Multistage Design of an Optimal Momentum Management Controller for the Space Station

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This paper presents a multistage design scheme for determining an optimal control moment gyro momentum management and attitude control system for the Space Station Freedom. First, the space station equations of motion are linearized and then block decomposed into two block decoupled subsystems using the matrix sign algorithm. Next, a sequential design procedure is utilized for designing a linear quadratic regulator for each subsystem, which optimally places the eigenvalues of the closed-loop subsystem in the region of an open sector, bounded by lines inclined at $\pm \pi/2k$ (for $k = 2$ or 3) from the negative real axis and the left hand side of a line parallel to the imaginary axis in the s plane. Simulation results are presented to compare the resultant designs.

Introduction

REALIZATIONS of aerospace systems often result in large-scale multivariable models that consist of many interacting subsystems with their own characteristics. This is certainly the case with the model of the space station's dynamics. The analysis and design of such a large-scale multivariable system itself is not an easy task because of complicated structures and computational difficulties. Therefore, a necessity arises for decomposing the original large-scale system having composited characteristics into decoupled subsystems, each with their own distinct characteristics, so that the resulting model has a completely decoupled multitime scale structure, thus facilitating the application of the well-known analysis and design methods in control theory to the subsystems of smaller order. This, in turn, would alleviate, to a certain extent, the computational and storage problems involved in the momentum management and attitude control design of large-scale space station models.

There are several ways to decompose the space station model into decoupled subsystems. These include aggregation,[1] singular perturbation,[2] and model analysis.[3] As an example of model analysis, small products of inertia in the space station equations of motion permit pitch motion to be uncoupled from roll/yaw motion. As a matter of fact, this was the approach followed by Wie et al. in Ref. 4 and the authors in Ref. 5. In this paper, however, we show how to decompose the system into block-decoupled subsystems using a technique based on the matrix sign function.

The Space Station Freedom will use control moment gyros (CMGs) to maintain attitude control during unpowered flight. One approach to CMG momentum management is the continuous approach, which integrates the momentum management and attitude control system design. Using the continuous approach, we develop a sequential design procedure that will optimally place the poles of the closed-loop subsystem within the common region of an open sector, bounded by the lines inclined at $\pm\pi/2k$ (for $k = 2$ or 3) from the negative real axis and the left side of a line parallel to the imaginary axis in the complex s plane.

Equations of Motion

The space station is expected to maintain an attitude within $\pm$ 5.0 deg of local vertical and local horizontal (LVLH) during normal unpowered flight phases. The nonlinear equations of motion in terms of components along the body-fixed control axes can be written as follows[4]:

space station dynamics ($I_{ij} = I_{ji}$ for $i \neq j$)

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = - \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + 3n^2 \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} -u_1 + w_1 \\ -u_2 + w_2 \\ -u_3 + w_3 \end{bmatrix} \tag{1}$$

where

$$c_1 \triangleq -\sin\theta_2 \cos\theta_3 \tag{2a}$$

$$c_2 \triangleq \cos\theta_1 \sin\theta_2 \sin\theta_3 + \sin\theta_1 \cos\theta_2 \tag{2b}$$

$$c_3 \triangleq -\sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2 \tag{2c}$$

Attitude kinematics (2-3-1 body-axis sequence)

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_3} \begin{bmatrix} \cos\theta_3 & -\cos\theta_1 \sin\theta_3 & \sin\theta_1 \sin\theta_3 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 \cos\theta_3 & \cos\theta_1 \cos\theta_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ n \\ 0 \end{bmatrix} \tag{3}$$

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CMG momentum

$$\begin{bmatrix} \dot h_1 \\ \dot h_2 \\ \dot h_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4}$$

where (1,2,3) are the roll, pitch, and yaw control axes whose origin is fixed at the mass center, with roll axis in the flight direction, the pitch perpendicular to the orbit plane, and the yaw toward the Earth; $(\theta_1, \theta_2, \theta_3)$ the roll, pitch, and yaw Euler angles of the central (body) axes with respect to LVLH axes that rotate with the orbital angular velocity n; $(\omega_1, \omega_2, \omega_3)$ the body-axis components of the absolute angular velocity of the station; (I_{11}, I_{22}, I_{33}) the moments of inertia; $I_{ij} = I_{ji} (i \neq j)$ the products of inertia; (h_1, h_2, h_3) the body-axis components of the CMG momentum; (u_1, u_2, u_3) the body-axis components of the control torque caused by the CMG momentum change; (w_1, w_2, w_3) the body-axis components of the external disturbance torque; and n the orbital rate of 0.0011 rad/s.

For the small attitude deviations from LVLH orientation with an assumption that $n \gg |\Delta\omega_i|$, $i = 1,2,3$ where $\Delta\omega = [\omega_1, \omega_2 + n, \omega_3]^T$, the linearized equations of motion can be written as follows:

space station dynamics ($I_{ij} = I_{ji}$ for $i \neq j$)

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \end{bmatrix}$$

$$= n \begin{bmatrix} I_{31} & 2I_{32} & I_{33}-I_{22} \\ -I_{32} & 0 & I_{12} \\ I_{22}-I_{11} & -2I_{12} & -I_{13} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$+ 3n^2 \begin{bmatrix} I_{33}-I_{22} & I_{21} & 0 \\ I_{12} & I_{33}-I_{11} & 0 \\ -I_{13} & -I_{23} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$+ n^2 \begin{bmatrix} -2I_{23} \\ 3I_{13} \\ -I_{12} \end{bmatrix} + \begin{bmatrix} -u_1 + w_1 \\ -u_2 + w_2 \\ -u_3 + w_3 \end{bmatrix} \tag{5}$$

Attitude kinematics

$$\dot\theta_1 - n\theta_3 = \omega_1 \tag{6a}$$

$$\dot\theta_2 - n = \omega_2 \tag{6b}$$

$$\dot\theta_3 + n\theta_1 = \omega_3 \tag{6c}$$

CMG momentum:

$$\dot h_1 - nh_3 = u_1 \tag{7a}$$

$$\dot h_2 = u_2 \tag{7b}$$

$$\dot h_3 + nh_1 = u_3 \tag{7c}$$

Equations (5–7) can be put together and written in the following state-space form:

$$\begin{bmatrix} \dot\omega \\ \dot\theta \\ \dot h \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{13} \end{bmatrix} \begin{bmatrix} \omega \\ \theta \\ h \end{bmatrix} + \begin{bmatrix} -B \\ 0 \\ Id_3 \end{bmatrix} u$$

$$+ \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} w + \begin{bmatrix} d_3 \\ d_2 \\ 0 \end{bmatrix} \tag{8}$$

where $\boldsymbol{\omega}$, $\boldsymbol{\theta}$, $\boldsymbol{h}$, $\boldsymbol{u}$, and $\boldsymbol{w}$ are column vectors consisting of ω_i, θ_i, h_i, u_i, and w_i, $i = 1,2,3$, respectively. Also,

$$A_{11} = n\hat{I}^{-1} \begin{bmatrix} I_{31} & 2I_{32} & I_{33}-I_{22} \\ -I_{32} & 0 & I_{12} \\ I_{22}-I_{11} & -2I_{12} & -I_{13} \end{bmatrix} \tag{9a}$$

$$A_{12} = 3n^2\hat{I}^{-1} \begin{bmatrix} I_{33}-I_{22} & I_{21} & 0 \\ I_{12} & I_{33}-I_{11} & 0 \\ -I_{13} & -I_{23} & 0 \end{bmatrix} \tag{9b}$$

$$A_{21} = Id_3, \qquad A_{22} = A_{13} = \begin{bmatrix} 0 & 0 & n \\ 0 & 0 & 0 \\ -n & 0 & 0 \end{bmatrix} \tag{9c}$$

$$d_3 = n^2\hat{I}^{-1}[-2I_{23},\ 3I_{13},\ -I_{12}]^T$$

$$d_2 = [0,\ n,\ 0]^T, \qquad B = \hat{I}^{-1} \tag{9d}$$

where $\hat{I}$ represents the inertia matrix with elements I_{ij} and Id_3 is an identity matrix of dimension 3. The external disturbances (aerodynamic disturbances) w_i are modeled as bias plus cyclic terms in the body-fixed control axes:

$$w_i(t) = \text{bias} + A_n \sin(nt + \phi_n) + A_{2n} \sin(2nt + \phi_{2n}) \tag{10}$$

The cyclic component at orbital rate is due to the diurnal bulge effect, whereas the cyclic torque at twice the orbital rate is caused by the rotating solar panels.

