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Limitations of Decentralized Control

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The limitations of decentralized control laws are investigated. Necessary conditions for the existence of a decentralized control law that achieves particular feedback system requirements are developed. If these conditions are violated, no decentralized control law can achieve the specified system requirements. The necessary conditions involve only properties of the plant and performance specifications. They are not functions of the control law and can be examined prior to control law synthesis. By evaluating the necessary conditions, it is shown that decentralized control may be unable to achieve a specified complementary sensitivity if large separation of subsystem bandwidths is required. It is also shown that if the system possesses certain distinctive pole/zero locations, then no decentralized control law can stabilize the system. These properties are interpreted in terms of special uncontrollability/unobservability characteristics between the subsystems.

Nomenclature

j	$= (-1)^{1/2}$
$K(s)$, $G(s)$	= compensator, plant transfer function matrices
$S(s)$, $T(s)$	= sensitivity, complementary sensitivity matrices
s	= Laplace transform variable
$\boldsymbol{u}(s)$, $\boldsymbol{y}(s)$	= vectors of plant controls, responses
UB, LB	= upper bound, lower bound
ω	= frequency

Introduction

THE trade between centralized vs decentralized control architecture arises frequently in practical situations. For example, the issue arises when considering integrated flight and propulsion control for aircraft design concepts such as advanced short takeoff/vertical landing (ASTOVL) and advanced supersonic and hypersonic vehicles. The dynamical interactions between the airframe and propulsion subsystems are expected to be more significant for these aircraft and are expected to require an integrated approach to the feedback control of the airframe/propulsion system.[1–4] At issue here is whether fully centralized control laws will be necessary. Such control laws can be more difficult to design and more costly to implement. Decentralized control laws are usually simpler, therefore cheaper, and hence the preferred architecture. Finally, one would like to determine when centralized control laws will be required prior to the synthesis of any particular control law, to avoid unnecessary effort and expense.

Therefore, the limitations of using decentralized control laws are to be investigated. The specific problem addressed is to determine necessary conditions for the existence of a decentralized control law that achieves 1) specified closed-loop performance and 2) closed-loop stability. Of practical interest are cases in which such conditions are not satisfied. Then no decentralized control law can achieve the stated feedback system requirements. Finally, the necessary conditions must be functions of only these requirements and properties of the plant, and not of any candidate control law.

Several authors have addressed stabilizing decentralized control laws for large-scale systems, which assume a large number of subsystems. For example, Wang and Davison[5] addressed eigenvalues of the closed-loop system unaffected by decentralized control, denoted herein as decentralized fixed eigenvalues. Methods for identifying such eigenvalues have been developed by, for example, Anderson and Clements.[6] In this paper, a method is presented for determining a critical subset of decentralized fixed eigenvalues. Systems possessing eigenvalues in this critical subset can be stabilized with centralized control laws, but cannot be stabilized with decentralized control laws. The characteristics of such systems are investigated, focusing on uncontrollability/unobservability conditions between the subsystems and the poles and zeros of the plant transfer functions.

Parametrizations of all stabilizing decentralized control laws have been developed in, for example, Ref. 7. Furthermore, several decentralized control-law synthesis procedures have been proposed, including, for example, optimal H_2 and H_∞ methods,[8] eigenstructure assignment,[9] and methods to approximate characteristics of desirable centralized control laws.[10] Iterative sequential loop closures utilizing structured singular value techniques are presented in Ref. 11, resulting in strictly diagonal control compensation matrices. The closed-loop stability of open-loop stable plants with actuator and/or sensor failures is investigated in Ref. 12 for diagonal decentralized control laws. Finally, the classical formulation of quantitative feedback theory (QFT)[13] also leads to strictly diagonal decentralized control laws. In this paper, however, no specific design methods are investigated, but instead limitations inherent to all decentralized control laws are sought, regardless of synthesis approach. Insightful examples are presented to illustrate the implications of the stability and performance limitations. Performance limitations of decentralized control laws for an integrated airframe/engine system are investigated in Ref. 2.

In the following, the class of systems addressed, and the centralized and decentralized control law architectures are first presented. Then, limitations of decentralized control laws in achieving acceptable performance are addressed. Decentralized fixed eigenvalues are then discussed, and an existence condition derived for a stabilizing decentralized control law. Finally, conclusions are drawn from the work.

System Description

The block diagram of a generic multivariable feedback system is presented in Fig. 1. The system dynamics are defined in terms of the transfer-function matrix $G(s)$, and the control compensation by the transfer-function matrix $K(s)$. The controlled responses, control inputs, and commanded responses are the vectors $\boldsymbol{y}(s)$, $\boldsymbol{u}(s)$, and $\boldsymbol{y}_c(s)$, respectively. Further note the precompensation matrix $P(s)$,

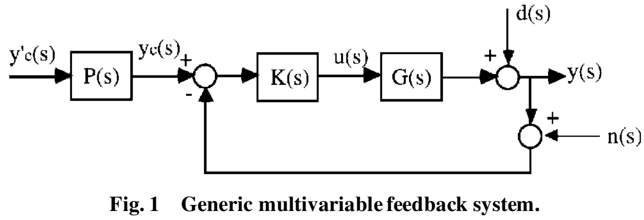

Fig. 1 Generic multivariable feedback system.

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the external disturbances $\boldsymbol{d}(s)$, and the measurement noise vector $\boldsymbol{n}(s)$. The quantities $P(s)$, $\boldsymbol{d}(s)$, and $\boldsymbol{n}(s)$ will be addressed later. For simplicity here, assume $\boldsymbol{d}(s) = 0$ and $\boldsymbol{n}(s) = 0$.

This multivariable system is taken to consist of two interacting subsystems, so that the plant transfer function matrix $G(s)$ may be partitioned in the following manner:

$$\boldsymbol{y}(s) = G(s)\boldsymbol{u}(s) \quad \text{or} \quad \begin{bmatrix} \boldsymbol{y}_1(s) \\ \boldsymbol{y}_2(s) \end{bmatrix} = \begin{bmatrix} G_1(s) & G_{12}(s) \\ G_{21}(s) & G_2(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1(s) \\ \boldsymbol{u}_2(s) \end{bmatrix} \tag{1}$$

The dynamics of subsystems 1 and 2 are defined in terms of the transfer function matrices $G_1(s)$ and $G_2(s)$, respectively, whereas the dynamic coupling between the two subsystems are reflected in the off-diagonal transfer function matrices $G_{12}(s)$ and $G_{21}(s)$.

