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Analytical Solutions for Exponentially Correlated Acceleration Tracking Filters

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I. Introduction

IN earlier papers, Gupta and Ahn[1] and Gupta[2] have presented an analytical solution of the steady-state exponentially correlated acceleration (ECA) filter with the target position as the only measurement. In radar applications, however, Doppler measurements are also available in addition. Fitzgerald[3] has presented computed steady-state data for a simple form of the ECA target tracking filter, which utilized both position and velocity measurements. In a recent paper, Ramachandra et al.[4] have presented the steady-state solution of a three-state Kalman tracking filter that utilized both position and velocity measurements and a constant-acceleration model.

The objective here is to extend the case of a closed-form steady-state solution for the discrete ECA tracking filter by using the MacFarlane–Potter–Fath eigenstructure method.[5] The ECA tracking filter estimates the target position, velocity, and acceleration in a track-while-scan system and utilizes the target position and velocity measurements as inputs to the tracking filter. When the variance of the velocity measurement error σ_2 goes to infinity, the results are shown to be in agreement with the analytic solution given by Gupta[2] and with the numerical solution given by Fitzgerald.[3]

II. Discrete Exponentially Correlated Acceleration Tracking Filter

The following discrete ECA tracking filter model is considered[6]:

$$x(k+1) = \Phi(T_s)x(k) + v(k) \qquad z(k) = Hx(k) + w(k)$$
$$\Phi(T_s) = \exp(FT_s) \tag{1}$$

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with

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau \end{bmatrix} \qquad Q(T_s) = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}$$

$$\Phi(T_s) = \begin{bmatrix} 1 & \tau\theta & \tau^2\phi_1 \\ 0 & 1 & \tau(1-\phi_3) \\ 0 & 0 & \phi_3 \end{bmatrix} \tag{2}$$

$$\Phi^{-1}(T_s) = \begin{bmatrix} 1 & -\tau\theta & -\tau^2\psi_1 \\ 0 & 1 & \tau(1-\psi_3) \\ 0 & 0 & \psi_3 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R(T_s) = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

and

$$\theta = T_s/\tau, \qquad \phi_1 = \theta - 1 + \phi_3, \quad \text{and} \quad \phi_3 = \exp(-\theta)$$
$$\psi_1 = \theta + 1 - \psi_3, \qquad \psi_3 = \exp(\theta), \quad \text{and} \quad \phi_3\psi_3 = 1 \tag{3}$$

Here $\Phi(T_s)$ is the dynamic state transition matrix of the system, H is the measurement matrix, $Q(T_s)$ is the process noise covariance matrix that is symmetric and nonnegative definite and q_{ij} given by Singer,[6] and $R(T_s)$ is the measurement covariance matrix and uncorrelated with $v(k)$.

It is well known that the steady-state Kalman filter for Eqs. (1–3) becomes

$$\hat{x}(k/k) = \hat{x}(k/k-1) + K[z(k) - H\hat{x}(k)] \tag{4}$$

where the Kalman gain matrix K is given by

$$K = PH^TR^{-1} = \begin{bmatrix} p_{11}/R_1 & p_{12}/R_2 \\ p_{21}/R_1 & p_{22}/R_2 \\ p_{31}/R_1 & p_{32}/R_2 \end{bmatrix} \tag{5}$$

where P is the a posteriori covariance matrix of the estimation errors.

The five parameters used to describe this problem[3] are rms target acceleration σ_a, correlation time of target acceleration τ, sampling time T_s, rms position measurement error σ_{mp}, and rms velocity measurement error σ_{mv}.