To avoid CMG momentum buildup, we also introduce an integral of the CMG momentum vector $\boldsymbol{h}$, i.e.,

$$\hat{h} = \int h\,\mathrm{d}t \qquad \text{or} \qquad \dot{\hat{h}} = h \tag{11}$$

Combining Eqs. (8) and (11), we obtain the state-space representation to be used for the control design:

$$\begin{bmatrix} \dot\omega \\ \dot\theta \\ \dot h \\ \dot{\hat h} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & A_{13} & 0 \\ 0 & 0 & Id_3 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ \theta \\ h \\ \hat h \end{bmatrix} + \begin{bmatrix} -B \\ 0 \\ Id_3 \\ 0 \end{bmatrix} u$$

$$+ \begin{bmatrix} B \\ 0 \\ 0 \\ 0 \end{bmatrix} w \tag{12}$$

In the following section, we will introduce optimal regional pole assignment for continuous time systems in a general state-space representation.

Continuous Time Optimal Quadratic Regulators with Pole Placement

Consider the linear controllable and observable continuous time system [similar to Eq. (12) without considering w] described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = \text{an initial vector} \tag{13}$$

where $x(t)$ and $u(t)$ are an $n \times 1$ state vector and an $m \times 1$ state vector, respectively, A and B are constant matrices

of appropriate dimensions. Let the quadratic cost function for the system in Eq. (13) be

$$J = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]\,dt \tag{14}$$

where the weighting matrices Q and R are $n \times n$ nonnegative definite and $m \times m$ positive definite symmetric matrices, respectively. The feedback control law that minimizes the performance index in Eq. (14) is given by Ref. 6

$$u(t) = -Kx(t) = -R^{-1}B^TPx(t) \tag{15}$$

where K is the feedback gain and P, an $n \times n$ non-negative definite symmetric matrix, is the solution of the Riccati equation,

$$PBR^{-1}B^TP - PA - A^TP - Q = 0_n \tag{16}$$

with (Q, A) detectable. The superscript T and the matrix 0_n denote the transpose and the $n \times n$ null matrix, respectively. Thus, the resulting closed-loop system becomes

$$\dot{x}(t) = (A - BK)x(t) \tag{17}$$

The eigenvalues of $A - BK$, denoted by $\lambda(A - BK)$, lie in the open left half-plane of the complex s plane. Our objective is to determine Q, R, and K so that the closed-loop system in Eq. (17) has its eigenvalues on or within the hatched region of Fig. 1.

Design of a Continuous Time Linear Quadratic Regulator (LQR)

The important results along with the design procedure based on the matrix Riccati equation are presented in the following.

Lemma 1[6,7]: Let (A,B) be the pair of the given open-loop system in Eq. (13). Also, let $\alpha \geq 0$ represent the prescribed degree of relative stability. Then, the eigenvalues of the closed-loop system $A - BR^{-1}B^TP$ lie to the left of the $-\alpha$ vertical line with the matrix P being the solution of the Riccati equation,

$$PBR^{-1}B^TP - P(A + \alpha Id_n) - (A + \alpha Id_n)^TP = 0_n \tag{18}$$

where the matrix Id_n is an $n \times n$ identity matrix. ■

Lemma 2[7,8]: Let $\{\lambda_i^-\}_{i=1}^{n^-}$ be the imaginary and pure left half-plane eigenvalues of A and $\{\lambda_i^+\}_{i=1}^{n^+}$ with $n = n^- + n^+$ as the pure right half-plane eigenvalues of A. Also, let ξ_i be the corresponding respective eigenvectors of λ_i^-. Consider the Riccati equation

$$PBR^{-1}B^TP - PA - A^TP - Q = 0_n \tag{19}$$

If the non-negative definite weighting matrix Q satisfies $Q\xi_i = 0_{n\times 1}$, for $i = 1,2, \ldots, n^-$, i.e., null Q = span $[\xi_1, \ldots, \xi_{n^-}]$, then the closed-loop system $A - BR^{-1}B^TP$ has the invariant eigenvalues λ_i^- with associated eigenvectors ξ_i and n^+ pure left half-plane eigenvalues. ■

Lemma 3[8]: Given matrix $A \in R^{n\times n}$ and define two matrices $A^- \in R^{n\times n}$ and $A^+ \in R^{n\times n}$ associated with A, and

$$\lambda(A^-) = \{\lambda_i^- \ (i = 1,2, \ldots, n^-) \text{ and } n^+ \text{ null eigenvalues}\}$$

$$\lambda(A^+) = \{\lambda_i^+ \ (i = 1,2, \ldots, n^+) \text{ and } n^- \text{ null eigenvalues}\}$$

where n^+, n^-, λ_i^+, and λ_i^- are defined in Lemma 2. If P is the solution of the Riccati equation in Eq. (19) with $Q = 0_n$, then

$$2\,\mathrm{tr}(A^+) = \mathrm{tr}(BR^{-1}B^TP)$$

where tr(·) represents the trace of (·). ■

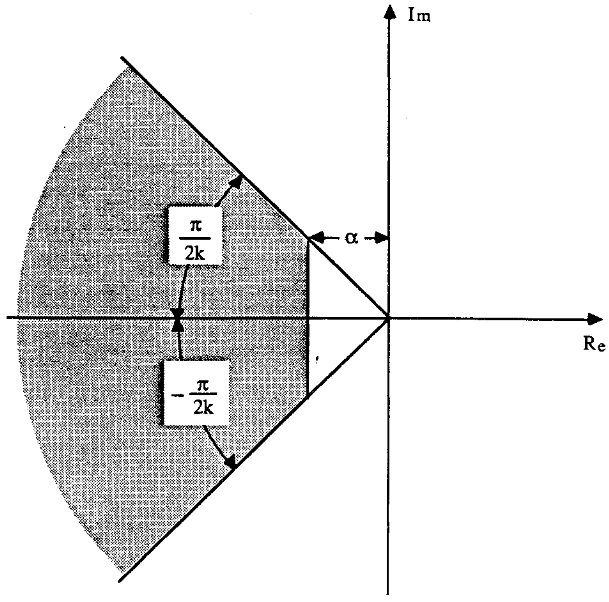


Fig. 1 Specified region for pole placement.

Lemma 4[7,9]: Consider the open sector whose boundaries are inclined at $\pm\pi/2k$ for $k = 2$ or 3, from the negative real axis in the s plane. The conformal mapping $\hat{A}[\triangleq (-1)^{k+1}A^k$, where A is asymptotically stable] maps the eigenvalues of A in the previously defined sector onto the left half-plane eigenvalues of $\hat{A}$ and the others onto the right half-plane eigenvalues of $\hat{A}$. ■

Theorem 1: Let (A, B) be the pair of the open-loop system given in Eq. (13), and the given asymptotically stable system matrix $A \in R^{n\times n}$ has eigenvalues $\{\lambda_i^-\}_{i=1}^{n^-}$ lying in the open sector of Fig. 1 with $\theta = \pi/2k$ for $k = 2$ or 3 and the eigenvalues $\{\lambda_i^+\}_{i=1}^{n^+}$ outside the open sector with $n = n^+ + n^-$. Consider two Riccati equations

$$QBR^{-1}B^TQ - Q[(-1)^{k+1}A^k] - [(-1)^{k+1}A^k]^TQ = 0_n \tag{20a}$$

$$PBR^{-1}B^TP - PA - A^TP - Q = 0_n \tag{20b}$$

where $R > 0$, the matrix $Q \geq 0$ in Eq. (20b) satisfying Eq. (20a), and the matrix P satisfying Eq. (20b). Let the state-feedback control law be

$$u(t) = -\gamma R^{-1}B^TPx(t) \tag{20c}$$

which minimizes the quadratic performance index

$$J = \int_0^\infty \{x(t)^T[\gamma(Q + (\gamma - 1)PBR^{-1}B^TP)]x(t) + u(t)^TRu(t)\}\,dt \tag{20d}$$

where γ is greater than or equal to the smallest positive real number satisfying

$$\gamma^k a_k + \gamma^{k-1}a_{k-1} + \cdots + a_0 \leq 0 \quad \text{for } k = 2 \text{ or } 3 \tag{20e}$$

with

$$a_k = \mathrm{tr}[(BR^{-1}B^TP)^k](-1)^{2k+1}$$

$$a_j = \mathrm{tr}[(BR^{-1}B^TP)^jA^{k-j}](-1)^{k+j+1}\binom{k}{j}$$

$$a_0 = \tfrac{1}{2}\mathrm{tr}[BR^{-1}B^TQ]$$

The optimal closed-loop system becomes

$$\dot{x}(t) = (A - \gamma BR^{-1}B^TP)x(t) \triangleq A_cx(t) \tag{20f}$$

and the optimal closed-loop system matrix A_c encloses the invariant eigenvalues $\{\lambda_i^-\}_{i=1}^{n^-}$ and at least two additional eigenvalues lying in the open sector of Fig. 1 with $\theta = \pi/2k$ for $k = 2$ or 3. If the value of a_0 ($= \frac{1}{2}\,\mathrm{tr}[BR^{-1}B^TQ]$) is equal to zero, all eigenvalues of A_c have been optimally placed in the desired open sector of Fig. 1.