A centralized control law is defined here as

$$\begin{bmatrix} \boldsymbol{u}_1(s) \\ \boldsymbol{u}_2(s) \end{bmatrix} = \begin{bmatrix} K_1(s) & K_{12}(s) \\ K_{21}(s) & K_2(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_{1c}(s) - \boldsymbol{y}_1(s) \\ \boldsymbol{y}_{2c}(s) - \boldsymbol{y}_2(s) \end{bmatrix} \tag{2}$$

with nonzero crossfeeds between the subsystems, reflected in the matrices $K_{12}(s)$ and $K_{21}(s)$. A decentralized control law is defined here with block diagonal $K(s)$, or one in which $K_{12}(s) = 0$ and $K_{21}(s) = 0$.

Algebraic Constraints of Decentralized Control Laws

Because decentralized control laws are block diagonal, they are subject to certain algebraic constraints. First, let the sensitivity $S(s)$ and complementary sensitivity $T(s)$ transfer function matrices be partitioned compatibly with the plant and control law as

$$S(s) = \begin{bmatrix} S_1(s) & S_{12}(s) \\ S_{21}(s) & S_2(s) \end{bmatrix}, \qquad T(s) = \begin{bmatrix} T_1(s) & T_{12}(s) \\ T_{21}(s) & T_2(s) \end{bmatrix} \tag{3}$$

Attention will be focused on the complementary sensitivity matrix, because a dual always exists for the sensitivity matrix, given that

$$T(s) + S(s) = I \tag{4}$$

Theorem 1: Algebraic constraints of decentralized control. For a plant under decentralized control, if $G_1(j\omega)$ and $G_2(j\omega)$ are square, and if $[K_2^{-1}(j\omega) + G_2(j\omega)]^{-1}$ and $[K_1^{-1}(j\omega) + G_1(j\omega)]^{-1}$ exist $\forall\omega \in [0, \omega_p]$, the following holds:

$$\begin{aligned} T_{12}(j\omega) &= [I - T_1(j\omega)]G_{12}(j\omega)[G_1(j\omega) \\ &\quad - T_{21}(j\omega)G_{12}(j\omega)]^{-1}T_2(j\omega) \qquad \forall\omega \in [0, \omega_p] \end{aligned} \tag{5}$$

$$\begin{aligned} T_{21}(j\omega) &= [I - T_2(j\omega)]G_{21}(j\omega) \\ &\quad \times [G_1(j\omega) - T_{12}(j\omega)G_{21}(j\omega)]^{-1}T_1(j\omega) \qquad \forall\omega \in [0, \omega_p] \end{aligned} \tag{6}$$

where ω_p is the largest frequency of practical interest.

Proof: Since

$$T(j\omega) = G(j\omega)K(j\omega)[I + G(j\omega)K(j\omega)]^{-1} \tag{7}$$

then

$$T(j\omega) = [I - T(j\omega)]G(j\omega)K(j\omega) \tag{8}$$

Solving for $T_{12}(j\omega)$ in the $(1, 2)$ partitioned matrix equation of Eq. (8) yields

$$T_{12}(j\omega) = [I - T_1(j\omega)]G_{12}(j\omega)\big[K_2^{-1}(j\omega) + G_2(j\omega)\big]^{-1} \tag{9}$$

Solving for $K_2^{-1}(j\omega)$ in the $(2, 2)$ partitioned matrix equation of Eq. (8) yields

$$K_2^{-1}(j\omega) = T_2^{-1}(j\omega)[G_2(j\omega) - T_{21}(j\omega)G_{12}(j\omega)] - G_2(j\omega) \tag{10}$$

Substitution of Eq. (10) into Eq. (9) results in Eq. (5). The proof of Eq. (6) is similar. Further details of this proof can be found in Ref. 14.

Therefore, for decentralized control laws only two degrees of freedom, $K_1(j\omega)$ and $K_2(j\omega)$, are available to achieve some desired complementary sensitivity matrix. However, with nonzero off-diagonal elements in $K(j\omega)$, there are four degrees of freedom. Hence, no algebraic constraints such as Eqs. (5) and (6) exist for centralized control laws.

Performance Limitations: Unbounded Gains

The closed-loop response for the feedback system of Fig. 1 is given by

$$\boldsymbol{y}(s) = T(s)P(s)\boldsymbol{y}_c'(s) - T(s)\boldsymbol{n}(s) + S(s)\boldsymbol{d}(s) \tag{11}$$

where again, $T(s)$ and $S(s)$ are the complementary sensitivity and sensitivity transfer function matrices, respectively. The command-following performance is determined by the matrix $T(s)P(s)$. However, note that the closed-loop responses from external disturbances $\boldsymbol{d}(s)$ and external noise inputs $\boldsymbol{n}(s)$ are unaffected by the precompensation matrix $P(s)$. Therefore, even with precompensation, closed-loop performance requirements on noise and/or disturbance rejection will lead to specifications on the complementary sensitivity or sensitivity transfer function matrices. Thus, for acceptable performance let it be required that the singular values of each partition of $T(j\omega)$ lie between upper and lower bounds $\forall\omega \in [0, \omega_p]$. Or

$$\begin{aligned} LB_1(\omega) &\le \underline{\sigma}[T_1(j\omega)], & \bar{\sigma}[T_1(j\omega)] &\le UB_1(\omega) \quad \forall\omega \in [0, \omega_p] \\ LB_{12}(\omega) &\le \underline{\sigma}[T_{12}(j\omega)], & \bar{\sigma}[T_{12}(j\omega)] &\le UB_{12}(\omega) \quad \forall\omega \in [0, \omega_p] \\ LB_{21}(\omega) &\le \underline{\sigma}[T_{21}(j\omega)], & \bar{\sigma}[T_{21}(j\omega)] &\le UB_{21}(\omega) \quad \forall\omega \in [0, \omega_p] \\ LB_2(\omega) &\le \underline{\sigma}[T_2(j\omega)], & \bar{\sigma}[T_2(j\omega)] &\le UB_2(\omega) \quad \forall\omega \in [0, \omega_p] \end{aligned} \tag{12}$$

where $\underline{\sigma}$ and $\bar{\sigma}$ denote the minimum and maximum singular values and LB_{ij} and UB_{ij} denote the specified lower and upper bounds on the singular values of $T_{ij}(j\omega)$, respectively. Note that no assumptions are made as to the shapes of these bounds.