In Eq. (3), $R_1 = \sigma_{mp}^2$, $R_2 = \sigma_{mv}^2$, and we define the three dimensionless parameters as

$$p_1 \equiv \tau/T_s \tag{6}$$

$$p_2 \equiv \frac{T_s^2\sigma_a}{\sigma_{mp}} \tag{7}$$

$$p_3 \equiv \frac{T_s\sigma_{mv}}{\sigma_{mp}} \tag{8}$$

We restrict p_1 (Ref. 3) to a few simple multiples of the critical value $p_{1c} = \tau_c/T_s$. The critical value maximizes the position and velocity errors of the filter. Values determined empirically are well approximated by the equation[3]

$$p_{1c} = \tau_c/T_s = \left[0.56 + 3.4p_2^{-0.86}\right]^{\frac{1}{2}} \tag{9}$$

III. MacFarlane–Potter–Fath Eigenstructure Method

The steady-state solution of the time-invariant matrix Riccati equation was discovered independently by MacFarlane et al.[5] The solution $P(\infty)$ of the steady-state matrix Riccati equation in discrete time is formalized as Lemma 1.

Lemma 1 (Ref. 5). If W_{11} and W_{21} are $n \times n$ matrices such that W_{21} is nonsingular and

$$H_f \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} D \tag{10}$$

for an $n \times n$ nonsingular matrix D, then $P_\infty = W_{11}W_{21}^{-1}$ satisfies the steady-state discrete time matrix Riccati equation and H_f is a Hamiltonian matrix:

$$H_f = \begin{bmatrix} \Phi_k + Q_k\Phi_k^{-T}H_k^T R_k^{-1}H_k & Q_k\Phi_k^{-T} \\ \Phi_k^{-T}H_k^T R_k^{-1}H_k & \Phi_k^{-T} \end{bmatrix} \tag{11}$$

$$P_\infty = \Phi\left\{P_\infty - P_\infty H^T\left[HP_\infty H^T + R\right]^{-1}HP_\infty\right\}\Phi^T + Q \tag{12}$$

Now we obtain the steady-state solutions of the ECA tracking filter as follows. First, construct the Hamiltonian matrix $(2n \times 2n)$ of the system $(n \times n)$:

$$H_f = \begin{bmatrix} U_1R_1^{-1}+1 & S_1R_2^{-1}+\tau\theta & \tau^2\phi_1 & U_1 & S_1 & \psi_3 q_{13} \\ U_2R_1^{-1} & S_2R_2^{-1}+1 & \tau(1-\phi_3) & U_2 & S_2 & \psi_3 q_{23} \\ U_3R_1^{-1} & S_3R_2^{-1} & \phi_3 & U_3 & S_3 & \psi_3 q_{33} \\ R_1^{-1} & 0 & 0 & 1 & 0 & 0 \\ -\tau\theta R_1^{-1} & R_2^{-1} & 0 & -\tau\theta & 1 & 0 \\ -\tau^2\psi_1 R_1^{-1} & \tau(1-\psi_3)R_2^{-1} & 0 & -\tau^2\psi_1 & \tau(1-\psi_3) & \psi_3 \end{bmatrix} \tag{13}$$

where

$$\begin{gathered} U_1 = q_{11} - \tau\theta q_{12} - \tau^2\psi_1 q_{13} \qquad U_2 = q_{12} - \tau\theta q_{22} - \tau^2\psi_1 q_{23} \\ U_3 = q_{13} - \tau\theta q_{23} - \tau^2\psi_1 q_{33} \\ S_1 = q_{12} + \tau(1-\psi_3)q_{13} \qquad S_2 = q_{22} + \tau(1-\psi_3)q_{23} \\ S_3 = q_{23} + \tau(1-\psi_3)q_{33} \end{gathered} \tag{14}$$

