modal state equations derived using Green's functions. Superimposing the transient and forced response produces the total structural response.

The transient structural response is constructed by first determining the general form of the homogeneous solution for the modal state equations [Eq. (23)]. These solutions have the form

$$ y_h(t) = y_h(0)\sum_{p=0}^{\infty}\frac{[-(at)^\beta]^p}{\Gamma(1+p\beta)} \quad (24) $$

which is a special case of the β-order Mittag-Leffler function defined as[13]

$$ E_\beta(x) \equiv \sum_{p=0}^{\infty}\frac{(x)^p}{\Gamma(1+p\beta)} \quad (25) $$

In Mittag-Leffler notation the homogeneous solution is

$$ y_h(t) = y_h(0)E_\beta[-(at)^\beta] \quad (26) $$

where this special Mittag-Leffler function has the property

$$ \hat{D}^\beta E_\beta[-(at)^\beta] = -a^\beta E_\beta[-(at)^\beta] \quad (27) $$

The property should come as no surprise because the Mittag-Leffler function has long been viewed as a generalized exponential function.[19] In related work[11,12] Koeller has shown that the quasistatic fractional calculus viscoelastic formulation leads to Mittag-Leffler functions.

Including the particular solution, we find that the total solution to each of the modified basis equations is

$$ y(t) = y_h(0)E_\beta[-(at)^\beta] + \int_0^t D^{1-\beta}\{E_\beta[-(a\tau)^\beta]\}g(t-\tau)\,\mathrm{d}\tau \quad (28) $$

which can be determined using Laplace transforms or other traditional solution techniques for integro-differential equations. The kernel in the convolution integral of Eq. (28) is the unit impulse solution (Green's function) for the modified basis equations and is singular. The singular nature of the kernel can be determined through a straightforward application of Eq. (1) to the β-order Mittag-Leffler function.

The singular nature of fractional order derivatives of $E_\beta[-(at)^\beta]$ is useful in resolving an apparent paradox in the overall initial value problem. Recall that there are $m \cdot N$ [Eqs. (23)] modal state equations needed to characterize the structure, where the solution for each equation has a homogeneous solution containing a different initial value. This paradox becomes apparent when Eq. (12) is used to solve for the $m \cdot N$ initial values of the homogeneous basis functions in terms of the structure's initial displacements $w_h(t)$ and their derivatives evaluated at time zero

$$ \left\{\begin{matrix} (D^\beta)^{m-1}w_h(t) \\ \vdots \\ (D^\beta)^2 w_h(t) \\ (D^\beta)^1 w_h(t) \\ w_h(t) \end{matrix}\right\}_{(t=0)} = \begin{bmatrix} \Phi \end{bmatrix} \left\{\begin{matrix} y_{m\cdot N}(t) \\ \vdots \\ y_3(t) \\ y_2(t) \\ y_1(t) \end{matrix}\right\}_{(t=0)} \quad (29) $$

The paradox is that at this point only $w_h(0)$ and $D^1 w_h(0)$ can be specified, whereas the remaining elements in the state vector on the left-hand side of Eq. (29) are undetermined. Note that the order of the differential equation of motion [Eq. (6)] is order $2+\alpha$, or equivalently, βm, and that the state vector in Eq. (31) calls for the initial values of derivatives up through $2+\alpha-\beta$, or equivalently, $\beta(m-1)$. In other words, when posing N, βm-order differential equations as a system of $m \cdot N$ differential equations of order β, the corresponding initial value problem calls for all of the initial values of the $p\beta$-order derivatives of the displacement vector, $w_h(t) : p = 0, 1, 2, \ldots, m-1$. These requirements appear to be analogous to the traditional initial value problem, but also leave one with the requirement for yet more initial conditions.