Theorem 2: Necessary conditions for acceptable complementary sensitivity. The necessary conditions for the existence of a decentralized control law that achieves acceptable performance, defined by Eq. (12), are the following:

$$UB_{12}(\omega) \ge \frac{\delta_1(\omega)\underline{\sigma}[G_{12}(j\omega)]LB_2(\omega)}{\bar{\sigma}[G_2(j\omega)] + UB_{21}(\omega)\bar{\sigma}[G_{12}(j\omega)]}, \qquad \forall\omega \in [0, \omega_p] \tag{13}$$

$$UB_{21}(\omega) \ge \frac{\delta_2(\omega)\underline{\sigma}[G_{21}(j\omega)]LB_1(\omega)}{\bar{\sigma}[G_1(j\omega)] + UB_{12}(\omega)\bar{\sigma}[G_{21}(j\omega)]}, \qquad \forall\omega \in [0, \omega_p] \tag{14}$$

$$LB_{12}(\omega) \le \frac{UB_2(\omega)[1 + UB_1(\omega)]\bar{\sigma}[G_{12}(j\omega)]}{\delta_{21}(\omega)}, \qquad \forall\omega \in [0, \omega_p] \tag{15}$$

$$LB_{21}(\omega) \le \frac{UB_1(\omega)[1 + UB_2(\omega)]\bar{\sigma}[G_{21}(j\omega)]}{\delta_{12}(\omega)}, \qquad \forall\omega \in [0, \omega_p] \tag{16}$$

where

$$\delta_1(\omega) \triangleq \max\{0, 1 - UB_1(\omega), LB_1(\omega) - 1\} \tag{17}$$

$$\begin{aligned} \delta_{21}(\omega) \triangleq \max\{&0, \underline{\sigma}[G_2(j\omega)] - UB_{21}(\omega)\,\bar{\sigma}[G_{12}(j\omega)], \\ &LB_{21}(\omega)\underline{\sigma}[G_{12}(j\omega)] - \bar{\sigma}[G_2(j\omega)]\} \end{aligned} \tag{18}$$

Also, $\delta_2(\omega)$ and $\delta_{12}(\omega)$ are duals of $\delta_1(\omega)$ and $\delta_{21}(\omega)$, respectively, and are obtained by simply interchanging 1 and 2 in the subscripts of Eqs. (17) and (18).

Proof: (For simplicity, the dependence on $j\omega$ has been dropped from the notation.) From Eq. (5),

$$\bar{\sigma}(T_{12}) = \bar{\sigma}\big\{[I - T_1]G_{12}[G_2 - T_{21}G_{12}]^{-1}T_2\big\} \tag{19}$$

Now, consider matrices X and Y. From Ref. 15, the following singular value properties can be derived. First, if X is $m \times n$ and Y is $n \times p$, then

$$\underline{\sigma}\{X\}\underline{\sigma}\{Y\} \le \underline{\sigma}\{XY\} \le \bar{\sigma}\{XY\} \le \bar{\sigma}\{X\}\bar{\sigma}\{Y\} \tag{20}$$

$$\underline{\sigma}\{XY\} \le \bar{\sigma}\{X\}\underline{\sigma}\{Y\}, \qquad \underline{\sigma}\{XY\} \le \underline{\sigma}\{X\}\bar{\sigma}\{Y\} \tag{21}$$

if $n \le p$,

$$\bar{\sigma}\{X\}\underline{\sigma}\{Y\} \le \bar{\sigma}\{XY\} \tag{22a}$$

if $n \le m$,

$$\underline{\sigma}\{X\}\bar{\sigma}\{Y\} \le \bar{\sigma}\{XY\} \tag{22b}$$

if X, Y are $m \times n$,

$$\bar{\sigma}\{X \pm Y\} \le \bar{\sigma}\{X\} + \bar{\sigma}\{Y\} \tag{23}$$

and

$$\max(0, \underline{\sigma}\{X\} - \bar{\sigma}\{Y\}, \underline{\sigma}\{Y\} - \bar{\sigma}\{X\}) \le \underline{\sigma}\{X \pm Y\} \tag{24}$$

If X $m \times m$ and X^{-1} exists,

$$\underline{\sigma}\{X^{-1}\} = 1/\bar{\sigma}\{X\} \tag{25}$$

Finally, note that $\underline{\sigma}\{I\} = \bar{\sigma}\{I\} = 1$. Applying these identities to Eq. (19) results in

$$\bar{\sigma}\{T_{12}\} \ge \frac{\max(0, 1 - \bar{\sigma}\{T_1\}, \underline{\sigma}\{T_1\} - 1)\underline{\sigma}\{G_{12}\}\underline{\sigma}\{T_2\}}{\bar{\sigma}\{G_2\} + \bar{\sigma}\{T_{21}\}\bar{\sigma}\{G_{12}\}} \tag{26}$$

To satisfy the bounds in Eq. (12), from Eq. (26) it is necessary that

$$UB_{12} \ge \bar{\sigma}\{T_{12}\} \ge \frac{\max(0, 1 - UB_1, LB_1 - 1)\underline{\sigma}\{G_{12}\}LB_2}{\bar{\sigma}\{G_2\} + UB_{21}\bar{\sigma}\{G_{12}\}} \tag{27}$$

Noting the definition of δ_1 in Eq. (17), see that the right-hand side of the preceding inequality is the right-hand side of the inequality of Eq. (13), proving that Eq. (13) is a necessary condition. Using the properties of Eqs. (20–25), the proofs of Eqs. (14–16) follow similarly. Further details of this proof are in Ref. 14.