By direct evaluation of the characteristic equation $|H_f - \lambda I| = 0$, the eigenvalue polynomial may be obtained as

$$f(\lambda) = f(\lambda^{-1}) = \lambda^6 - a\lambda^5 + b\lambda^4 - c\lambda^3 + b\lambda^2 - a\lambda + 1 = 0 \tag{15}$$

where

$$\begin{aligned} a &= 4 + 2\cosh\theta + U_1R_1^{-1} + S_2R_2^{-1} \\ b &= 7 + 8\cosh\theta + \big[2U_1(1+\cosh\theta) - \tau\theta(U_2 - S_1) \\ &\quad - \tau^2(\phi_1U_3 - \psi_1)\psi_3q_{13}\big]R_1^{-1} + [2S_2(1+\cosh\theta) \\ &\quad - \tau S_3(1-\phi_3) - z\tau(1-\psi_3)\psi_3q_{23}]R_2^{-1} \\ &\quad + [U_1S_2 - U_2S_1]R_1^{-1}R_2^{-1} \\ c &= 8 + 12\cosh\theta - \big\{\tau^2\theta^2S_2 + \tau^3\theta\phi_1S_3 + \psi_1\psi_3q_{23} \\ &\quad + \tau^4\phi_1\psi_1\psi_3q_{33} + (2U_1 - \tau\theta U_2 + \tau\theta S_1)(1+2\cosh\theta) \\ &\quad + \tau^2U_3[\theta(1-\phi_3) - \phi_1(2+\psi_3)] \\ &\quad + \tau^2\psi_3q_{13}[\psi_1(2+\phi_3) - \theta(1-\psi_3)]\big\}R_1^{-1} \\ &\quad + \big[2S_2(1+2\cosh\theta) - \tau S_3(1-\phi_3)(2+\psi_3) \\ &\quad - \tau\psi_3q_{23}(2+\phi_3)(1-\psi_3) + \tau^2\psi^3q_{33}(1-\psi_3)(1-\phi_3)\big]R_2^{-1} \\ &\quad + \big\{2\cosh\theta(U_1S_2 - U_2S_1) - \tau(1-\phi_3)(U_1S_3 - U_3S_1) \\ &\quad - \tau^2\phi_1(U_3S_2 - U_2S_3) + \tau\psi_3q_{13}[\tau\psi_1S_2 + (1-\psi_3)U_2] \\ &\quad - \tau\psi_3q_{23}[\tau\psi_1S_1 + (1-\psi_3)U_1]\big\}R_1^{-1}R_2^{-1} \end{aligned} \tag{16}$$

Let[1]

$$m_i \equiv \lambda_i + \lambda_i^{-1}, \qquad i = 1, 2, 3 \tag{17}$$

so that

$$\lambda_i = \frac{m_i \pm \sqrt{\left(m_i^2 - 4\right)}}{2}, \qquad |\lambda_i| > 1 \tag{18}$$

Factorizing the eigenvalue Eq. (15) such that

$$f(\lambda) = \prod_{i=1}^{3}(\lambda - \lambda_i)\left(\lambda - \lambda_i^{-1}\right) \tag{19}$$

we find

$$\begin{gathered} m_1 + m_2 + m_3 = a \qquad m_1m_2 + m_2m_3 + m_3m_1 = b - 3 \\ m_1m_2m_3 = c - 2a \end{gathered} \tag{20}$$

From Eq. (20) we can obtain

$$m_1, m_2 = \frac{(a - m_3) \pm \sqrt{\left[(a-m_3)^2 - 4(c-2a)m_3^{-1}\right]}}{2} \tag{21}$$

and a cubic equation for m_3,

$$m_3^3 - am_3^2 + (b-3)m_3 - (c-2a) = 0 \tag{22}$$

The roots of Eq. (22) can be determined as follows.

Let $\alpha = -a$, $\beta = b - 3$, and $\gamma = 2a - c$; then the cubic equation is obtained as

$$m^3 + \alpha m^2 + \beta m + \gamma = 0 \tag{23}$$

The roots of the cubic equation (23) can be obtained by algebraic processes as follows: Eq. (23) is transformed to the reduced equation

$$\zeta^3 + \eta\zeta + \rho = 0 \tag{24}$$

with

$$\eta = \beta - (\alpha^2/3) \qquad \text{and} \qquad \rho = \gamma - (\alpha\beta/3) + (2\alpha^3/27) \tag{25}$$

and through the substitution $m = \zeta - \alpha/3$.