Reference 20 proves that all of the noninteger derivatives of $w_h(t)$ of order less than two appearing in the state vector have zero initial value. The initial values for acceleration and the accompanying higher-order derivations appearing in the state vector can be determined by returning to the original equation of motion [Eq. (6)] and using successive applications of

Eq. (22) to determine the singular terms in the equation of motion. The resulting equation of motion for the response to turning off the previous forcing function is

$$
\begin{aligned}
&-b\underline{M}\frac{\ddot{\tilde{w}}(0^-)t^{-\alpha}}{\Gamma(-\alpha)} - b\underline{M}\sum_{\ell=1}^{m-2n-1}\frac{t^{-\ell\beta}}{\Gamma(1-\ell\beta)}\hat{D}^{(m-2n-\ell)\beta}\ddot{\tilde{w}}(0^-) \\
&+(1+b\hat{D}^{\alpha})\underline{M}\ddot{\tilde{w}}(t) - \underline{k}_1\frac{\tilde{w}(0^-)t^{-\alpha}}{\Gamma(1-\alpha)} + (\underline{k}_0+\underline{k}_1\hat{D}^{\alpha})\tilde{w}(t) \\
&= -b\frac{\tilde{F}(0^-)t^{-\alpha}}{\Gamma(1-\alpha)} + \tilde{G}(t)
\end{aligned}
\tag{30}
$$

The fractional derivatives in this equation of motion are evaluated for $t = 0^-$, or equivalently, $\tilde{t} = t_0^-$ (see Fig. 1). The variables $\tilde{G}(t)$ are the pseudoforces needed to produce the residual motion associated with the previous loading history, already accounted for in the modified basis equations. The singular forcing function is the result of the α-order derivative of the step function turning off $\tilde{F}(t)$. The remaining singular behavior is the result of repeatedly applying Eq. (22) to separate out the singular behavior of the fractional derivatives of acceleration.

The corresponding equation of motion for the response to the new loads at $t = 0^+$ in Fig. 1 is

$$
\begin{aligned}
&b\underline{M}\frac{\ddot{\tilde{\tilde{w}}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + b\underline{M}\sum_{\ell=1}^{m-2n-1}\frac{t^{-\ell\beta}}{\Gamma(1-\ell\beta)}\hat{D}^{(m-2n-\ell)\beta}\ddot{\tilde{\tilde{w}}}(0^+) \\
&+(1+b\hat{D}^{\alpha})\underline{M}\ddot{\tilde{\tilde{w}}}(t) + \underline{k}_1\frac{\tilde{\tilde{w}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + (\underline{k}_0+\underline{k}_1\hat{D}^{\alpha})\tilde{\tilde{w}}(t) \\
&= \frac{b\tilde{\tilde{F}}(0^+)t^{-\alpha}}{\Gamma(1-\alpha)} + (1+b\hat{D}^{\alpha})\tilde{\tilde{F}}(t)
\end{aligned}
\tag{31}
$$

where the singular forcing function results from again using Eq. (22) to express the effects of the step function turning on $\tilde{\tilde{F}}(t)$. The remaining singular behavior is also the result of using Eq. (22) to separate out the singular behavior of the fractional derivatives of acceleration. Again the tilde and double tilde notation identify motion due to previous forces, $\tilde{F}(t)$, and present forces, $\tilde{\tilde{F}}(t)$, respectively, as in Eqs. (16) and (17).

Equating the coefficients of the strongest singularities (order α) in Eq. (30) and then in Eq. (31) yields two equations needed to establish the initial conditions of acceleration:

$$
-b\underline{M}\ddot{\tilde{w}}(0^-) - \underline{k}_1\tilde{w}(0^-) = -b\tilde{F}(0^-) \tag{32}
$$

$$
b\underline{M}\ddot{\tilde{\tilde{w}}}(0^+) + \underline{k}_1\tilde{\tilde{w}}(0^+) = b\tilde{\tilde{F}}(0^+) \tag{33}
$$

Adding Eqs. (32) and (33) produces the relationship needed to establish changes in the initial conditions due to stopping and starting of the load histories:

$$
\begin{aligned}
&\underline{M}[\ddot{\tilde{\tilde{w}}}(0^+) - \ddot{\tilde{w}}(0^-)] + b^{-1}\underline{k}_1[\tilde{\tilde{w}}(0^+) - \tilde{w}(0^-)] \\
&= \tilde{\tilde{F}}(0^+) - \tilde{F}(0^-)
\end{aligned}
\tag{34}
$$

Because this relationship is based on step loading, which is incapable of instantaneously changing the displacement or velocity time history between time 0^- and 0^+, one can conclude that

$$
\tilde{\tilde{w}}(0^+) = \tilde{w}(0^-) \tag{35}
$$

$$
\dot{\tilde{\tilde{w}}}(0^+) = \dot{\tilde{w}}(0^-) \tag{36}
$$

and Eq. (36) can now be re-expressed as

$$
\ddot{\tilde{\tilde{w}}}(0^+) - \ddot{\tilde{w}}(0^-) = \underline{M}^{-1}[\tilde{\tilde{F}}(0^+) - \tilde{F}(0^-)] \tag{37}
$$

Thus, we see that the change in the initial accelerations is proportional to any instantaneous changes (steps) in the magnitudes of the applied loads at $t = 0$. It is reassuring to note that Eq. (37) is strongly reminiscent of Newton's second law.