Proof: From Eq. (20f) we can write

$$\mathrm{tr}[(-1)^{k+1}A_c^k] = \mathrm{tr}[(-1)^{k+1}(A - \gamma BR^{-1}B^TP)^k]$$
$$= \gamma^k a_k + \gamma^{k-1}a_{k-1} + \cdots + \gamma a_1 + \mathrm{tr}[(-1)^{k+1}A^k] \tag{21a}$$

Since

$$\mathrm{tr}[(-1)^{k+1}A^k] = \mathrm{tr}\{[(-1)^{k+1}A^k]^-\} + \mathrm{tr}\{[(-1)^{k+1}A^k]^+\}$$
$$= (-1)^{k+1}\sum_{i=1}^{n^-}(\lambda_i^-)^k + (-1)^{k+1}\sum_{i-1}^{n^+}(\lambda_i^+)^k \tag{21b}$$

We obtain from Eq. (21a)

$$\mathrm{tr}[(-1)^{k+1}A_c^k] = \mathrm{tr}\{[(-1)^{k+1}A^k]^-\} + \gamma^k a_k + \cdots + \gamma a_1 + a_0 \tag{21c}$$

where

$$a_0 = \mathrm{tr}\{[(-1)^{k+1}A^k]^+\}$$
$$a_j = \mathrm{tr}[(BR^{-1}B^TP)^jA^{k-j}](-1)^{k+j+1}\binom{k}{j}$$
$$a_k = \mathrm{tr}[(BR^{-1}B^TP)^k](-1)^{2k+1}$$

From Lemma 2, Eq. (20a) and Eq. (20b), the closed-loop system A_c contains the invariant eigenvalues $\{\lambda_i^-\}_{i=1}^{n^-}$ in the hatched region of Fig. 1. From Lemma 4, we can see this implies that the matrix $[(-1)^{k+1}A_c^k]$ contains the eigenvalues associated with the invariant eigenvalues $\{\lambda_i^-\}_{i=1}^{n^-}$, which lie in the left half-plane of the $[(-1)^{k+1}A_c^k]$ mapping. Thus

$$\mathrm{tr}[(-1)^{k+1}A_c^k] = \mathrm{tr}\{[(-1)^{k+1}A_c^k]^-\} + \mathrm{tr}\{[(-1)^{k+1}A_c^k]^+\}$$
$$= \mathrm{tr}\{[(-1)^{k+1}A^k]^-\} + \mathrm{tr}\{[(-1)^{k+1}A_c^k]^+\} \tag{21d}$$

where $\{[(-1)^{k+1}A_c^k]^+\}$ would contain the rest $n^+ (= n - n^-)$ eigenvalues associated with the remaining eigenvalues of A_c required to be placed in the open sector of Fig. 1. (This comes from Lemma 4.) Comparing Eq. (21c) with Eq. (21d), we get

$$\mathrm{tr}\{[(-1)^{k+1}A_c^k]^+\} = \gamma^k a_k + \cdots + a_0 \tag{21e}$$

When $\mathrm{tr}\{[(-1)^{k+1}A_c^k]^+\} \leq 0$, the matrix $\{[(-1)^{k+1}A_c^k]^+\}$ has at least two additional eigenvalues lying in the left half-plane of the $(-1)^{k+1}A_c^k$ mapping. This implies, from Lemma 4, that at least two additional eigenvalues of A_c lie in or on the boundaries of the sector in Fig. 1. If $a_0 (= \mathrm{tr}\{[(-1)^{k+1} \times A^k]^+\}) = 0$, then all eigenvalues of A_c have been placed in the desired open sector. It can be shown, from Lemma 3, that

$$a_0 = \mathrm{tr}\{[(-1)^{k+1}A^k]^+\} = \tfrac{1}{2}\mathrm{tr}[BR^{-1}B^TQ] \tag{21f}$$

where Q satisfies Eq. (20a). Also, it is known[7] that a positive real solution of Eq. (21e) exists. Thus, any γ greater than or equal to the smallest positive real root of Eq. (21e) would satisfy the inequality given in Eq. (20e) for $k = 2$ or 3. The proof is completed. ∎

Remark: The steady-state solutions of the Riccati equations in Eqs. (20a) and (20b) can be found using the matrix sign function[10,11] (see the Appendix).

Design Procedure

1) Let the given continuous time system be as in Eq. (13). Specify α so that the $-\alpha$ vertical line on the negative real axis would represent the line beyond which the eigenvalues have to be placed in the sector of Fig. 1. Also, assign the positive definite matrix R. If the given system is asymptotically stable, then proceed to step 2 with $A_1 = A$, $P_0 = 0_n$, $Q_0 = 0_n$, and $i = 1$, or else, stabilize the system by solving for P_0 from Eq. (18) with $Q_0 = 0_n$. The immediate optimal closed-loop system becomes $A_1 = A - \gamma_0 BR^{-1}B^TP_0$, which has all its eigenvalues in the left half-plane beyond the $-\alpha$ vertical line, with $\gamma_0 = 1$. Set $i = 1$.

2) Select $k = 2$ or 3 in Theorem 1. Solve the following Riccati equation

$$Q_iBR^{-1}B^TQ_i - Q_i[(-1)^{k+1}A_i^k] - [(-1)^{k+1}A_i^k]^TQ_i = 0_n \tag{22a}$$

and obtain Q_i. Check if $\frac{1}{2}\,\mathrm{tr}[BR^{-1}B^TQ_i]$($= \mathrm{tr}\{[(-1)^{k+1} \times A^k]^+\}$) is zero. If it is equal to zero, go to step 5, or else continue. Note that, when $\mathrm{tr}\{[(-1)^{k+1}A^k]^+\} = 0$, all eigenvalues of the matrix A_i have been placed in the desired region.

3) Solve for the following Riccati equation

$$P_iBR^{-1}B^TP_i - P_iA_i - A_i^TP_i - Q_i = 0_n \tag{22b}$$

Obtain P_i and then solve for the constant gain γ_i from Eq. (20e). The closed-loop system matrix is

$$A_{i+1} = A_i - \gamma_iBR^{-1}B^TP_i = A_i - \gamma_iK_i \tag{22c}$$

4) Set $i = i + 1$ and go to step 2.

5) Check if $\mathrm{tr}[(A_i + \alpha Id_n)^+] = 0$. If it is equal to zero, let $P_{i+1} = 0_n$ and go to step 6, or else, solve for P_{i+1} from Eq. (18) with $A = A_i$ and $Q = 0_n$. The desired optimal closed-loop system becomes

$$A_{i+1} = A_i - \gamma_{i+1}BR^{-1}B^TP_{i+1} \text{ with } \gamma_{i+1} = 1 \tag{22d}$$

6) Algorithm is completed. The eigenvalues of the matrix

$$A_c = A - BR^{-1}B^T(P_0 + \gamma_1P_1 + \cdots + \gamma_jP_j) \tag{22e}$$

exist in the hatched region in Fig. 1. The optimal state-feedback control law is

$$u(t) = -\left(R^{-1}B^T\sum_{j=0}^{i+1}\gamma_jP_j\right)x(t) = -Kx(t) \tag{22f}$$

and the weighting matrix is

$$Q = 2\alpha P_0 + \sum_{j=1}^{i+1}[Q_j + (\gamma_j - 1)P_jBR^{-1}B^TP_j]\gamma_j \tag{22g}$$

Block Diagonalization via the Matrix Sign Function

Theorem 2[10,12,13]: Let $A \in R^{n\times n}$ and $\{\mathrm{Re}[\lambda(A)]\} \cap \alpha = 0$, where $\lambda(A)$ represents the eigenspectrum of A, $\alpha \in R$ is a scalar. Let $A_0 \triangleq A - \alpha Id_n$, where Id_n is an identity matrix of dimension n. Define sign (A_0)as the matrix sign function of A_0 (see the Appendix). Let two other matrix sign functions be

$$\mathrm{sign}^+(A_0) \triangleq \tfrac{1}{2}[Id_n + \mathrm{sign}(A_0)] \tag{23a}$$

$$\mathrm{sign}^-(A_0) \triangleq \tfrac{1}{2}[Id_n - \mathrm{sign}(A_0)] \tag{23b}$$

Define a right block modal matrix

$$M_s = [S_1, S_2] \tag{24}$$

where

$$S_1 \triangleq \mathrm{ind}[\mathrm{sign}^+(A_0)] \in R^{n\times n_1}$$

and

$$S_2 \triangleq \text{ind}[\text{sign}^+(A_0)] \in R^{n \times n_2}$$

where ind$(\cdot)$ represents the collection of linearly independent column vectors of $(\cdot)$ and $n_1 + n_2 = n$. Then, the matrix A can be block diagonalized by

$$M_s^{-1}AM_s = A_R = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tag{25}$$

where $A_1 \in R^{n_1 \times n_1}$ and $A_2 \in R^{n_2 \times n_2}$ contain the eigenvalues of A lying within the left and right sides of the $-\alpha$ vertical line on the negative real axis in the s plane, respectively. ■

Two-Stage Design Procedure

To reduce the computational load and to alleviate the numerical difficulty in solving the Riccati equations in Eqs. (20) using the high-order original system in Eq. (13), we decompose the original system having composited characteristics into a two-time scale structure with their own characteristics, using the matrix sign function and then we design each decomposed subsystem using optimal pole placement in Theorem 1.