If $d = (\rho/2)^2 + (\eta/3)^3 \geq 0$, then roots ζ_1, ζ_2, and ζ_3 of the reduced cubic equation (24) are

$$\zeta_1 = \mu + \nu, \qquad \zeta_{2,3} = -[(\mu+\nu)/2] \pm i[(\mu-\nu)/2]\sqrt{3} \qquad i \equiv \sqrt{-1} \tag{26}$$

with

$$\mu = \sqrt[3]{(-\rho/2 + \sqrt{d})}, \qquad \nu = \sqrt[3]{(-\rho/2 - \sqrt{d})} \tag{27}$$

and if $d = (\rho/2)^2 + (\eta/3)^3 < 0$, there are three distinct real solutions:

$$\begin{gathered} \zeta_1 = 2\sqrt{(-\rho/3)}\cos(\vartheta/3) \\ \zeta_2 = 2\sqrt{(-\rho/3)}\cos(\vartheta/3 + 120\,\text{deg}) \\ \zeta_3 = 2\sqrt{(-\rho/3)}\cos(\vartheta/3 + 240\,\text{deg}) \\ \cos\vartheta = \frac{-\rho/2}{\sqrt{-\eta^3/27}} \end{gathered} \tag{28}$$

where for $d > 0$ one root is real and the other two are complex conjugate; for $d = 0$ there are three real roots, and at least two are equal; and for $d < 0$ there are three real roots that are unequal to one another. Let the largest real value of the cube roots be ζ_3. We can then obtain the values of $m_3 = \zeta_3 - \alpha/3$; thereafter m_1 and m_2 are obtained from Eq. (21). With these values in Eq. (18), we can determine the three roots $\lambda_{1,2,3}$.

As our system model is of order 3, H_f is of order 6. If λ is an eigenvalue of H_f, then λ^{-1} is also an eigenvalue of H_f, and hence the eigenvalue problem is of third order only.[4] The eigenvectors w_i corresponding to the eigenvalues λ_i are obtained by direct calculations as

$$W_{11} = \begin{bmatrix} 1 \\ w_{2i} \\ w_{3i} \end{bmatrix}, \qquad W_{21} = \begin{bmatrix} w_{4i} \\ w_{5i} \\ w_{6i} \end{bmatrix} \tag{29}$$

where

$$w_{2i} = \frac{N_{2i}}{D_{2i}} \qquad w_{3i} = \frac{N_{3i}}{D_{3i}} \qquad w_{4i} = \frac{1}{(\lambda_i - 1)R_1}$$
$$w_{5i} = \frac{N_{5i}}{D_{5i}} \qquad w_{6i} = \frac{N_{6i}}{D_{6i}} \tag{30}$$

and

$$\begin{aligned}
N_{2i} &= \tau\theta\lambda_i\Big\{(\psi_3 - \lambda_i)\big[\tau(1-\phi_3)S_1 - \tau^2\phi_1 S_2\big] \\
&\quad - \tau(1-\psi_3)\big[q_{13}\psi_3\tau(1-\phi_3) - q_{23}\psi_3\tau^2\phi_1\big]\Big\} \\
&\quad - (\lambda_i - 1)\Big(\tau(\psi_3 - \lambda_i)\Big\{(1-\phi_3)\big[U_1\lambda_i - R_1(\lambda_i - 1)^2\big] \\
&\quad - U_2\lambda_i\tau\phi_1\Big\} + \tau^2\lambda_i(\theta + 1 - \psi_3) \\
&\quad \times \big[q_{13}\psi_3\tau(1-\phi_3) - q_{23}\psi_3\tau^2\phi_1\big]R_1\Big) \\
D_{2i} &= (\lambda_i - 1)R_1 R_2^{-1}\Big\{(\psi_3 - \lambda_i)\big[\tau(1-\phi_3)S_1 - \tau^2\phi_1 S_2\big] \\
&\quad - \big[\tau(1-\psi_3)(q_{13}\psi_3\tau(1-\phi_3) - q_{23}\psi_3\tau^2\phi_1\big]\Big\} \\
&\quad + (\lambda_i - 1)^2 R_1\Big\{\tau(\psi_3 - \lambda_i)\Big[\big(S_1 R_2^{-1} + \tau\theta\big) \\
&\quad \times (1-\phi_3) - \tau\phi_1\big(S_2 R_2^{-1} + 1 - \lambda_i\big)\Big] \\
&\quad - \tau^2\psi_3(1-\psi_3)R_2^{-1}[q_{13}(1-\phi_3) - q_{23}\tau\phi_1]\Big\} \\
N_{3i} &= [S_3\lambda_i(\lambda_i - \psi_3) + q_{33}\tau\psi_3(\lambda_i - 2)(\lambda_i - 1)]R_1 N_{2i} \\
&\quad + \Big\{\lambda_i(\lambda_i - \psi_3)[U_3(\lambda_i - 1) + S_3\tau\theta] \\
&\quad + q_{33}\tau^2\psi_3[\theta\lambda_i(1-\psi_3) - \psi_1(\lambda_i - 1)]\Big\}R_2 D_{2i}
\end{aligned}$$