To determine the initial accelerations at time 0^+, one needs to determine the accelerations at time 0^- and then add to them the additional component of acceleration from the change in load histories. Should there be a continuous transition from one load history to the other, then

$$
\ddot{\tilde{\tilde{w}}}(0^+) = \ddot{\tilde{w}}(0^-) \tag{38}
$$

and the accelerations at time 0^- are the accelerations used in the initial value problem. Satisfying the initial conditions on acceleration in this manner effectively removes the α-order singular terms on both sides of Eqs. (30) and (31).

The remaining singular terms in these equations do not have corresponding terms on the respective force sides of the equations. To preserve the equality, one must conclude that the coefficients of these singular terms are zero. Note that setting these coefficients to zero in effect generates the remaining initial conditions needed in Eq. (29). From Eq. (30),

$$
\hat{D}^{(m-2n-\ell)\beta}\left[\ddot{\tilde{w}}(0^-)\right] = 0 \qquad \ell = 1,2,3,\ldots,m-2n-1 \tag{39}
$$

and from Eq. (31),

$$
\hat{D}^{(m-2n-\ell)\beta}\left[\ddot{\tilde{\tilde{w}}}(0^+)\right] = 0 \qquad \ell = 1,2,3,\ldots,m-2n-1 \tag{40}
$$

Proof is given in Ref. 20. One can see that the initial values of the fractional derivatives of displacement greater than second order and less than order βm must be zero to preserve the equation of motion.

Adding the two equations of motion and recalling that

$$
\tilde{\tilde{w}}(t) + \tilde{w}(t) = w(t), \qquad t \geq 0 \tag{41}
$$

yields

$$
\underline{M}(1+b\hat{D}^{\alpha})\ddot{w}(t) + (\underline{k}_0+\underline{k}_1\hat{D}^{\alpha})w(t) = (1+b\hat{D}^{\alpha})\tilde{\tilde{F}}(t) + \tilde{G}(t) \tag{42}
$$

which is identical to Eq. (8) except for one very important detail: the fractional derivative operator has changed from the original definition [Eq. (1)] to the modified definition [Eq. (22)]. Recall that the modified basis functions use this modified definition as well.

In fact, the entire initial value problem, consisting of Eqs. (42), (10), (23), and (29), and its solutions, Eq. (28), can be cast in terms of the modified definition of fractional differentiation. The composition property for the modified operator,

$$
\hat{D}^{\alpha}\{\hat{D}^{\gamma}[w(t)]\} = \hat{D}^{\alpha+\gamma}[w(t)] \tag{43}
$$

holds when the initial values of the fractional derivatives are zero, as stipulated in the initial value problem. One can now demonstrate in a straightforward manner that Eq. (42) leads to a corresponding form of Eq. (10) where the D^{β} operator is replaced by $\hat{D}^{\beta}$. The D^{β} operators in Eq. (29) can now be replaced by $\hat{D}^{\beta}$ as well. Noting that the particular solution in Eq. (28) is independent of the initial value and may be viewed as an excitation from a quiescent state, one can show that the solution of the modal state equation has the form

$$
\begin{aligned}
y_j(t) &= y_j(0)E_{\beta}[-(a_jt)^{\beta}] \\
&+ \int_0^t (-a_j^{\beta})^{m-1}\hat{D}^{-1-\alpha}\{E_{\beta}[-(a_jt)^{\beta}]\}g_j(t-\tau)\,\mathrm{d}\tau
\end{aligned}
\tag{44}
$$

Proof is given in Ref. 20. Note that the kernel is now nonsingular. One can now conclude that Eq. (44) is the

solution of a well-posed problem. The uniqueness of the solution follows immediately from Laplace transforms. Multiplying the initial value and the modal forcing function $g_j(t)$ by $(1+\epsilon)$ and taking ϵ small demonstrates continuous dependence on the data, as long as the convolution integral is bounded.

