3) There exists a Lyapunov function when $x \in S$.
Consider

$$V = \omega_n^2[(k_1/k) - 1]x_1^2 + x_2^2, \qquad k_1 > k \tag{19}$$

then $x \in S$ implies

$$U(x) = k_1(x_1 - X_1) + \text{sat}(k_1 k_2 x_2) \tag{20}$$

Computing $\dot{V}$ from Eq. (19) and substituting Eqs. (3) and (20) into the $\dot{V}$ equation yields

$$\dot{V} = -2k_1\left(\omega_n^2/k^2\right)x_2 \,\text{sgn}(x_2)(\Lambda - 1) - 2\left(\omega_n^2/k\right)x_2 \,\text{sat}(k_1 k_2 x_2)$$

$$\Lambda = \sqrt{1 + (kk_2/\omega_n)^2} \tag{21}$$

Because $\Lambda \geq 1$, it follows from Eq. (21) that $\dot{V} \leq 0$ and $\dot{V} = 0$ if and only if $x_2 = 0$. Because $\dot{V} = 0$ for $x_2 = 0$, we must show that $x = 0$ is the only invariant subset of $\{x : x_2 = 0\}$. Substituting $x_2 = 0$ into Eqs. (9) and (14), we obtain $U = u = k_1 x_1$. Thus, for $x_1 \neq 0$, then $U = u \neq 0$, which implies from Eq. (3) that $\dot{x}_2 = -\omega_n^2[(k_1/k) - 1]x_1 \neq 0$ and x_2 becomes nonzero. Hence, $x = 0$ is the only invariant set, and La Salle's theorem[11] completes the proof.

VII. Conclusions

We have presented a nonlinear feedback law for an aerodynamically unstable rocket. This is proximate time-optimal in the sense it marches near minimal-time behavior. Unlike the proximate time-optimal laws of Pao and Franklin,[12] it is continuous and, thus, produces a continuous control function that is likely to produce little response in the presence of small plant imperfections. For states near the origin, the nonlinear law approximates the linear law $k_1(x_1 + k_2 x_2)$, where k_1 and k_2 can be chosen from a satisfactorily large set of values. Finally, it has been shown that the system is asymptotically stable as long as the initial state is within the RZ of the unstable plant.

CPTO laws for two more second-order plants, including the double-integrator plant, have been obtained.[6] CPTO laws for third- and higher-order systems have also been obtained.[6]

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Evaluation of Practical Solutions for Onboard Aircraft Four-Dimensional Guidance

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Introduction

ONE of the principal goals of flight management systems (FMS) is to optimize the flight parameters to minimize fuel- and time-related costs. With the sustained increase of commercial air traffic during the last decades, capacity and congestion problems have become relevant for airspace and terminal traffic controllers, and time constraints have been imposed on particular flights. Such time constraints can be found at a terminal airspace, where it can be necessary to arrive at specified time to be allowed direct access to the runway.

In earlier versions of FMS, no time constraints were explicitly taken into account and the time of crossing at any waypoint was the result of the aircraft trajectory optimization process, minimizing a global criterion mixing fuel- and time-related costs and, therefore, did not necessarily match any desired arrival time constraint.

Recently, new functionalities have been made available on some modern aircraft to allow the control of crossing time at any waypoint of the flight plan. Practical solutions typically seek to adjust the so-called cost index, which is a parameter used by the airlines to combine fuel and time costs to obtain a new speed profile that is consistent with the time constraint while the altitude cruise profile is either frozen or adjusted in a limited way.[1] This solution approach has been considered by some authors as near optimal with respect to fuel costs.[1,2]

In this Note, the optimal control theory is used to show that this is not necessarily the case. The cost-index-based solution is compared, on theoretical and quantitative bases, to the optimality conditions resulting from the formulation of a cruise optimization problem with an explicit crossing time constraint.

Practical Solutions for Time-Constrained Flight

This study is restricted to the cruise phase, which represents the main part of a long-range flight. During cruise, the main constraints are to fly at selected legal air traffic control levels and to match an imposed time constraint T_c at some given crossing point.

The objective of the cruise flight optimization consists here in determining the best cruise flight altitudes and speed profile to satisfy the given time constraint while minimizing total fuel burn.

Fixed-Cost-Index-Based Solutions

A typical class of solutions for this problem is that of the cost-index-based approaches. In this case, the cruise phase is discretized into a succession of constant altitude and constant Mach number segments. The solution is generated by computing online, over each segment, the optimum cruise speed that minimizes the mean flight cost over the given flight segment. Here total flight cost is defined as

$$\text{cost} = \text{fuel} + \text{CI} \times \text{time} \tag{1}$$

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where fuel and time represent fuel burn and flight duration over the given flight distance. The cost index (CI) is a fuel equivalent cost for time, which allows the airlines to balance, for each flight, fuel and time costs in the cost function (1).