In other words, if Eq. (13) is violated, for example, then for any decentralized control law for which the singular values of $T_1(j\omega)$, $T_{21}(j\omega)$, and $T_2(j\omega)$ lie within their allowable bounds, the upper bound on $\bar{\sigma}[T_{12}(j\omega)]$ will be violated.

Finally, note that the necessary conditions of Eqs. (13–16) are functions only of the properties of the plant and performance requirements. They are, however, dependent on the units selected for the plant, except in the case of a 2×2 plant. A scaling algorithm has been developed to remove the dependence on the units, and this algorithm is presented in Ref. 14. The algorithm attempts to find a set of units such that all inequalities of interest are satisfied for all frequencies $\omega \in [0, \omega_p]$.

Example: Let the multivariable plant $G(s)$ in Eq. (1) be 2×2, so that the partitioned elements of $G(s)$ are scalars. (All scalars will be denoted in lower case.) Let the magnitudes of the elements of $G(j\omega)$ be as shown in Fig. 2. Note that the coupling in one direction, $g_{21}(j\omega)$, is comparatively significant, with $|g_{21}(j\omega)|$ roughly 40 dB larger than $|g_1(j\omega)|$. Furthermore, let performance specifications give rise to bounds on $|t_1(j\omega)|$ and $|t_2(j\omega)|$ as presented in Fig. 3. Note that the required closed-loop bandwidth of subsystem 2 is approximately one decade greater than that of subsystem 1. Finally, let performance specifications also lead to upper bounds on $|t_{21}(j\omega)|$ and $|t_{12}(j\omega)|$ as shown in Figs. 4 and 5, respectively. Further discussion and examples of how such bounds may be specified are given in Refs. 2 and 14. Note also that deriving such bounds is a fundamental part of QFT control law synthesis.[13]

Evaluating the right-hand sides of Eqs. (14) and (13) yields the results also shown in Figs. 4 and 5, respectively. For simplicity of notation, $\gamma_{12}(\omega)$ denotes the right-hand side of Eq. (13), and $\gamma_{21}(\omega)$ is the dual of $\gamma_{12}(\omega)$, the right-hand side of Eq. (14). Note from Fig. 4 that the inequality of Eq. (14) is violated for frequencies greater than 6 rad/s. Therefore, no decentralized control law can meet the given performance requirements on $T(j\omega)$. This is due, in large part, to the significant coupling reflected in $|g_{21}(j\omega)|$ as compared to $|g_1(j\omega)|$ (see Fig. 2). Increased coupling increases the right-hand side of Eq. (14), and it is, therefore, more likely that the inequality is violated.

Likewise, Fig. 5 shows that Eq. (13) is violated for $0.7 \le \omega \le 15$ rad/s. In this range, $LB_2(\omega) \approx 1$ and $UB_1(\omega) \ll 1$ (see Fig. 3). Hence, $\delta_1(\omega) \approx 1$ [see Eq. (17)]. Because of this, and because the upper bound on $|t_{21}(j\omega)|$ is small, the numerator of the right-hand side of Eq. (13) is approximately $|g_{12}(j\omega)|/|g_2(j\omega)|$. However, the inequality is violated, even with modest coupling $g_{12}(j\omega)$, since

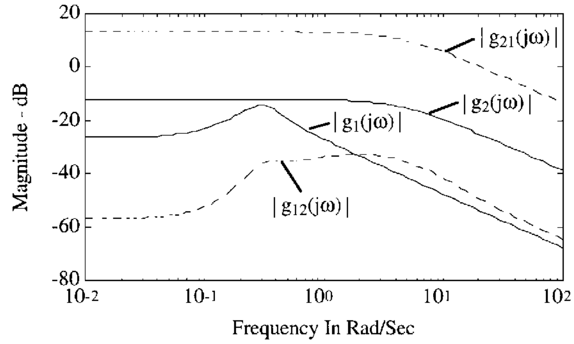

Fig. 2 Magnitudes of elements in $G(j\omega)$.

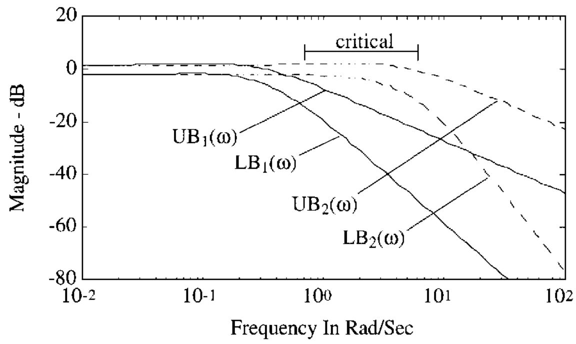

Fig. 3 Allowable bounds on $|t_1(j\omega)|$ and $|t_2(j\omega)|$.

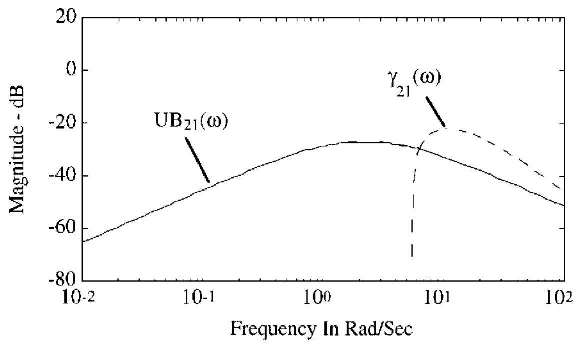

Fig. 4 Necessary condition for acceptable performance [Eq. (14)].

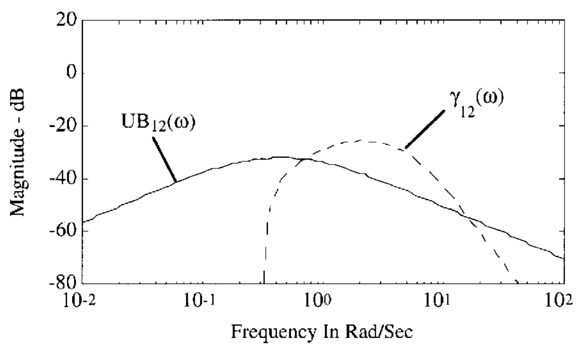

Fig. 5 Necessary condition for acceptable performance [Eq. (13)].

it requires the (small) upper bound on $|t_{12}(j\omega)|$ to be greater than $|g_{12}(j\omega)|/|g_2(j\omega)|$. Hence, if the bandwidth of subsystem 1 must be much lower than the bandwidth of subsystem 2, and $|t_{12}(j\omega)|$ and $|t_{21}(j\omega)|$ must be small, then Eq. (13) will be difficult to satisfy. For this reason, $0.7 \le \omega \le 7$ rad/s is indicated as critical in Fig. 3. Conversely, if the bandwidth of subsystem 2 must be much lower than the bandwidth of subsystem 1, then Eq. (14) would be likely to fail.