Step 1

Specify a positive real scalar α and find a transformation matrix $T^{(1)}$ such that the matrix A can be block diagonalized into the following form:

$$A^{(1)} = (T^{(1)})^{-1}AT^{(1)} = \begin{bmatrix} A_{11}^{(1)} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & A_{22}^{(1)} \end{bmatrix} \tag{26a}$$

where $A_{11}^{(1)} \in R^{n_1 \times n_1}$ and $A_{22}^{(1)} \in R^{n_2 \times n_2}$, with $n_1 + n_2 = n$, contain the eigenvalues with real parts less than and greater than $-\alpha$, respectively. The transformation matrix $T^{(1)}$ is given by

$$T^{(1)} = [S_1, S_2] \tag{26b}$$

where $S_1 \in R^{n \times n_1}$ and $S_2 \in R^{n \times n_2}$ are defined in Eq. (24) with respect to the matrix sign function of the matrix A. Using $T^{(1)}$, we can transform B as

$$B^{(1)} = (T^{(1)})^{-1}B = \begin{bmatrix} B_1^{(1)} \\ B_2^{(1)} \end{bmatrix} \tag{26c}$$

The dimensions of the matrices $B_1^{(1)}$ and $B_2^{(1)}$ are $n_1 \times m$ and $n_2 \times m$, respectively.

Step 2: Two-Stage Design

Stage 1

The subsystem considered for design at this stage is $(A_{22}^{(1)}, B_2^{(1)})$. Let $K_2^{(1)} \in R^{m \times n_2}$ and $Q_2^{(1)} \in R^{n_2 \times n_2}$ represent the feedback gain matrix and weighting matrix of the designed subsystem, respectively. Then, the overall feedback gain matrix and weighting matrix for $(A^{(1)}, B^{(1)})$ are

$$K^{(1)} = [0_{m \times n_1}, K_2^{(1)}] \tag{27a}$$

$$Q^{(1)} = \text{block diag}[0_{n_1}, Q_2^{(1)}] \tag{27b}$$

The overall designed system matrix for $(A^{(1)}, B^{(1)})$ is

$$\hat{A}^{(1)} = A^{(1)} - B^{(1)}K^{(1)} = \begin{bmatrix} A_{11}^{(1)} & -B_1^{(1)}K_2^{(1)} \\ 0_{n_2 \times n_1} & A_{22}^{(1)} - B_2^{(1)}K_2^{(1)} \end{bmatrix} \tag{27c}$$

Stage 2

For this stage of design, first we block diagonalize the partially designed system $\hat{A}^{(1)}$ and move the last block of $\hat{A}^{(1)}$ in Eq. (27c) to the first block, via a transformation matrix $T^{(2)}$ given by

$$T^{(2)} = \begin{bmatrix} L & Id_{n_1} \\ Id_{n_2} & 0_{n_2 \times n_1} \end{bmatrix}, \qquad (T^{(2)})^{-1} = \begin{bmatrix} 0_{n_2 \times n_1} & Id_{n_2} \\ Id_{n_1} & -L \end{bmatrix} \tag{28a}$$

where the matrix $L \in R^{n_1 \times n_2}$ can be solved from the following Lyapunov equation

$$A_{11}^{(1)}L - L(A_{22}^{(1)} - B_2^{(1)}K_2^{(1)}) - B_1^{(1)}K_2^{(1)} = 0_{n_1 \times n_2} \tag{28b}$$

The transformed system is

$$A^{(2)} = (T^{(2)})^{-1}\hat{A}^{(1)}T^{(2)} = \begin{bmatrix} A_{11}^{(2)} & 0_{n_2 \times n_1} \\ 0_{n_1 \times n_2} & A_{22}^{(2)} \end{bmatrix} \tag{29a}$$

$$B^{(2)} = (T^{(2)})^{-1}B^{(1)} = \begin{bmatrix} B_1^{(2)} \\ B_2^{(2)} \end{bmatrix} \tag{29b}$$

where $A_{11}^{(2)} = A_{22}^{(1)} - B_2^{(1)}K_2^{(1)}$ and $A_{22}^{(2)} = A_{11}^{(1)}$. Now, the subsystem considered for design is $(A_{22}^{(2)}, B_2^{(2)})$. Let $K_2^{(2)} \in R^{m \times n_1}$ and $Q_2^{(2)} \in R^{n_1 \times n_1}$ represent the feedback gain matrix and weighting matrix of the designed subsystem, respectively, then the overall feedback gain matrix and weighting matrix for $(A^{(2)}, B^{(2)})$ are

$$K^{(2)} = [0_{m \times n_2}, K_2^{(2)}] \tag{30a}$$

$$Q^{(2)} = \text{block diag}[0_{n_2}, Q_2^{(2)}] \tag{30b}$$

The overall designed system matrix for $(A^{(2)}, B^{(2)})$ is

$$\hat{A}^{(2)} = A^{(2)} - B^{(2)}K^{(2)} = \begin{bmatrix} A_{11}^{(2)} & -B_1^{(2)}K_2^{(2)} \\ 0_{n_1 \times n_2} & A_{22}^{(2)} - B_2^{(2)}K_2^{(2)} \end{bmatrix} \tag{30c}$$

After the two-stage design, the accumulated feedback gain matrix K_c and the weighting matrix Q for the original system (A,B) are

$$K_c = [K^{(1)} + K^{(2)}(T^{(2)})^{-1}](T^{(1)})^{-1} \tag{31a}$$

and

$$Q = (T^{(1)})^{-T}[Q^{(1)} + (T^{(2)})^{-T}Q^{(2)}(T^{(2)})^{-1}](T^{(1)})^{-1} \tag{31b}$$

The eigenvalues of the closed-loop system matrix $(A - BK_c)$ lie in the hatched region of Fig. 1, and the weighting matrix Q in Eq. (31b) is symmetric nonnegative definite.

Block Decomposition of the Space Station Model

The inertial properties of the phase 1 space station are shown in Table 1. From Eq. (12) we know that

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & A_{13} & 0 \\ 0 & 0 & Id_3 & 0 \end{bmatrix} \tag{32}$$

The resulting eigenvalues of matrix A are located at

$$0.0, \quad 0.0, \quad 0.0, \quad 0.0, \quad \pm nj, \quad \pm 1.52n$$

$$(1.05 \pm 0.7j)n, \quad (-1.05 \pm 0.7j)n$$

Set $\alpha = 0.001$. From Eqs. (26), we compute a transformation matrix $T^{(1)}$, which block diagonalizes the matrix A into the following form:

$$A^{(1)} = (T^{(1)})^{-1} A T^{(1)} = \begin{bmatrix} A_{11}^{(1)} & 0_{3\times 9} \\ 0_{9\times 3} & A_{22}^{(1)} \end{bmatrix} \tag{33}$$

where

$$A_{11}^{(1)} = \begin{bmatrix} -1.6191e-3 & 4.3279e-6 & 1.5099e-3 \\ -1.3796e-6 & -1.6692e-3 & -3.3553e-6 \\ -5.3840e-4 & 1.2301e-5 & -6.9117e-4 \end{bmatrix}$$

Table 1 Phase 1 space station parameters

Inertia, slug-ft^2		Aerodynamic torque, ft-lb	
I_{11}	50.28e6	w_1	$1+\sin(nt)+0.5\sin(2nt)$
I_{22}	10.80e6	w_2	$4+2\sin(nt)+0.5\sin(2nt)$
I_{33}	58.57e6	w_3	$1+\sin(nt)+0.5\sin(2nt)$
I_{12}	−0.39e6	—	—
I_{13}	0.16e6	—	—
I_{23}	0.16e6	—	—

$$A_{22}^{(1)} = \begin{bmatrix} 1.6321e-3 & 9.3766e-6 & 1.5089e-3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4.9940e-6 & 1.6693e-3 & -3.1058e-7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5.5305e-4 & -2.6628e-6 & 6.7808e-4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.1e-3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.1e-3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B^{(1)} = \begin{bmatrix} -1.9894e-8 & -7.1924e-10 & 5.6312e-11 \\ -7.1924e-10 & -9.2622e-8 & 2.5499e-10 \\ 5.6312e-11 & 2.5499e-10 & -1.7074e-8 \\ -1.9894e-8 & -7.1924e-10 & 5.6312e-11 \\ -7.1924e-10 & -9.2622e-8 & 2.5499e-10 \\ 5.6312e-11 & 2.5499e-10 & -1.7074e-8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The next step is to design each decomposed subsystem using optimal pole placement.