Mathematical Justification of CI-Based Methods

The mean total cost C_s is defined as

$$C_s = \frac{\text{cost}}{\text{flight distance}} = \frac{P(m, M, z)}{V} + \frac{\text{CI}}{V} \tag{2}$$

where V is aircraft ground speed; P is the fuel flow, which is a function of aircraft weight m, Mach number M, and cruise flight altitude z.

For a given aircraft weight, flight altitude, and CI, the optimal cruise speed is the speed that minimizes mean cost over the considered flight distance. A necessary condition for a local minimum is

$$\frac{\mathrm{d}C_s}{\mathrm{d}V} = 0 \tag{3}$$

Equations (2) and (3) lead to the necessary condition of optimality

$$M\frac{\partial P(m, M, z)}{\partial M} - P(m, M, z) - \text{CI} = 0 \tag{4}$$

where, for a no-wind flight, the Mach number $M = V/a$, with a the speed of sound at the current cruise altitude z.

The solution retained on existing aircraft has been to compute optimal cruise speeds from Eq. (4) for discrete values of aircraft weight, flight altitude, and CI and to store them in the performance database of the FMS.

CI Adaptation Process for Crossing Time Constraints

It appears that, when the CI is higher (or lower), the minimization of total mean flight cost induces a shorter (or longer) flight duration. Therefore, when explicit time constraints have been introduced in the flight optimization process, the CI has been used to balance fuel and time costs to determine a speed profile that achieves the crossing time constraint. Note that, in this case, the practical solution freezes the current altitude profile and can be referred to as a fixed-altitude-profile (FAP) solution.

Cruise Optimization with Explicit Crossing Time Constraint

Formulation of the Optimization Problem

The terminal time-constrained trajectory optimization problem can be formulated as an optimal control problem whose performance index to be minimized is fuel cost during cruise flight:

$$J = \int_{x_0}^{x_f} c_f \frac{P(m, M, z)}{aM}\,\mathrm{d}x \tag{5}$$

with

$$t(xf) = T_c \tag{6}$$

where c_f is the cost for fuel and aircraft ground distance and x is chosen as the independent variable.

The aircraft dynamics of interest are reduced to the evolution of aircraft weight and crossing time. Because cruise altitude is considered only at discrete values, the state variables evolve following the equations

$$\frac{\mathrm{d}m}{\mathrm{d}x} = -\frac{P_i(m, M)}{a_i M} \tag{7}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{a_i M} \tag{8}$$

where i defines the current flight altitude.

The initial conditions are

$$m(x_0) = m_0 \qquad t(x_0) = 0 \tag{9}$$

Necessary Conditions of Optimality

Without loss of generality, we can consider the simple case of a two-flight-level cruise with a flight-level change point at x_1 and a crossing time constraint at the end of the cruise portion.

The Hamiltonian is defined as

$$H_i = (1/a_i M)[(c_f - \lambda_m)P_i + \lambda_t] \qquad i = 1, 2 \tag{10}$$

with the following final conditions on the costates:

$$\lambda_m(xf) = 0 \qquad \text{and} \qquad \lambda_t(xf) = \nu_t \tag{11}$$

where ν_t is a constant selected so as to satisfy the terminal constraint (6).

Furthermore, the following necessary condition must also be satisfied[3]:

$$\frac{\partial H_i}{\partial M} = 0$$

or

$$M\frac{\partial P_i(m, M)}{\partial M} - P_i(m, M) - \frac{\nu_t}{c_f - \lambda_m} = 0 \tag{12}$$

Defining $\Psi(x) = \nu_t/(c_f - \lambda_m)$, Eq. (12) can be written as

$$M\frac{\partial P_i(m, M)}{\partial M} - P_i(m, M) - \Psi(x) = 0 \tag{13}$$

Theoretical Comparison of the Two Approaches

CI-based solutions offer a relatively simple approach to the problem of meeting a crossing time constraint. However, this approach is based on the satisfaction of Eq. (4) on every segment of the discretized cruise flight and assumes that CI is constant over the entire cruise portion.

It appears that the optimality conditions (4) and (13) for the practical and theoretical solutions are very similar. In fact, these two expressions would coincide exactly if $\Psi(x)$ was constant. This would imply that either

$$\nu_t = 0$$

or

$$\lambda_m(x) = 0 \qquad \text{for} \qquad 0 \le x \le xf$$

The case $\nu_t = 0$ implies that no crossing time constraint is imposed. In fact, the time constraint is chosen as the crossing time given by the unconstrained problem where fuel is minimized over cruise. This is the minimum-fuel problem solved by the practical solution when CI = 0 in the expression of flight cost (1).