Finally note (Figs. 4 and 5) that Eqs. (14) and (13) are trivially satisfied at low frequencies. From Fig. 3, below 0.3 rad/s, $LB_1(\omega) \le 1 \le UB_1(\omega)$, and below 5 rad/s, $LB_2(\omega) \le 1 \le UB_2(\omega)$. Thus, $\delta_1(\omega) = 0$ and $\delta_2(\omega) = 0$ below 0.3 and 5 rad/s, respectively [see Eq. (17)]. Consistent with this result, Figs. 3–5 indicate that $T(j\omega)$ is required to approximate the identity matrix at lower frequencies. This can be achieved by decentralized control if $k_1(j\omega)$ and $k_2(j\omega)$ are sufficiently large [see Eq. (7)].

Performance Limitations: Bounded Gains

In most cases, the magnitudes of feedback gains are limited due to, for example, actuator rate and deflection limits. The effect of such limitations will now be addressed. Assume the following upper bounds on $\bar{\sigma}[K_1(j\omega)]$ and $\bar{\sigma}[K_2(j\omega)]$ are specified:

$$\bar{\sigma}[K_1(j\omega)] \le UB_{K1}(\omega) \qquad \text{and} \qquad \bar{\sigma}[K_2(j\omega)] \le UB_{K2}(\omega) \qquad \forall\omega \in [0, \omega_p] \tag{28}$$

Such bounds lead to additional necessary conditions.

Theorem 3: Necessary conditions for acceptable complementary sensitivity with bounded feedback gains. Necessary conditions for the existence of a decentralized control law that achieves acceptable performance, defined by Eq. (12), and does not violate the bounds defined by Eq. (28) are the following:

$$UB_{12}(\omega) \ge \gamma'_{12}(\omega), \qquad \forall\omega \in [0, \omega_p] \tag{29}$$

$$UB_{21}(\omega) \ge \gamma'_{21}(\omega), \qquad \forall\omega \in [0, \omega_p] \tag{30}$$

where

$$\begin{aligned} \gamma'_{12}(\omega) &\triangleq \frac{\underline{\sigma}[G_{12}(j\omega)]LB_2(\omega)}{d} \\ d &\triangleq \{1 + \bar{\sigma}[G_1(j\omega)]UB_{K1}(\omega)\} \\ &\quad \times \{\bar{\sigma}[G_2(j\omega)] + UB_{21}(\omega)\bar{\sigma}[G_{12}(j\omega)]\} \\ &\quad + \underline{\sigma}[G_{12}(j\omega)]\,\bar{\sigma}[G_{21}(j\omega)]LB_2(\omega)UB_{K1}(\omega) \end{aligned} \tag{31}$$

with $\gamma'_{21}(\omega)$ the dual of $\gamma'_{12}(\omega)$.

Proof: Solving for $I - T_1$ using the (1, 1) partition of Eq. (8) yields

$$I - T_1 = [I + T_{12}G_{21}K_1][I + G_1K_1]^{-1} \tag{32}$$

Substituting this into Eq. (5) and rearranging gives

$$T_{12} = [I + T_{12}G_{21}K_1][I + G_1K_1]^{-1}G_{12}[G_2 - T_{21}G_{12}]^{-1}T_2 \tag{33}$$

Again, applying the singular value properties of Eqs. (20–25) to Eq. (33) results in

$$\bar{\sigma}(T_{12}) \ge \frac{\underline{\sigma}(G_{12})\underline{\sigma}(T_2)}{[1 + \bar{\sigma}(G_1)\bar{\sigma}(K_1)][\bar{\sigma}(G_2) + \bar{\sigma}(T_{21})\bar{\sigma}(G_{12})] + \underline{\sigma}(G_{12})\bar{\sigma}(G_{21})\underline{\sigma}(T_2)\bar{\sigma}(K_1)} \tag{34}$$

To satisfy the bounds in Eqs. (12) and (28), from Eq. (34) it is necessary that the inequality of Eq. (29) be satisfied. The proof of Eq. (30) is similar, and further details of this proof are in Ref. 14.

Consider again the previous example, and let an upper bound on $|k_1(j\omega)|$ be

$$UB_{k1}(\omega) = 1/|j\omega|, \qquad \forall\omega \in [0, \omega_p], \qquad \omega_p = 100\,\text{rad/s} \tag{35}$$

Evaluating Eq. (29) leads to the results shown in Fig. 6. Here, from approximately 0.3 to 15 rad/s the inequality is violated. Also, $\gamma_{12}(\omega)$ from Fig. 5 is replotted for comparison. This result shows how bounded feedback gains make acceptable performance more difficult to achieve.

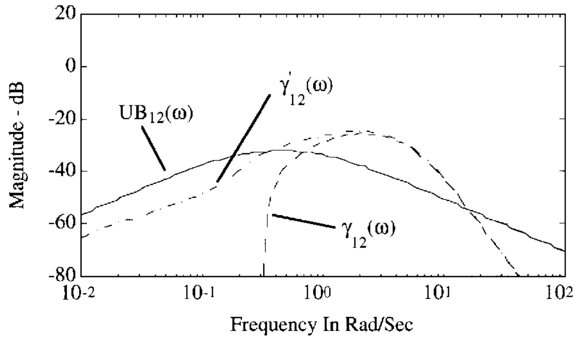

Fig. 6 Necessary condition with bounded feedback gain.