To test the robustness of the modal state equations, one needs to ascertain their ability to generate the structural response to impulsive loading. This method entails solving the initial value problem for a step response [using initial accelerations, Eq. (37)] from a quiescent state and noting that the impulse response is the first derivative of the step response. The structural response for a unit impulse load at the zth degree of freedom of the structure is

$$
\begin{aligned}
w_{\delta z}(t) = {} & b \sum_{j=1}^{m \cdot N} \boldsymbol{\phi}_{1j}[1+b(-a_j)^q] \\
& \times (-a_j^{\beta})^{2n+q-1} \hat{D}^{-1-\alpha}\{E_\beta[-(a_j t)^\beta]\} \boldsymbol{\phi}_{1j}^T \boldsymbol{z} \\
& + b \sum_{j=1}^{m \cdot N} \boldsymbol{\phi}_{1j}(-a_j)^{2n+q-1} \boldsymbol{\phi}_{1j}^T \boldsymbol{z} \cdot t
\end{aligned} \tag{45}
$$

where $\boldsymbol{z}$ is a N-order column vector of zeros except the zth element, which is 1. Here $\boldsymbol{\phi}_{1j}$ are the structure's mode shapes, which constitute the lowest N terms of the jth eigenvector of the expanded equations of motion [Eq. (10)]. Again the solution is seen to be continuous and is expressed in terms of the modified operator and the Mittag-Leffler function. Derivation of this expression is given elsewhere.[20]

At this point one might be tempted to assert that the original definition of fractional order differentiation [Eq. (1)] is somehow inappropriate for the initial value problem. This is not true. Recall that the initial value problem has insufficient numbers of physically motivated initial values to determine uniquely the overall homogeneous solution as a superposition of solutions to the modified basis equations. The additional auxiliary initial conditions, developed by suppressing singular behavior at time zero, provided precisely the number of needed initial conditions for a unique solution. Recall that the original definition [Eq. (1)] produced this singular behavior without which the initial value problem would flounder for lack of initial information.[20]

Moreover, having derived the structural equations of motion in terms of the modified fractional derivative operator [Eq. (42)] and having established the robustness of the formulation through the existence of its impulse response, one can now proceed to construct the structure's fractional order state equations. Casting Eq. (42) in terms of two real, square, and symmetrical matrices, as shown in Eq. (10), produces the fractional order state equations:

$$
D^\beta \left[\begin{array}{c|ccc}
\underline{0} & & & \\
\vdots & & \boldsymbol{H} & \\
\underline{0} & & & \\
\underline{0} & & & \\
\hline
\underline{c}_m & \cdots & & \underline{c}_3 \underline{c}_2 \underline{c}_1
\end{array}\right]
\left\{\begin{array}{c}
(\hat{D}^\beta)^{m-1} w(t) \\
\vdots \\
(\hat{D}^\beta)^2 w(t) \\
(\hat{D}^\beta)^1 w(t) \\
w(t)
\end{array}\right\}
+ \left[\begin{array}{ccc|c}
& & & \underline{0} \\
& & & \vdots \\
& -\boldsymbol{H} & & \underline{0} \\
& & & \underline{0} \\
\hline
\underline{0} & \cdots & \underline{0} \ \ \underline{0} & \underline{c}_0
\end{array}\right]
\left\{\begin{array}{c}
(\hat{D}^\beta)^{m-1} w(t) \\
\vdots \\
(\hat{D}^\beta)^2 w(t) \\
(\hat{D}^\beta)^1 w(t) \\
w(t)
\end{array}\right\}
= \left\{\begin{array}{c}
\mathbf{0} \\
\vdots \\
\mathbf{0} \\
\mathbf{0} \\
\tilde{\tilde{f}}^*(t) + \hat{G}(t)
\end{array}\right\} \tag{46}
$$

With straightforward matrix manipulation, these state equations take the form

$$
\hat{D}^\beta \boldsymbol{x}(t) = \underline{A}\boldsymbol{x}(t) + \underline{B}\boldsymbol{u}(t) + \underline{B}\tilde{\boldsymbol{G}}(t)
$$

Note that applying the orthogonal transformation given in Eq. (12) to the state equations yields the modal state equations [Eqs. (23)]. In effect, one has come full circle and derived equations of motion capable of describing the hereditary viscoelastic damping effects as well as characterizing the system in terms of its initial states.