Multistage Design for $\theta = 45$ deg ($k = 2$)

The problem is to design the following subsystem:

$$\dot{x}_2^{(1)} = A_{22}^{(1)} x_2^{(1)} + B_2^{(1)} u \tag{34}$$

This subsystem is unstable with poles at

$$0.0, \quad 0.0, \quad 0.0, \quad \pm nj, \quad 1.52n, \quad (1.05 \pm 0.7j)n$$

To relocate the unstable poles at the origin to the stable location at $-1.0n$, which has a damping constant of $1.0n$, we set the degree of relative stability α as $\alpha = 0.5n$ and go to step 1 of the design procedure to solve for

$$A_1 = A_{22}^{(1)} - \gamma_0 B_2^{(1)} R^{-1} B_2^{(1)T} P_0$$

where $\gamma_0 = 1.0$ and $R = Id_3$. The resulting eigenvalues of matrix A_1 are located at

$$-1.0n, \quad -1.0n, \quad -1.0n, \quad -1.0n, \quad -2.52n$$

$$(-1.0 \pm 1.0j)n, \quad (-2.05 \pm 0.7j)n$$

which lie within or on the open sector of Fig. 1 for $k = 2$.

The control gain is

$$K_2^{(1)} = \left[\begin{array}{ccc} -6.1269e5 & 7.0398e3 & 2.0464e5 \\ 1.4384e3 & -1.3188e5 & 2.2111e3 \\ -1.4159e5 & 1.3955e2 & -4.9443e5 \\ \\ -5.4927e-3 & 1.8141e-4 & 4.5865e-3 \\ -4.4140e-5 & -5.5766e-3 & 3.7422e-5 \\ -1.7293e-3 & 3.2814e-6 & -3.9163e-3 \\ \\ 2.2861e-6 & 5.1147e-8 & 2.5831e-6 \\ 8.6374e-9 & -2.0071e-6 & 3.8640e-8 \\ -2.6062e-6 & 3.1370e-9 & 8.1048e-7 \end{array}\right] \tag{35}$$

The overall feedback gain matrix is

$$K^{(1)} = [0_{3\times 3}, K_2^{(1)}] \tag{36a}$$

The overall designed system for $(A^{(1)}, B^{(1)})$ is

$$\dot{x}^{(1)} = (A^{(1)} - B^{(1)} K^{(1)}) x^{(1)} + B^{(1)} r_1 = \hat{A}^{(1)} x^{(1)} + B^{(1)} r_1$$

$$\begin{bmatrix} \dot{x}_1^{(1)} \\ \dot{x}_2^{(1)} \end{bmatrix} = \begin{bmatrix} A_{11}^{(1)} & -B_1^{(1)} K_2^{(1)} \\ 0_{9\times 3} & A_{22}^{(1)} - B_2^{(1)} K_2^{(1)} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} + \begin{bmatrix} B_1^{(1)} \\ B_2^{(1)} \end{bmatrix} r_1 \tag{36b}$$

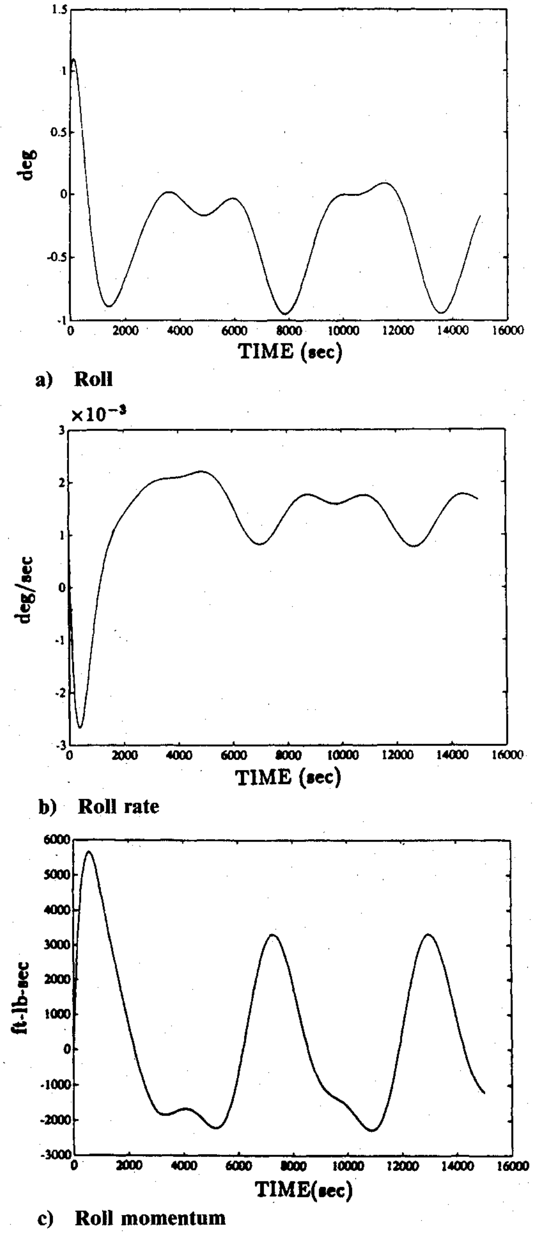


Fig. 2 Roll response for $k=2$.

Using the transformation matrix $T^{(2)}$ from Eq. (28), let

$$x^{(1)} = T^{(2)}x^{(2)} \tag{37}$$

The transformed system is

$$\dot{x}^{(2)} = A^{(2)}x^{(2)} + B^{(2)}r_1 \tag{38}$$

where

$$A^{(2)} = (T^{(2)})^{-1}\hat{A}^{(1)}T^{(2)} = \begin{bmatrix} A_{11}^{(2)} & 0_{9\times 3} \\ 0_{3\times 9} & A_{22}^{(2)} \end{bmatrix} \tag{39a}$$

$$B^{(2)} = (T^{(2)})^{-1}B^{(1)} = \begin{bmatrix} B_1^{(2)} \\ B_2^{(2)} \end{bmatrix} \tag{39b}$$

and $A_{11}^{(2)} = A_{22}^{(1)} - B_2^{(1)}K_2^{(1)}$, $A_{22}^{(2)} = A_{11}^{(1)}$. The subsystem to be designed is $(A_{22}^{(2)}, B_2^{(2)})$. Let $K_2^{(2)} \in R^{3\times 3}$ and $Q_2^{(2)} \in R^{3\times 3}$ represent the feedback gain matrix and weighting matrix of the designed subsystem, respectively. The eigenvalues of $A_{22}^{(2)}$are located at $-1.52n$ and $(-1.05 \pm 0.7j)n$, which places them within the design region.