Note that the necessary conditions of Eqs. (13–16) and Eqs. (29) and (30) can be optimistic in that for some systems no decentralized control law may exist that can achieve the required performance, yet all necessary conditions are satisfied. Approaches to reducing this optimism are currently being addressed. However, in spite of the optimism, some of the conditions were still violated for an integrated airframe/engine model, investigated in Ref. 2.

Finally, note that additional necessary conditions for the existence of a decentralized control law that achieves an acceptable loop transfer matrix were derived, and these conditions are presented in Ref. 14.

Stability Limitations

Although stated differently, the following definition is consistent with that given by Wang and Davison.[5]

Definition of decentralized fixed eigenvalues: Here p is a decentralized fixed eigenvalue (DFE) of the system if p is an eigenvalue of the open-loop system and p is an eigenvalue of the closed-loop system for all decentralized control laws.

Note that eigenvalues associated with uncontrollable and/or unobservable modes are a subset of all DFEs, because if these eigenvalues are invariant under centralized control, they are clearly invariant under decentralized control. However, some DFEs may be both (centralized) controllable and observable. Such eigenvalues are of particular interest here and are denoted as controllable and observable decentralized fixed eigenvalues (CODFEs).

Theorem 4: Necessary condition for stability. A necessary condition for the existence of a stabilizing decentralized control law is that the system possess no unstable CODFEs. The proof follows from the definition of a CODFE.

Note that a stabilizing centralized control law does exist for such a system, by definition. The focus here, then, is to determine whether a system possesses CODFEs and the implications of such a property.

Theorem 5: Existence of distinct CODFEs. Let the n-dimensional system be described by

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} = A\boldsymbol{x} + [B_1 \quad B_2]\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix} \\ \boldsymbol{y} &= \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix} = C\boldsymbol{x} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\boldsymbol{x} \end{aligned} \tag{36}$$

where the matrices in Eq. (36) are partitioned compatibly with the plant $G(s)$ in Eq. (1). That is,

$$\begin{aligned} G_1(s) &= C_1(sI - A)^{-1}B_1, \qquad G_{12}(s) = C_1(sI - A)^{-1}B_2 \\ G_{21}(s) &= C_2(sI - A)^{-1}B_1, \qquad G_2(s) = C_2(sI - A)^{-1}B_2 \end{aligned} \tag{37}$$

Let p be a distinct eigenvalue of A associated with a controllable and observable mode. Then p is a distinct CODFE if and only if either

$$\text{rank}(H_{12}) = n - 1, \qquad \text{or} \qquad \text{rank}(H_{21}) = n - 1 \tag{38}$$

where

$$H_{12} = \begin{bmatrix} pI - A & B_2 \\ C_1 & 0 \end{bmatrix} \quad \text{and} \quad H_{21} = \begin{bmatrix} pI - A & B_1 \\ C_2 & 0 \end{bmatrix} \tag{39}$$

Proof (Abbreviated) 1) *Necessity:* Assume p is a distinct CODFE. By definition, p is an eigenvalue of the closed-loop system for all real constant decentralized control laws. Hence,

$$\text{rank}[pI - A_{\text{cl}}] = \text{rank}[pI - A + B_1K_1C_1 + B_2K_2C_2] \leq n - 1 \tag{40}$$

for all K_1 and K_2. Here, A_{cl} denotes the closed-loop system dynamic matrix for a real constant decentralized control law. [Note that the reason for the inequality in Eq. (40) is that some K_1 and K_2 may exist such that other open-loop eigenvalues are moved to the location of p in the closed-loop system.]

In Ref. 14 it is shown that because p is associated with a (centralized) controllable and observable mode, and because Eq. (40) holds for all K_1 and K_2, then matrices M_B and M_C exist such that

$$\begin{gathered} B_2 = [pI - A]M_B, \qquad C_1 = M_C[pI - A] \\ M_C[pI - A]M_B = 0 \end{gathered} \tag{41}$$

or the dual case, with B_1 and C_2 substituted in for B_2 and C_1. For the moment, assume the former case is true, and define the following matrices:

$$R_C \triangleq \begin{bmatrix} I & 0 \\ M_C & I \end{bmatrix}, \quad Z_P \triangleq \begin{bmatrix} pI - A & 0 \\ 0 & 0 \end{bmatrix}, \quad R_B \triangleq \begin{bmatrix} I & M_B \\ 0 & I \end{bmatrix} \tag{42}$$

Using the relationships in Eq. (41), it can be shown that

$$R_C Z_P R_B = H_{12} \tag{43}$$

where H_{12} is defined in Eq. (39). It can also be shown that both R_C and R_B are square, full rank matrices, and this implies that

$$\text{rank}(Z_P) = \text{rank}(H_{12}) \tag{44}$$

Now, from Eq. (42), it is evident that $\text{rank}(Z_P) = \text{rank}[pI - A]$. Since p is a distinct eigenvalue of the system, $\text{rank}[pI - A] = n - 1$. Thus, Eq. (44) proves that $\text{rank}(H_{12}) = n - 1$. If the dual of Eq. (41) is true, it can be shown that $\text{rank}(H_{21}) = n - 1$.

2) *Sufficiency:* First, assume $\text{rank}(H_{12}) = n - 1$. Then, for real constant decentralized control laws, see that

$$[pI - A_{\text{cl}}] = [I \quad B_1K_1]H_{12}\begin{bmatrix} I \\ K_2C_2 \end{bmatrix} \tag{45}$$

Now, for any matrices, say, W and X, $\text{rank}(WX) \leq \min\{\text{rank}(W), \text{rank}(X)\}$ (Ref. 16). Therefore, Eq. (45) implies

$$\text{rank}[pI - A_{\text{cl}}] \leq \min\left\{\text{rank}[I \quad B_1K_1], \text{rank}(H_{12}), \text{rank}\begin{bmatrix} I \\ K_2C_2 \end{bmatrix}\right\} \tag{46}$$

Since it is assumed that $\text{rank}(H_{12}) = n - 1$, Eq. (46) implies Eq. (40) is satisfied, which implies p is an eigenvalue of the closed-loop system for all real constant decentralized control laws. In a dual manner, it can be shown that if $\text{rank}(H_{21}) = n - 1$, then Eq. (40) is also satisfied. By similar means, it can be shown that p is an eigenvalue of the closed-loop system for all dynamic decentralized control laws as well. This theorem, along with the more general case in which p is of multiplicity ≥ 1, are proven in greater detail in Ref. 14.