$$D_{3i} = (\lambda_i - 1)^2(\lambda_i - \phi_3)(\lambda_i - \psi_3)R_1 R_2 D_{2i}$$

$$N_{5i} = \tau\theta\lambda_i R_2 D_{2i} - (\lambda_i - 1)R_1 N_{2i}$$

$$D_{5i} = (\lambda_i - 1)^2 R_1 R_2 D_{2i}$$

$$\begin{aligned}
N_{6i} &= \big[\tau^2\theta(1-\psi_3)\lambda_i - \tau^2\psi_3(\lambda_i - 1)\big]R_2 D_{2i} \\
&\quad + \tau(1-\psi_3)(\lambda_i - 1)(\lambda_i - 2)R_1 N_{2i}
\end{aligned}$$

$$D_{6i} = (\lambda_i - 1)^2(\lambda_i - \psi_3)R_1 R_2 D_{2i}$$

The steady-state P matrix is

$$P = W_{11}W_{21}^{-1} \tag{31}$$

where W_{11} and W_{21} are determined by the eigenvectors as

$$W_{11} = \begin{bmatrix} 1 & 1 & 1 \\ N_{21}/D_{21} & N_{22}/D_{22} & N_{23}/D_{23} \\ N_{31}/D_{31} & N_{32}/D_{32} & N_{33}/D_{33} \end{bmatrix}$$

$$W_{21}^{-1} = \begin{bmatrix} 1/R_1(\lambda_1 - 1) & 1/R_1(\lambda_2 - 1) & 1/R_1(\lambda_3 - 1) \\ N_{51}/D_{51} & N_{52}/D_{52} & N_{53}/D_{53} \\ N_{61}/D_{61} & N_{62}/D_{62} & N_{63}/D_{63} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} w_{I11} & w_{I12} & w_{I13} \\ w_{I21} & w_{I22} & w_{I23} \\ w_{I31} & w_{I32} & w_{I33} \end{bmatrix} \tag{32}$$

with

$$\begin{aligned}
D_I &= \frac{1}{R_1(\lambda_1 - 1)}\left[\frac{N_{52}N_{63}}{D_{52}D_{63}} - \frac{N_{53}N_{62}}{D_{53}D_{62}}\right] \\
&\quad + \frac{1}{R_1(\lambda_2 - 1)}\left[\frac{N_{53}N_{61}}{D_{53}D_{61}} - \frac{N_{51}N_{63}}{D_{51}D_{63}}\right] \\
&\quad + \frac{1}{R_1(\lambda_3 - 1)}\left[\frac{N_{51}N_{62}}{D_{51}D_{62}} - \frac{N_{52}N_{61}}{D_{52}D_{61}}\right]
\end{aligned}$$