The feedback gain matrix is

$$r_1 = [0_{3\times 9}, K_2^{(2)}]x^{(2)} + r^2 \tag{40a}$$

where $K_2^{(2)} = 0_{3\times 3}$. The overall designed system for $(A^{(2)}, B^{(2)})$ is

$$\begin{bmatrix} \dot{x}_1^{(2)} \\ \dot{x}_2^{(2)} \end{bmatrix} = \begin{bmatrix} A_{11}^{(2)} & 0_{9\times 3} \\ 0_{3\times 9} & A_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} + \begin{bmatrix} B_1^{(2)} \\ B_2^{(2)} \end{bmatrix} r_2 \tag{40b}$$

From Eq. (31), we can compute the accumulated feedback gain matrix K_c for the original system (A, B)

$$K_c = [K^{(1)} + K^{(2)}(T^{(2)})^{-1}](T^{(1)})^{-1}$$

$$= \left[\begin{matrix} -6.1269e5 & 7.0398e3 & 2.0464e5 \\ 1.4384e3 & -1.3188e5 & 2.2111e3 \\ -1.4159e5 & 1.3955e2 & -4.9443e5 \end{matrix}\right.$$

$$\begin{matrix} -9.5786e2 & 6.7283e0 & -1.4418e2 \\ 2.7418e0 & -2.2018e2 & 1.9703e0 \\ -3.2344e2 & 8.4401e0 & -4.0383e2 \end{matrix}$$

$$\begin{matrix} -5.4927e-3 & 1.8141e-4 & 4.5865e-3 \\ -4.4140e-5 & -5.5766e-3 & 3.7422e-5 \\ -1.7293e-3 & 3.2814e-6 & -3.9163e-3 \end{matrix}$$

$$\left.\begin{matrix} 2.2861e-6 & 5.1147e-8 & 2.5831e-6 \\ 8.6374e-9 & -2.0071e-6 & 3.8640e-8 \\ -2.6062e-6 & 3.1370e-9 & 8.1048e-7 \end{matrix}\right] \tag{41}$$

The eigenvalues of the closed-loop system matrix $(A - BK_c)$ are located at

$$-1.0n,\quad -1.0n,\quad -1.0n,\quad -1.0n,\quad -1.52n\quad -2.52n,$$

$$(-1.0 \pm 1.0j)n,\quad (-1.05 \pm 0.7j)n,\quad (-2.05 \pm 0.7j)n$$

and are seen to lie in the hatched region of Fig. 1 for $k=2$.

The closed-loop response of the system to the aerodynamic disturbance torque shown in Table 1 is shown for roll, pitch, and yaw in Figs. 2–4, respectively. Initial conditions are $\theta_1(0) = \theta_2(0) = \theta_3(0) = 1.0$ deg, $\omega_1(0) = \omega_2(0) = \omega_3(0) = 0.001$ deg/s, and $h_1(0) = h_2(0) = h_3(0) = 0.0$ ft-lb-s. The disturbance torque causes the periodic responses.

Multistage Design for $\theta = 30$ deg ($k = 3$)

Again, the problem is to design the following subsystem:

$$\dot{x}_2^{(1)} = A_{22}^{(1)}x_{22}^{(1)} + B_2^{(1)}u \tag{42}$$

We stabilize the system as before using step 1 of the design procedure and then iterate one time through steps 2–6 to place the eigenvalues within the region defined by $k=3$ and $\alpha = 0.5n$. The sequence occurs as follows:

$A_{22}^{(1)}$ eigenvalues

$$0.0,\quad 0.0,\quad 0.0,\quad 0.0,\quad (1.05 \pm 0.7j)n,\quad \pm nj,\quad 1.52n$$

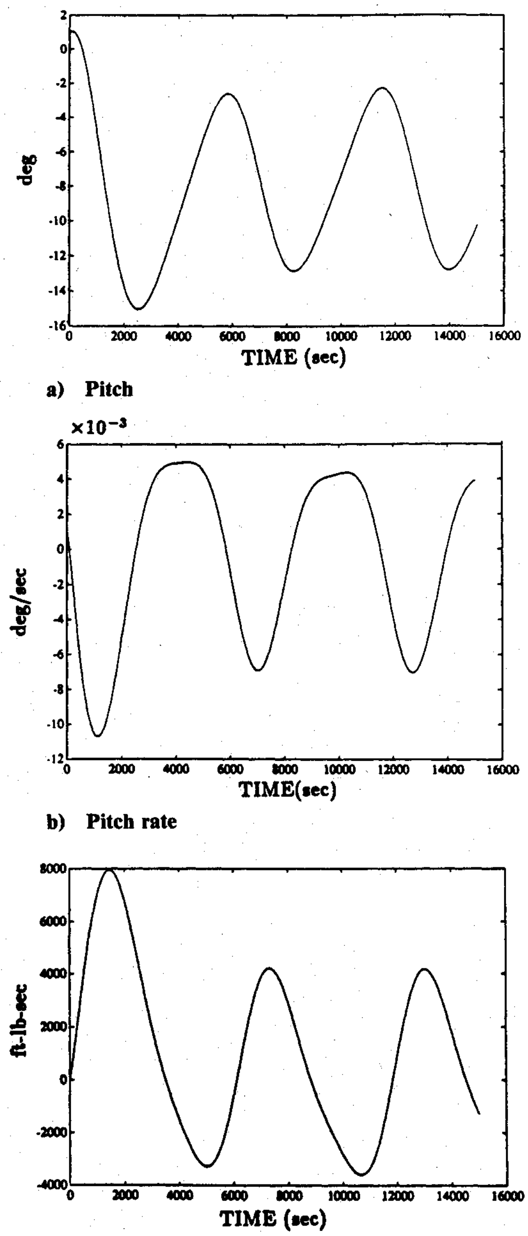
$$\downarrow \text{ step 1} \quad A_1 = A_{22}^{(1)} - B_2^{(1)}K_2^{(1)}$$

A_1 eigenvalues

$$-1.0n,\quad -1.0n,\quad -1.0n. -1.0n,\quad -2.52n$$

$$(-1.0 \pm 1.0j)n,\quad (-2.05 \pm 0.7j)n$$

$$\downarrow \text{ step 2–6} \quad A_2 = A_1 - B_2^{(1)}K_{22}^{(1)}$$

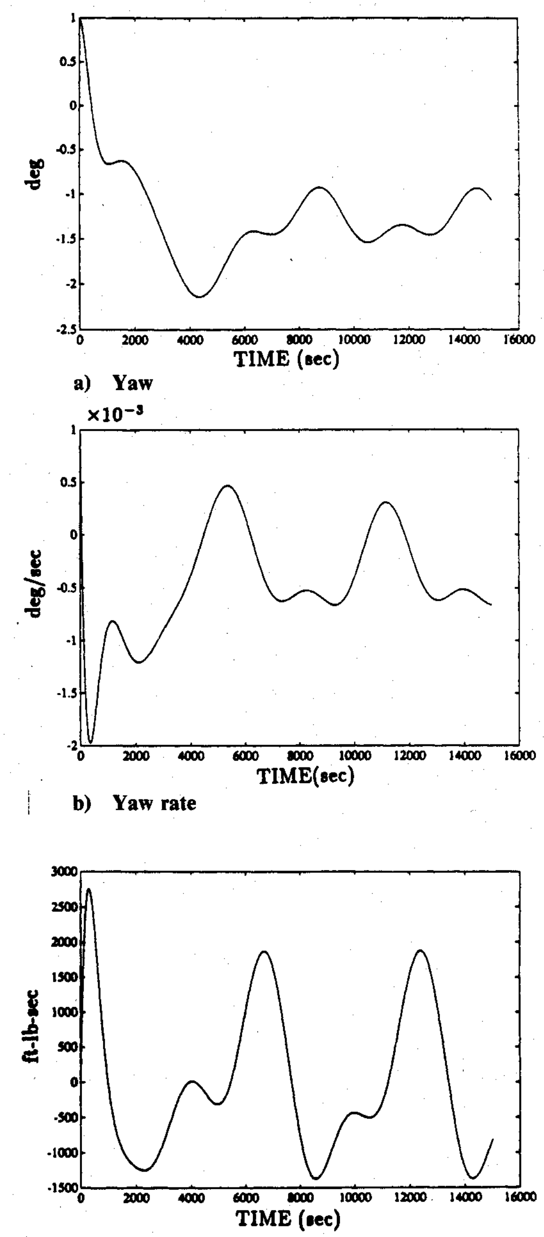


a) Pitch

b) Pitch rate

c) Pitch momentum

Fig. 3 Pitch response for $k = 2$.



a) Yaw

b) Yaw rate

c) Yaw momentum

Fig. 4 Yaw response for $k = 2$.