For the remainder of this section, attention will be focused on the case in which $\text{rank}(H_{12}) = n - 1$. Dual arguments and examples are apparent for the case in which $\text{rank}(H_{21}) = n - 1$.

Corollary 1: Uncontrollability/unobservability implications. If p is a CODFE due to $\text{rank}(H_{12}) = n - 1$, then the mode associated with p is unobservable in (A, C_1), and uncontrollable in (A, B_2).

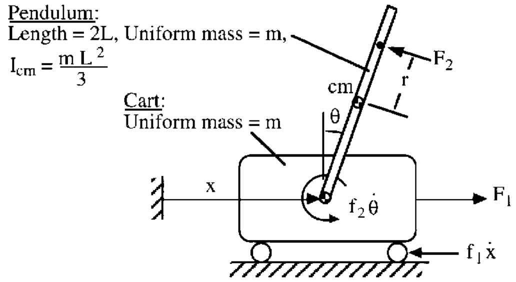

Fig. 7 Cart/inverted pendulum system.

Proof (Abbreviated): If $\text{rank}(H_{12}) = n - 1$, then $\text{rank}[pI - A \quad B_2] \leq n - 1$, since a subset of the rows of H_{12} cannot have rank greater than H_{12}. But, since $\text{rank}[pI - A] = n - 1$, then

$$\text{rank}\,[pI - A \quad B_2] = n - 1 \tag{47}$$

By the same arguments

$$\text{rank}\begin{bmatrix} pI - A \\ C_1 \end{bmatrix} = n - 1 \tag{48}$$

Therefore, p is associated with an uncontrollable mode in (A, B_2), and an unobservable mode in (A, C_1) (Ref. 17).

Hence, from Eq. (36), the mode associated with p cannot be observed in $\boldsymbol{y}_1(\boldsymbol{s})$, and cannot be affected by $\boldsymbol{u}_2(\boldsymbol{s})$. The implications of this property are demonstrated in the following example.

Example: The venerable cart with inverted pendulum is illustrated in Fig. 7. As indicated in the figure, friction at the cart wheels is $f_1\dot{x}$, and at the pin connection is $f_2\dot{\theta}$ (f_1 and $f_2 > 0$). The force F_2 is applied perpendicular to the pendulum, and its point of application is located a distance r from the center of mass (or midpoint) of the pendulum.

Linearizing the equations of motion about the fixed, inverted ($\theta = 0$) condition yields

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + [B_1 \quad B_2]\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{49a}$$

or

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -4af_1L & -bL & 3af_2 \\ 0 & 0 & 0 & 1 \\ 0 & 3af_1 & 2b & -6af_2/L \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4aL & a(3r - L) \\ 0 & 0 \\ -3a & -3a(2r/L + 1) \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \tag{49b}$$

where $a \triangleq 1/(5\,mL)$ and $b \triangleq 3g/(5L)$. It can be shown that this system possesses one real unstable eigenvalue, denoted as p in the following discussion.

Let the measurement y_1 be a function only of cart position and velocity and let y_2 be the pendulum angle θ. That is, let

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{50}$$

Further, let all parameters of the system be fixed except r, c_{11}, and c_{12}. It can then be shown that a physically realizable A, B_2, and C_1 exist such that $\text{rank}(H_{12}) = n - 1$. In this case,

$$r = r^* = -(h/p + L), \quad \text{where} \quad h = \frac{3(f_2p - mgL)}{L(5mp + 4f_1)} \tag{51}$$

Table 1 Example transfer functions

Case	Specifications	$g_{12}(s)$	$g_{21}(s)$
A: p controllable and observable, but not a CODFE	$b_{12} \neq 0, c_{22} \neq 0$	$\dfrac{R_{12}(s-z_{12})(s-p)}{(s-q)(s-p)(s-r)}$	$\dfrac{R_{21}(s-z_{211})(s-z_{212})}{(s-q)(s-p)(s-r)}$
B: p is a CODFE	$b_{12} \neq 0, c_{22} \neq 0$, $c_{11} = \dfrac{c_{13}b_{23}(p-q)}{b_{21}(r-p)}$	$\dfrac{R_{12}(s-p)(s-p)}{(s-q)(s-p)(s-r)}$	$\dfrac{R_{21}(s-z_{211})(s-z_{212})}{(s-q)(s-p)(s-r)}$
C: p uncontrollable and/or unobservable	$b_{12} = 0$ and/or $c_{22} = 0$	$\dfrac{R_{12}(s-z_{12})(s-p)}{(s-q)(s-p)(s-r)}$	$\dfrac{R_{21}(s-z_{211})(s-p)}{(s-q)(s-p)(s-r)}$

and

$$y = y_1^* = [\alpha p \quad -\alpha \quad 0 \quad 0]\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{52}$$

where α is an arbitrary real, nonzero constant. Additionally, with B_1 as in Eq. (49) and C_2 as in Eq. (50), it can be shown that p is associated with a controllable and observable mode. Therefore, by Theorem 5, p is a CODFE.

Note that the point on the pendulum at r^* remains at its undeflected position if one examines the mode shape (eigenvector) associated with p. Therefore, this mode is uncontrollable from a force applied at this point. Furthermore, under a decentralized control law, $F_1 = -k_1 y_1^*$, and it can be shown that the mode associated with p is unobservable in y_1^*. Thus, the force on the pendulum cannot affect the instability because of where it is applied, and the force on the cart cannot stabilize the system because the unstable mode cannot be detected in the cart measurement.

Corollary 2: Uncontrollability/unobservability implications in $G(s)$. If p is a CODFE due to rank$(H_{12}) = n - 1$, then p is unobservable in $G_1(s)$, uncontrollable in $G_2(s)$, and both uncontrollable and unobservable in $G_{12}(s)$.

Proof: The proof is straightforward from Eq. (37) and Corollary 1. Note, however, that if p is unobservable in $G_1(s)$, uncontrollable in $G_2(s)$, and both uncontrollable and unobservable in $G_{12}(s)$, this does not, in general, guarantee that p is a CODFE.