It is reassuring that the fractional order state equations predict a response that is, strictly speaking, a function of all of its previous states, as it should be for a system that exhibits hereditary behavior. The pseudoforces $\tilde{\boldsymbol{G}}(t)$ describe the effects of previous internal viscoelastic deformation, and the initial states (taken at some time t_0 after the onset of motion) describe the effects of immediately previous motion. To predict an accurate short-term system response, records of previous motion must be kept to construct the pseudoforces. This is necessary for heavily damped structures. In lightly damped structures the hereditary effects are much smaller, and the pseudoforces may be ignored.

With or without the pseudoforces included in the fractional order state equations one can predict open- or closed-loop system response. The closed-loop feedback relationship between the state vector and the applied forces is

$$
\tilde{\tilde{\boldsymbol{F}}}(t) = -\overset{*}{\underline{G}}\boldsymbol{x}_r \tag{47a}
$$

$$
\tilde{\tilde{f}}^*(t) = -(1+b\hat{D}^\alpha)\overset{*}{\underline{G}}\boldsymbol{x}_r = -\underline{G}\boldsymbol{x} \tag{47b}
$$

where $\boldsymbol{x}_r$ is the reduced state vector containing the displacements $\boldsymbol{w}(t)$ and all derivatives (including fractional order) of $\boldsymbol{w}(t)$ up to, but not including, the second derivative. When the stress operator takes the α-order derivative of $\boldsymbol{x}_r$, this generates the higher-order derivatives of $\boldsymbol{w}(t)$ in the full state vector. Here the $-\overset{*}{\underline{G}}$ is the matrix of actual gain coefficients, $-\underline{G}$ is the matrix of effective gain coefficients, and $\boldsymbol{x}$ is the full state vector appearing in Eq. (46).

Note that when b is zero in the stress operator the reduced state vector is the full state vector, and no distinction is necessary between actual gains and effective gains. Substituting Eq. (47) into Eq. (46) produces the equations for the closed-loop response:

$$
\hat{D}^\beta \boldsymbol{x} = (\underline{A} - \underline{B}\underline{G})\boldsymbol{x} \tag{48}
$$

This equation includes the feedback of fractional order derivatives of the structure's response. Recall that the fractional derivatives actually being fed back are those of order less than two, namely, those in the reduced state vector $\boldsymbol{x}_r$. The fact that the full state vector appears in Eq. (48) is a consequence of the mathematics in Eq. (47). However, Eq. (48) is in fact the closed-loop state equations. There is no a priori reason to exclude the fractional derivatives from feedback.

In fact, Oldham and Zoski[21] have developed RLC circuits that generate the fractional order derivatives and integrals of input signals over limited frequency ranges. It is possible to take signals proportional to structural deflections and accelerations and produce signals proportional to their fractional derivatives and feed them back.

Fractional Order Matrix Exponential Function

Although the modal equations are an effective tool in deriving the fractional order state equations, solution formats for these state equations are not limited to modal analysis. When modal analysis is unwarranted, the fractional order analog of the matrix exponential function can serve as an alternate solution format.

The development begins with the open-loop state equations without the pseudoforce:

$$\hat{D}^{\beta}x = \underset{\sim}{A}x \tag{49}$$

One can use the following approach to determine the closed-loop response by substituting $\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{G}$ into Eq. (48) for $\underset{\sim}{A}$ here and replacing the orthogonal transformation that follows with a similarity transformation for the asymmetric matrix $\underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{G}$. For simplicity of notation the open-loop case is considered.

One assumes a time series solution of the form

$$x(t) = x_0 + x_1 t^{\beta} + x_2 t^{2\beta} + \cdots + x_p t^{p\beta} + \cdots \tag{50}$$