The condition $\lambda_m(x) = 0$ implies that the mass adjoint λ_m does not appear in Eq. (10). This assumption has been made in most past studies that investigated the optimal control formulation of aircraft trajectory optimization.[2] This assumption is also made in CI approaches where $\lambda_m(x) = 0$ implies that $\Psi(x) = \text{CI}$. CI-based approaches, therefore, are solutions that treat weight as a parameter but do not include the influences of changing weight on cruise trajectory.

The $\Psi(x)$ does alter the aircraft cruise speed profile, and the influence of weight evolutions can be analyzed through the study of the function λ_m. In past studies, the values of λ_m were generally found to be small. However, the behavior of λ_m depends on aircraft and engine characteristics and should be studied for each aircraft.

Quantitative Comparison of the Two Approaches

Computer simulations have been performed on several case studies to evaluate the performance of CI-based approaches for cruise phase optimization. These trajectories have been compared to near-optimal solutions obtained using a dynamic programming (DP) approach, which is used to solve the full optimal control problem. This approach does not require CI iterations but does require extensive computation. The speed profile is determined by the optimization process itself and is no longer restricted to constant-CI profiles. Furthermore, the flight-level-shift points are also optimized.

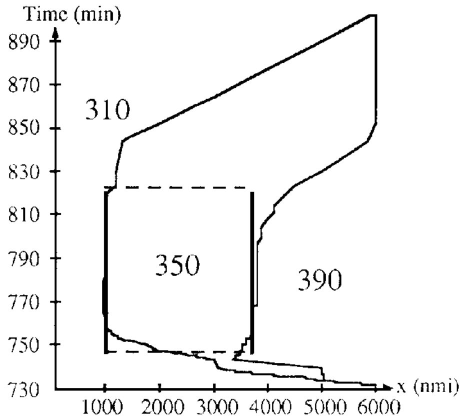

Fig. 1 Step point location for different time constraints.

Case Study

The following case study considers a long-range cruise flight with several possible flight levels and a time constraint at the end of this cruise phase. The FAP and DP solutions have been implemented on a work station to determine the altitude profiles, speed profiles, and associated fuel burn based on the aerodynamic, engine, and performance models of an Airbus A340.

In this case study, we have considered a 6000-n mile-long cruise flight with no wind at all flight levels. The other data for this example are as follows: possible cruise flight levels are FL310, FL350, and FL390, and aircraft weight at start of cruise is 250 tons.

Step Point Location

In Fig. 1, cruise distance is represented along the horizontal axis, whereas the possible values of the time constraint are given along the vertical axis. The curves represent, for the different values of the time constraint, the location of the step points generated by the FAP and DP solutions.

For the classical FAP algorithm, the step points are frozen and correspond to the initial assumptions of a no-wind flight condition and a CI value of 50. The step point locations are represented in Fig. 1 by two vertical bold lines. The achievable time constraints are determined by using the range of possible CI values [−100, 999], which determines minimum and maximum speed schedules. With the FAP approach, time constraints between 747 and 817 min can be met. This time window is defined (in Fig. 1) by two horizontal dashed lines.

The locations of the step points generated by the DP solution are also represented for all achievable time constraints. In the time window [757 min, 800 min], both solutions define comparable step points and speed profiles. Outside this time window, the step points are located differently. For a time constraint of 747 min, representing the lower range of the FAP approach, the DP solution defines a first step point that is located about 1500-n miles farther down the path. It also appears that the additional degrees of freedom of the DP approach provide a much larger achievable time window (as can be seen in Fig. 1).

Fuel Burn

In the time frame [757 min, 800 min] where both solutions define the same step points, the fuel burn is similar. However, outside this time window, fuel consumption performance can be very different. For a time constraint of 747 min, for example, the fuel saving associated with the shift of the step point locations is about 1800 kg, which represents a total cruise fuel saving of 2%.

Conclusions

Today, FMS optimization functions only deal with a single time constraint at a specific waypoint of the flight plan. The optimization algorithms typically seek to adjust the CI to satisfy the required time constraint.

The theoretical analysis of the time-constrained cruise optimization problem shows that existing solutions are based on two main assumptions. The first one is to consider that the optimal speed profile can be approximated by a constant-CI-speed profile. The second is to consider that the altitude profile is frozen when a time constraint is introduced.

A DP-based approach shows that the optimization of the altitude-shift points can lead to additional fuel savings of several percent for particular time constraints. The constant-CI assumption, on the other hand, seems to be valid because the speed profiles generated by both approaches on identical altitude profiles lead to similar total fuel burn.

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