A_2 eigenvalues

$$-1.0n,\quad -1.0n,\quad -1.0n, -1.0n,\quad -2.52n$$

$$(-1.77 \pm 1.02j)n,\quad (-2.05 \pm 0.7j)n$$

The overall feedback gain matrix is

$$K^{(1)} = [0_{3\times 3}, K_2^{(1)} + K_{22}^{(1)}] \tag{43a}$$

The overall designed subsystem is given by

$$\begin{bmatrix} \dot{x}_1^{(1)} \\ \dot{x}_2^{(1)} \end{bmatrix} = \begin{bmatrix} A_{11}^{(1)} & -B_1^{(1)}(K_2^{(1)} + K_{22}^{(1)}) \\ 0_{9\times 3} & A_{22}^{(1)} - B_2^{(1)}(K_2^{(1)} + K_{22}^{(1)}) \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} + \begin{bmatrix} B_1^{(1)} \\ B_2^{(1)} \end{bmatrix} r_1$$

$$= \hat{A}^{(1)} x^{(1)} + B^{(1)} r_1 \tag{43b}$$

Using the transformation from Eqs. (28) results in

$$\dot{x}^{(2)} = A^{(2)} x^{(2)} + B^{(2)} r_1 \tag{44}$$

where as before

$$A^{(2)} = (T^{(2)})^{-1} \hat{A}^{(1)} T^{(2)} = \begin{bmatrix} A_{11}^{(2)} & 0_{9\times 3} \\ 0_{3\times 9} & A_{22}^{(2)} \end{bmatrix} \tag{45a}$$

$$B^{(2)} = (T^{(2)})^{-1} B^{(1)} = \begin{bmatrix} B_1^{(2)} \\ B_2^{(2)} \end{bmatrix} \tag{45b}$$

The problem is to design the following subsystems:

$$\dot{x}_2^{(2)} = A_{22}^{(2)} x_2^{(2)} + B_2^{(2)} r_1 \tag{46}$$

We step through the design procedure one time to obtain the following results:

$A_{22}^{(2)}$ eigenvalues

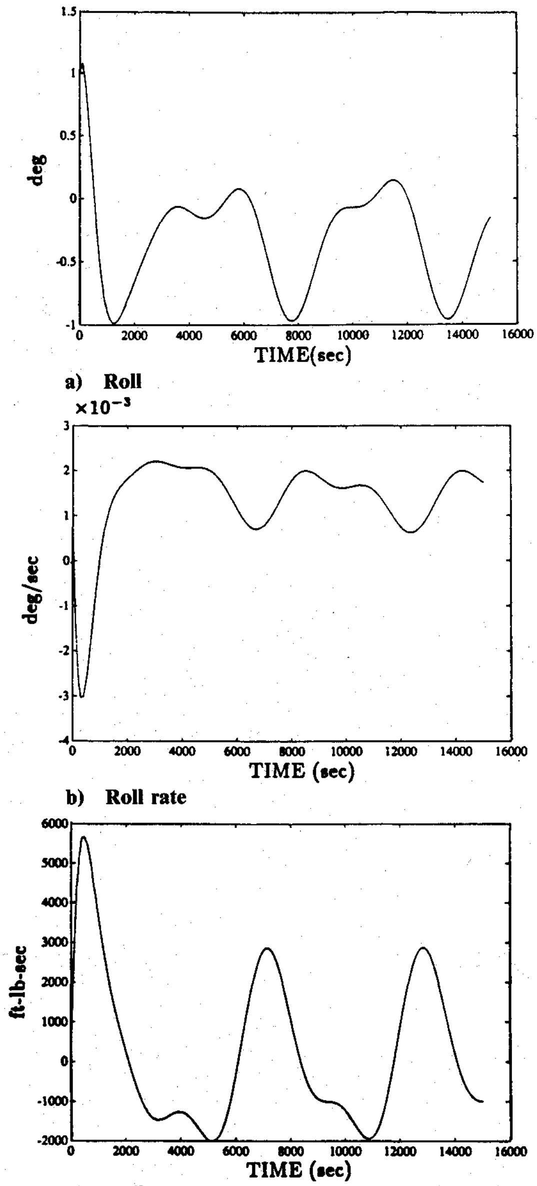
$$(-1.05 \pm 0.7j)n, \qquad -1.52n$$

$$\downarrow \text{ steps 1-6}$$

$$A_2 = A_{22}^{(2)} - B_2^{(2)} K_2^{(2)}$$

A_2 eigenvalues

$$(-1.22 \pm 0.7j)n, \qquad -1.52n$$

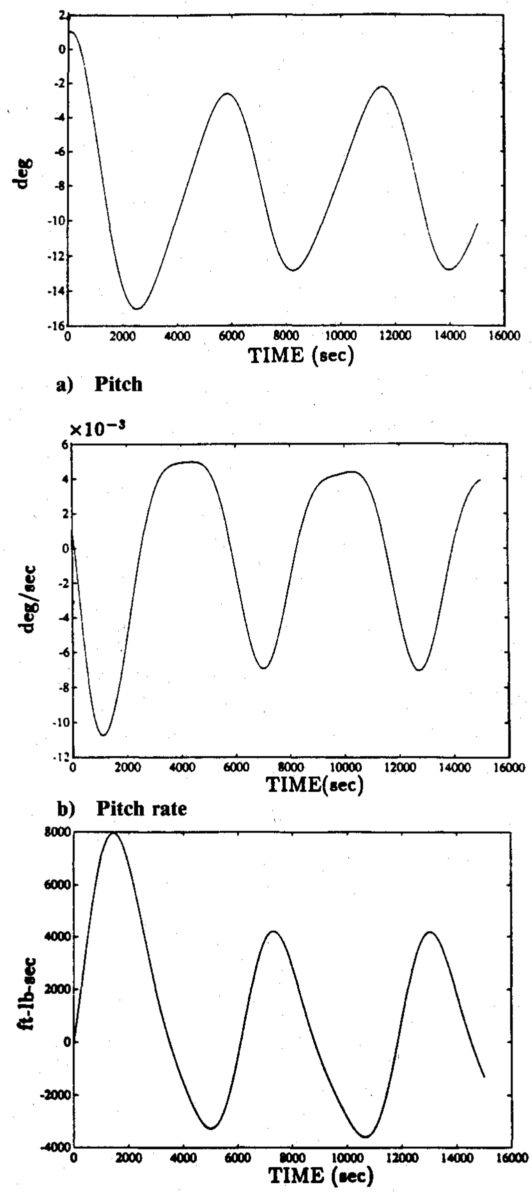


a) Roll

b) Roll rate

c) Roll momentum

Fig. 5 Roll response for $k = 3$.



a) Pitch

b) Pitch rate

c) Pitch momentum

Fig. 6 Pitch response for $k = 3$.

The feedback gain matrix is $K^{(2)} = [0_{3\times 9}, K_{22}^{(2)}]$ (47a)

The designed feedback gain for the subsystem is given by

$$r_1 = -[0_{3\times 9}, K_{22}^{(2)}]x_2^{(2)} + r_2 \tag{47b}$$

Therefore, the designed overall system is given by

$$\begin{bmatrix} \dot{x}_1^{(2)} \\ \dot{x}_2^{(2)} \end{bmatrix} = \begin{bmatrix} A_{11}^{(2)} & -B_1^{(2)}K_{22}^{(2)} \\ 0_{3\times 9} & A_{22}^{(2)} - B_2^{(2)}K_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} + \begin{bmatrix} B_1^{(2)} \\ B_2^{(2)} \end{bmatrix} r_2 \tag{47c}$$

The desired feedback gain in original coordinates is given by Eqs. (31)

$$\begin{aligned} \boldsymbol{u} &= -[0_{3\times 3}, K_2^{(1)} + K_{22}^{(1)}]\boldsymbol{x}^{(1)} + \boldsymbol{r}_1 = -K^{(1)}\boldsymbol{x}^{(1)} + \boldsymbol{r}_1 \\ &= -K^{(1)}\boldsymbol{x}^{(1)} - [0_{3\times 9}, K_{22}^{(2)}]\boldsymbol{x}^{(2)} + \boldsymbol{r}_2 = -K^{(1)}\boldsymbol{x}^{(1)} \\ &\quad - K^{(2)}\boldsymbol{x}^{(2)} + \boldsymbol{r}_2 \\ &= -K^{(1)}\boldsymbol{x}^{(1)} - K^{(2)}(T^{(2)})^{-1}\boldsymbol{x}^{(1)} + \boldsymbol{r}_2 \\ &= -(K^{(1)} + K^{(2)}(T^{(2)})^{-1})\boldsymbol{x}^{(1)} + \boldsymbol{r}_2 \\ &= -(K^{(1)} + K^{(2)}(T^{(2)})^{-1})T^{(1)}\boldsymbol{x} + \boldsymbol{r}_2 = -K_c\,\boldsymbol{x} + \boldsymbol{r}_2 \end{aligned} \tag{48}$$

The feedback gain is given by

$$K_c = \left[\begin{matrix} -9.1720e5 & 1.1545e4 & 5.7516e5 \\ -2.5905e3 & -1.3183e5 & 7.3875e3 \\ -8.8597e4 & -1.7309e3 & -7.9081e5 \end{matrix}\right.$$

$$\begin{matrix} -1.3841e3 & 7.1231e0 & -3.3995e1 \\ -2.8180e0 & -2.2019e2 & 3.5283e0 \\ -2.8851e2 & 1.0505e1 & -5.8369e2 \end{matrix}$$