Substituting this solution into Eq. (49), evaluating the fractional derivatives term by term using the modified operator defined in Eq. (22), and equating terms of like power in time yields the following solution:

$$x(t) = \left[\underset{\sim}{I} + \frac{\underset{\sim}{A}t^{\beta}}{\Gamma(1+\beta)} + \frac{\underset{\sim}{A}^2 t^{2\beta}}{\Gamma(1+2\beta)} + \cdots + \frac{\underset{\sim}{A}^p t^{p\beta}}{\Gamma(1+p\beta)} + \cdots\right]x_0 \tag{51}$$

or

$$x(t) = E_{\beta}(\underset{\sim}{A}t^{\beta})x_0 \tag{52}$$

where $E_{\beta}(\underset{\sim}{A}t^{\beta})$ is the fractional order matrix exponential function. It may be viewed as the generalized matrix form of the scalar Mittag-Leffler function given in Eq. (24). Similar to its scalar counterpart, the fractional order matrix exponential function has the property

$$\hat{D}^{\beta}[E_{\beta}(\underset{\sim}{A}t^{\beta})] = \underset{\sim}{A}E_{\beta}(\underset{\sim}{A}t^{\beta}) \tag{53}$$

One can relate this form of the solution back to the modal solutions [Eq. (22)] by using the orthogonal transformation given in Eq. (12):

$$x = \underset{\sim}{\phi}y \tag{54}$$

to decouple the homogeneous form of Eq. (46). The result is the homogeneous modal state equations,

$$\hat{D}^{\beta}y = -\underset{\sim}{a}^{\beta}y \tag{55}$$

where $-\underset{\sim}{a}^{\beta}$ is a diagonal matrix containing the system's eigenvalues. Solutions of this equation take the form

$$y_j(t) = E_{\beta}[-(a_j t)^{\beta}]y_j(0) \tag{56}$$

which are identical to those in Eq. (26). However, using the orthogonal transformation to construct the structure's response from Eq. (56) produces

$$x(t) = \underset{\sim}{\phi}E_{\beta}(-\underset{\sim}{a}^{\beta}t^{\beta})\underset{\sim}{\phi}^{T}x_0 \tag{57}$$

This result is equivalent to that shown in Eq. (52).

Example Problems

To demonstrate the solution techniques developed for the fractional order state equations, one will first apply them to two simple cases. The first case is a homogeneous first-order differential equation with constant coefficients. The second example is a second-order differential equation for a single-degree-of-freedom viscoelastically damped oscillator.

If one is to view the fractional order state formulation as a generalization of the initial value problem, its solution techniques should apply to ordinary equations with constant coefficients. The first-order differential equation is

$$\hat{D}^{1}w + a^2 w = 0, \qquad w(0) = w_0$$

which, using the composition property, can also be expressed as

$$\hat{D}^{2/2}w + a^2 w = 0$$

Posed in fractional order state form this equation becomes

$$\hat{D}^{1/2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} \hat{D}^{1/2}w \\ w \end{Bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & a^2 \end{bmatrix}\begin{Bmatrix} \hat{D}^{1/2}w \\ w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The associated eigenvalue problem is

$$\lambda\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} \phi \end{Bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & a^2 \end{bmatrix}\begin{Bmatrix} \phi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which has eigenvalues

$$\lambda = \pm ia$$

and associated eigenvectors of the form

$$\begin{Bmatrix} \phi \end{Bmatrix} = \begin{Bmatrix} \pm ia \\ 1 \end{Bmatrix}$$

The solution takes the form

$$\begin{Bmatrix} \hat{D}^{1/2}w(t) \\ w(t) \end{Bmatrix} = \begin{bmatrix} ia & -ia \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} y_1(0)E_{1/2}[-(iat^{1/2})] \\ y_2(0)E_{1/2}[-(-iat^{1/2})] \end{Bmatrix}$$

To determine the initial values $y_1(0)$ and $y_2(0)$, one evaluates this expression at $t = 0$:

$$\begin{Bmatrix} 0 \\ w_0 \end{Bmatrix} = \begin{bmatrix} ia & -ia \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} y_1(0) \\ y_2(0) \end{Bmatrix}$$

and solves for $y_1(0)$ and $y_2(0)$:

$$\begin{Bmatrix} y_1(0) \\ y_2(0) \end{Bmatrix} = \frac{1}{2ia}\begin{bmatrix} 1 & ia \\ -1 & ia \end{bmatrix}\begin{Bmatrix} 0 \\ w_0 \end{Bmatrix} = \begin{Bmatrix} w_0/2 \\ w_0/2 \end{Bmatrix}$$

Substituting these values into the solution for $w(t)$ given earlier yields

$$w(t) = \frac{w_0}{2}E_{1/2}[-(iat^{1/2})] + \frac{w_0}{2}E_{1/2}[-(-iat^{1/2})]$$

Using the series representation of the Mittag-Leffler function given in Eq. (25) and summing the two series, the terms having fractional order powers of time add out, and one is left with

$$w(t) = w_0 \sum_{p=0}^{\infty}\frac{(-a^2 t)^p}{\Gamma(1+p)}$$

or

$$w(t) = w_0\, e^{-a^2 t}$$

as expected.