$$w_{I11} = \frac{1}{D_I}\left[\frac{N_{52}N_{63}}{D_{52}D_{63}} - \frac{N_{53}N_{62}}{D_{53}D_{62}}\right]$$

$$w_{I12} = \frac{1}{D_I}\left[\frac{N_{62}}{R_1(\lambda_3 - 1)D_{62}} - \frac{N_{63}}{R_1(\lambda_2 - 1)D_{63}}\right]$$

$$w_{I13} = \frac{1}{D_I}\left[\frac{N_{53}}{R_1(\lambda_2 - 1)D_{53}} - \frac{N_{52}}{R_1(\lambda_3 - 1)D_{52}}\right]$$

$$w_{I21} = \frac{1}{D_I}\left[\frac{N_{53}N_{61}}{D_{53}D_{61}} - \frac{N_{51}N_{63}}{D_{51}D_{63}}\right]$$

$$w_{I22} = \frac{1}{D_I}\left[\frac{N_{63}}{R_1(\lambda_1 - 1)D_{63}} - \frac{N_{61}}{R_1(\lambda_3 - 1)D_{61}}\right]$$

$$w_{I23} = \frac{1}{D_I}\left[\frac{N_{51}}{R_1(\lambda_3 - 1)D_{51}} - \frac{N_{53}}{R_1(\lambda_1 - 1)D_{53}}\right]$$

$$w_{I31} = \frac{1}{D_I}\left[\frac{N_{51}N_{62}}{D_{51}D_{62}} - \frac{N_{52}N_{61}}{D_{52}D_{61}}\right]$$

$$w_{I32} = \frac{1}{D_I}\left[\frac{N_{61}}{R_1(\lambda_2 - 1)D_{61}} - \frac{N_{62}}{R_1(\lambda_1 - 1)D_{62}}\right]$$

$$w_{I33} = \frac{1}{D_I}\left[\frac{N_{52}}{R_1(\lambda_1 - 1)D_{52}} - \frac{N_{51}}{R_1(\lambda_2 - 1)D_{51}}\right]$$

The steady-state covariance $P = W_{11}W_{21}^{-1}$ then yields

$$P_{11} = \sum_{i=1}^{3} w_{Ii1} \tag{33}$$

$$P_{12} = \sum_{i=1}^{3} w_{Ii2} = \sum_{i=1}^{3}\frac{N_{2i}}{D_{2i}}w_{Ii1} \tag{34}$$

$$P_{13} = \sum_{i=1}^{3} w_{Ii3} = \sum_{i=1}^{3}\frac{N_{3i}}{D_{3i}}w_{Ii1} \tag{35}$$

$$P_{22} = \sum_{i=1}^{3}\frac{N_{2i}}{D_{2i}}w_{Ii2} \tag{36}$$

$$P_{23} = \sum_{i=1}^{3}\frac{N_{2i}}{D_{2i}}w_{Ii3} = \sum_{i=1}^{3}\frac{N_{3i}}{D_{3i}}w_{Ii2} \tag{37}$$

$$P_{33} = \sum_{i=1}^{3}\frac{N_{3i}}{D_{3i}}w_{Ii3} \tag{38}$$

IV. Conclusions

In this Note steady-state covariance matrices have been derived for the discrete ECA track filter when the measurement matrix is composed of position and velocity measurements. The MacFarlane–Potter–Fath eigenstructure method gives an answer to the discrete ECA filter in an elegant analytic fashion.

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Robust Controller Design with Damping and Stability Specifications

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Introduction

In many physical systems, controllers must be designed to operate within a nominal domain that covers different stages of operation. Multiple models or a model with parametric uncertainties must be established to represent the dynamics. In dealing with systems characterized by multiple models, there are two methods for designing controllers by state feedback: 1) controllers based on pole assignment[1,2] and 2) controllers based on linear quadratic design.[2,3] Both techniques are well suited for tradeoffs between eigenvalue locations and requirements of robustness against model changes. In dealing with systems characterized by models with parametric uncertainties, robust controller design has gained new interest. By use of the Riccati-equation approach,[4,5] robust controllers have been proposed to ensure the stability of the overall system for all admissible parametric uncertainties. In this Note, the Riccati-equation approach is extended to design robust controllers with damping and stability characteristics.