Corollary 3: Transmission zero implications. Assume that B_2 and C_1 are full rank, and let B_2 be $n \times m_2$ and C_1 be $p_1 \times n$. If p is a CODFE due to rank$(H_{12}) = n - 1$, then n_Z transmission zeros of $G_{12}(s)$ are located at p, where $n_Z = \min\{m_2, p_1\} + 1$.

Proof: First note that H_{12} is $(n + p_1) \times (n + m_2)$. If B_2 and C_1 are full rank and p is a distinct eigenvalue, then n_Z transmission zeros of $G_{12}(s)$ are located at p if[17]

$$\text{rank}(H_{12}) = n + \min\{m_2, p_1\} - n_Z, n_Z > 0 \tag{53}$$

Therefore, if p is a CODFE due to rank$(H_{12}) = n - 1$, then this result implies $n_Z = \min\{m_2, p_1\} + 1$, which proves the corollary.

The properties addressed in Corollaries 2 and 3 will be demonstrated in the following example.

Example: With reference to the system description of Eq. (36), consider

$$A = \begin{bmatrix} q & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & r \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{21} \\ 0 \\ b_{23} \end{bmatrix} \tag{54}$$
$$C_1 = [c_{11} \quad 0 \quad c_{13}], \qquad C_2 = [c_{21} \quad c_{22} \quad c_{23}]$$

This system is in modal coordinates, it is assumed that the eigenvalues are distinct, and the focus of this example will be the eigenvalue p. See that the mode associated with the eigenvalue p is unobservable in y_1 and uncontrollable from u_2 (Ref. 18). The elements in the B and C matrices are assumed nonzero, but otherwise arbitrary, except for the specifications given in Table 1.

From Eq. (37) one obtains

$$g_1(s) = \frac{R_1(s - z_1)(s - p)}{(s - q)(s - p)(s - r)}$$
$$g_2(s) = \frac{R_2(s - z_2)(s - p)}{(s - q)(s - p)(s - r)} \tag{55}$$

where R_1, R_2, z_1, and z_2 are functions of the elements in the matrices in Eq. (54). The transfer functions for $g_{12}(s)$ and $g_{21}(s)$ are given in Table 1 for three cases.

Under a centralized control law, the subsystem 1 loop transfer function (i.e., one loop broken at u_1) can be shown to be

$$l(s) = \frac{g_1(s)k_1(s) + [g_1(s)g_2(s) - g_{12}(s)g_{21}(s)]\mathcal{K} + g_{12}(s)k_{21}(s)}{1 + g_2(s)k_2(s) + g_{21}(s)k_{12}(s)} \tag{56}$$

where $\mathcal{K} = k_1(s)k_2(s) - k_{12}(s)k_{21}(s)$. [It is assumed here that $(s - p)$ is not a factor in either the numerators or denominators of $k_1(s)$, $k_{12}(s)$, $k_{21}(s)$, or $k_2(s)$.] Furthermore, letting $k_{12}(s) = k_{21}(s) = 0$ in Eq. (56) yields the loop transfer function under decentralized control.

In case A in Table 1, note that all of the factors $(s - p)$ are not canceled in the product $g_{12}(s)g_{21}(s)$. Because of this, there is no pole-zero cancellation at p in the loop transfer function for decentralized control [Eq. (56)], and p is not a root of the closed-loop characteristic polynomial.[19] Thus, p is not invariant under decentralized control even though the mode associated with p is unobservable in $y_1(s)$ and uncontrollable from $u_2(\mathbf{s})$.

In case B, the additional specification on c_{11} causes rank$(H_{12}) = n - 1$, thus rendering p a CODFE by Theorem 5. Note from Eq. (55) and Table 1 that the pole-zero cancellations at p in the transfer functions are consistent with Corollary 2. Further, $g_{12}(s)$ has two zeros at p, consistent with Corollary 3. Because of this, the factor $(s - p)$ is now canceled in the product $g_{12}(s)g_{21}(s)$, and it can be shown that this results in a pole-zero cancellation at p in the loop transfer function for decentralized control [Eq. (56)]. Therefore, the eigenvalue p cannot be affected by decentralized control. However, under centralized control, the factor $(s - p)$ is not canceled in $g_{21}(s)k_{12}$ in the denominator of Eq. (56), and p is, therefore, not a root of the closed-loop characteristic polynomial. A critical observation can be made here for this 2×2 $G(s)$. If p is an unstable CODFE due to rank$(H_{12}) = n - 1$, then p can be stabilized by adding only the one crossfeed, $k_{12}(s)$. The crossfeed $k_{21}(s)$ can remain zero. Likewise, in the dual situation [rank$(H_{21}) = n - 1$], p can be stabilized by adding only the crossfeed $k_{21}(s)$.

Finally, in case C, the factor $(s - p)$ is canceled in each plant transfer function. Hence, it is canceled in all terms in Eq. (56) including $g_{21}(s)k_{12}$. This results in a pole-zero cancellation at p in the loop transfer function for either centralized or decentralized control, rendering p uncontrollable and/or unobservable.

Conclusions

Limitations of decentralized control laws were investigated. Necessary conditions for the existence of a decentralized control law that meets specified feedback system requirements were developed. If these necessary conditions are violated, centralized control laws

are required. It was shown that if certain frequency dependent inequalities are violated, no decentralized control law can achieve specified complementary sensitivity functions. These inequalities are functions only of properties of the plant and the performance specifications and can be evaluated prior to control law synthesis. It was seen that decentralized control can be limited in terms of achieving acceptable performance if the required bandwidths of the closed-loop subsystems greatly differ. Additional necessary conditions were developed to include specifications on the magnitudes of the feedback gains, and it was noted that such specifications further limit decentralized controllers.

Some systems cannot be stabilized under decentralized control. It was demonstrated that such systems exhibit special uncontrollability and unobservability properties, and the implications of these lead to distinctive properties of the poles and zeros of the plant subsystems. It was shown that zeros of the off-diagonal (or coupling) transfer functions play a key role in determining whether decentralized control laws can stabilize the system. It was also observed that for a 2×2 system, the addition of only one off-diagonal control element (or crossfeed) is required to stabilize a system that cannot be stabilized by decentralized control.

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