In the second example the fractional order time behavior does not add out, but instead describes the decaying motion of a damped oscillator:

$$\hat{D}^{2}w(t) + 2\hat{D}^{1/2}w(t) + w(t) = 0$$

For simplicity, the coefficient of the half-order derivative in the stress operator is taken to be zero. The remaining half-order derivative describes the low-frequency viscoelastic damping, and the mass and stiffness coefficients are taken to be 1. Again using the composition property the equation may be posed as

$$\hat{D}^{4/2}w(t) + 2\hat{D}^{1/2}w(t) + w(t) = 0$$

Table 1 Eigenvalues and eigenvectors of the fractional order state equation for the damped oscillator

$\lambda_1 = -0.5437$	$\lambda_2 = -1.0$	$\lambda_3 = 0.7718 \pm 1.1151i$	$\lambda_4 = 0.7718 - 1.1151i$
$\phi_1 = \begin{Bmatrix} \lambda_1^3 \\ \lambda_1^2 \\ \lambda_1 \\ 1 \end{Bmatrix}$	$\phi_2 = \begin{Bmatrix} \lambda_2^3 \\ \lambda_2^2 \\ \lambda_2 \\ 1 \end{Bmatrix}$	$\phi_3 = \begin{Bmatrix} \lambda_3^3 \\ \lambda_3^2 \\ \lambda_3 \\ 1 \end{Bmatrix}$	$\phi_4 = \begin{Bmatrix} \lambda_4^3 \\ \lambda_4^2 \\ \lambda_4 \\ 1 \end{Bmatrix}$

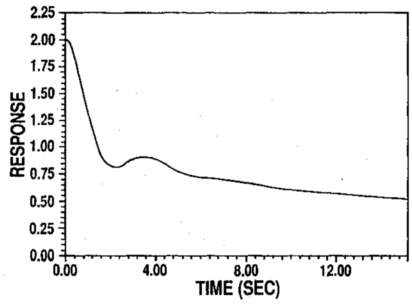

Fig. 2 Response of the damped oscillator.

In expanded form the equations become

$$\hat{D}^{1/2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\begin{Bmatrix} \hat{D}^{3/2}w(t) \\ \hat{D}^{2/2}w(t) \\ \hat{D}^{1/2}w(t) \\ w(t) \end{Bmatrix} + \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \hat{D}^{3/2}w(t) \\ \hat{D}^{2/2}w(t) \\ \hat{D}^{1/2}w(t) \\ w(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The eigenvalues and eigenvectors for this system appear in Table 1. Applying the initial conditions,

$$x(0) = 2.0, \quad \hat{D}^{1/2}w(0) = 0, \quad \hat{D}^{2/2}w(0) = 0, \quad \hat{D}^{3/2}w(0) = 0$$

and solving for the coefficients of the Mittag-Leffler functions as was performed earlier yields the response of the heavily damped oscillator. A plot of the response is given in Fig. 2.

Conclusions

The fractional derivative model of viscoelastic damping appears to be a useful tool in constructing state equations that describe the motion of damped structures. Aside from its accuracy, the essential value of this viscoelastic model lies in its use of generalized derivative operators. Because these state equations contain fractional order time derivatives of structural motion in the state vector, this formulation suggests the feedback of rational order time derivatives of structural response.

Moreover, this formulation appears to be a strong candidate for the general description of linear systems exhibiting strong hereditary behavior with weak frequency dependence. The advantages for the controls engineer are numerous. First, one avoids the use of time-dependent coefficients in the state equations. Second, the fractional derivative models are compact, making least-squares fits to damping data tractable and manipulation of the model practical. The resulting state equations have analytical solutions, and the solution techniques are similar to classical approaches. Finally, the inclusion of the fractional order derivatives in the state vector provides additional forms of feedback to improve system performance. Given that a fractional derivative model accurately captures the hereditary effects, the fractional order state equations appear to be a useful tool in the design and analysis of a feedback control system.

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