Statement of the Problem

The linear systems described by the following dynamic equation are considered:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \left[A_0 + \sum_{i=1}^{p} r_i(t)\Delta A_i\right]x(t) + \left[B_0 + \sum_{i=1}^{p} r_i(t)\Delta B_i\right]u(t) \tag{1}$$

where $x \in \Re^n$ and $u \in \Re^m$, the nominal system matrix A_0 and the nominal input connection matrix B_0 are controllable, and each of the parametric uncertainties is modeled by an uncertain variable $r_i(t) \in \Re$ along with given constant matrices ΔA_i and ΔB_i. We assume that all uncertain variables are bounded for all time t such that

$$|r_i(t)| \leq 1 \qquad i = 1, 2, \ldots, p \tag{2}$$

Then, matrices ΔA_i and ΔB_i designate the ranges of deviation of parametric uncertainties.

For systems with state-space models, linear feedback control laws can be written as

$$u(t) = -Kx(t) \tag{3}$$

Thus, closed-loop dynamics are

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_c x(t)$$

$$A_c = A_0 - B_0K + \sum_{i=1}^{p} r_i(t)\{\Delta A_i - \Delta B_i K\} \tag{4}$$

In the following, we denote the closed-loop eigenvalues by

$$\lambda_k(A_c) \qquad k = 1, 2, \ldots, n \tag{5}$$

The eigenvalues are located within the design sector $D(\alpha, \theta)$ in the complex plane if, for a particular choice of real positive numbers α and θ, we have

$$\mathrm{Re}(\lambda_k) < -\alpha \qquad \alpha \geq 0 \tag{6a}$$

$$|\mathrm{Im}(\lambda_k)|\tan(\theta) < -\mathrm{Re}(\lambda_k) - \alpha \qquad \theta \in [0, \pi/2) \tag{6b}$$

Our problem is to determine linear feedback control laws (3) for uncertain linear systems (1) such that, for all admissible uncertainties, the closed-loop poles (5) are located within the design sector $D(\alpha, \theta)$.

The design sector $D(\alpha, \theta)$ has been presented in the following well-known theorem.[6]

Relative Stability Theorem: Given matrix $A_c \in \Re^{n\times n}$, $\lambda_k(A_c)$ lie within $D(\alpha, \theta)$ if and only if the eigenvalues of matrix $H(\theta) \otimes (A_c + \alpha I) \in \Re^{2n\times 2n}$ lie in the left-half complex plane (see Ref. 6 for proof). Here, the matrix $H(\theta) \in \Re^{2\times 2}$ is given by

$$H(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{7}$$

and the $\otimes$ denotes the Kronecker product such that

$$H(\theta) \otimes (A_c + \alpha I) = \begin{bmatrix} \cos(\theta)(A_c + \alpha I) & -\sin(\theta)(A_c + \alpha I) \\ \sin(\theta)(A_c + \alpha I) & \cos(\theta)(A_c + \alpha I) \end{bmatrix} \tag{8}$$

In this Note, a new formulation of the Riccati equation is proposed. The robust controller design is formulated as a linear quadratic state feedback problem with prescribed damping and stability characteristics. Thus, given the bounds of the system parameters, the proposed controller can ensure that the overall system is asymptotically stable with a prescribed degree of damping and stability for all admissible parametric uncertainties.

Method of Controller Design

In this section, the design of robust controllers involves the determination of matrices $P, Q > 0$ (i.e., matrices P and Q are positive definite) such that the following Riccati equation is fulfilled:

$$(A_0 + \alpha I)^T P + P(A_0 + \alpha I) - PB_0(R/2)^{-1}B_0^T P + 2Q = 0 \tag{9}$$

where matrix $R > 0$ is chosen by the designer. The following result can be obtained.

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