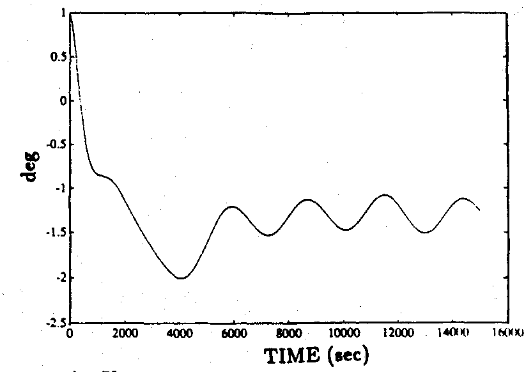
$$\begin{matrix} -1.0027e-2 & 3.0307e-4 & 1.0618e-2 \\ -1.0672e-4 & -5.5756e-3 & 1.2221e-4 \\ -9.7071e-4 & -6.0081e-5 & -8.4560e-3 \end{matrix}$$

$$\left.\begin{matrix} 3.9985e-6 & 8.5191e-8 & 5.3340e-6 \\ 2.7998e-8 & -2.0069e-6 & 7.7404e-8 \\ -4.6964e-6 & -1.5477e-8 & -7.1248e-7 \end{matrix}\right] \tag{49}$$

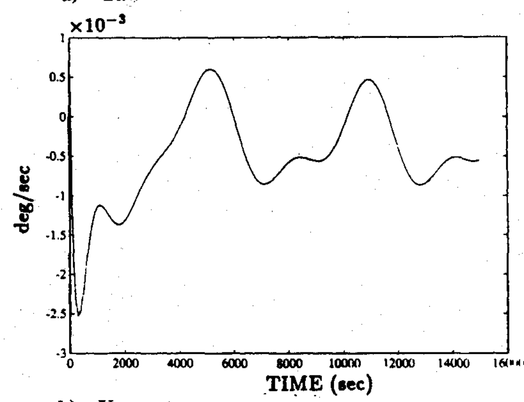
The eigenvalues of the closed-loop system $(A - BK_c)$ are located at

$$-1.0n,\ -1.0n,\ -1.0n,\ -1.0n,\ -1.52n,\ -2.52n,$$

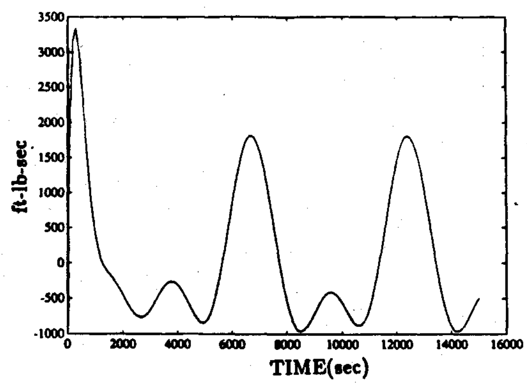
$$(-1.22 \pm 0.7j)n,\ (-1.77 \pm 1.02j)n,\ (-2.05 \pm 0.7j)n$$



a) Yaw



b) Yaw rate



c) Yaw momentum

Fig. 7 Yaw response for $k = 3$.

and can be seen to lie on or within the hatched region of Fig. 1 for $k = 3$.

The closed-loop response is shown in Figs. 5-7 for roll, pitch, and yaw, respectively. Initial conditions are the same as for the previous case.

Conclusions

A multistage design technique for determining an optimal momentum management and attitude control system for the space station Freedom has been presented. The matrix sign function was used throughout the design process. It was used first to decompose the space station equations of motion and then to solve the Riccati equations associated with the optimal pole placement problem. The proposed method enables the decomposition of a large-scale multivariable system, which does not exhibit a two-time scale structure explicitly, into a completely decoupled two-time scale structure using the techniques based on the matrix sign function and the fast stable matrix sign algorithm, without explicitly utilizing the open-loop eigenvalues of the given system. Also, the proposed method requires the solution of Riccati equations of smaller order only at each stage of the design via the fast stable matrix sign algorithm. Thus, the computational load, numerical difficulty, and storage problem can be greatly alleviated. Moreover, the proposed design method enables the optimal placement of the closed-loop poles within the specified sectors. As a result, the designed system responses converge at appropriate speed and any existing vibrating modes are well damped.

Appendix

Matrix Sign Function

Let the eigenspectrum of a matrix $A \in C^{n\times n}$ be $\sigma(A) = \{\lambda_i, i = 1, \ldots, n\}$, $\lambda_i \neq 0$, and $\arg(\lambda_i) \neq \pi$. The matrix sign function of the matrix $A \in C^{n\times n}$[13,14] is defined as

$$\mathrm{sign}(A) = A(\sqrt{A^2})^{-1} = A^{-1}(\sqrt{A^2}) \tag{A1}$$

where the matrix $\sqrt{A^2} \in C^{n\times n}$ denotes the principal square root of A^2, which is defined as[15]

$$(\sqrt{A})^2 = A \qquad \text{and} \qquad \arg[\sigma(\sqrt{A^2})] \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

A fast and stable algorithm[14] to compute the matrix sign function is listed below. For $k = 0,1,2, \ldots$

$$P_j(k) = P_{j-1}(k) + S^{-2}(k)Q_{j-1}(k), \qquad P_1(k) = Id_n \tag{A2}$$

$$Q_j(k) = P_{j-1}(k) + Q_{j-1}(k), \qquad Q_1(k) = Id_n \tag{A3}$$

with $j = 2, \ldots, r$.

$$S(k+1) = S(k)Q_r^{-1}(k)P_r(k), \qquad S(0) = A$$

$$\lim_{k\to\infty} S(k) = \mathrm{sign}(A) \tag{A4}$$

where r is the order of the desired rate of convergence.

Solving Riccati Equation via the Matrix Sign Function

The Riccati equation for the controllable continuous time system (A, B) with weighting matrices $Q(\geq 0)$ and $R(>0)$ is given by

$$PBR^{-1}B^TP - A^TP - PA - Q = 0 \tag{A5}$$

The steady-state solution of this Riccati equation, $P(\geq 0)$ with (Q,A) detectable, can be computed easily using the properties of the matrix sign function[10,11] and the eigenvalue-eigenvector approach.[16] Consider the Hamiltonian associated with the given system

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \tag{A6}$$

The following algorithm with $k = 0,1,2,\ldots$ can be utilized to obtain the solution P

$$H_{k+1} = \tfrac{1}{2}[H_k + H_k^{-1}], \quad H_0 = H, \quad \lim_{k\to\infty} H_k = \mathrm{sign}(H) \tag{A7}$$

Let

$$\mathrm{sign}^+(H) \triangleq \tfrac{1}{2}[Id_{2n} + \mathrm{sign}(H)] \triangleq \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{A8}$$

Then, we have

$$P = -(X_{22})^{-1}X_{21} = -(X_{12})^{-1}X_{11} \tag{A9}$$

To alleviate the problems of computing H_k^{-1}, the Hamiltonian can be transformed into a symmetric form as follows[11]

$$\hat{H} = \hat{J}H = \begin{bmatrix} 0_n & -Id_n \\ Id_n & 0_n \end{bmatrix} H = \begin{bmatrix} Q & A^T \\ A & -BR^{-1}B^T \end{bmatrix} \tag{A10}$$

Then, the algorithm in Eq. (A7) becomes

$$\hat{H}_{k+1} = \tfrac{1}{2}[\hat{H}_k + \hat{J}\hat{H}_k^{-1}\hat{J}], \qquad \hat{H}_0 = \hat{J}H$$

$$\lim_{k\to\infty}(-\hat{J}\hat{H}_k) = \mathrm{sign}(H) \tag{A11}$$

The computation of the inverse of the symmetric matrix $\hat{H}_k$ is much simpler than computing the inverse of H_k. The Riccati solution P is again given by Eq. (A9).

Solving Lyapunov Equation via the Matrix Sign Function[17]

Given a Lyapunov equation

$$AX + XB = C \tag{A12}$$

where $A \in R^{n_1\times n_1}$, $B \in R^{n_2\times n_2}$,$X \in R^{n_1\times n_2}$, and $C \in R^{n_1\times n_2}$. If Re $[\lambda(A)]<0$ and Re$[\lambda(B)]<0$, the algorithm for finding X with $k = 0,1,2,\ldots$ is

$$A_{k+1} = \tfrac{1}{2}(A_k + A_k^{-1}) \qquad B_{k+1} = \tfrac{1}{2}(B_k + B_k^{-1})$$

$$C_{k+1} = -\tfrac{1}{2}(C_k + A_k^{-1}C_kB_k^{-1})$$

$$X = \lim_{k\to\infty} C_k \tag{A13}$$

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