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Fuel-Optimal Periodic Control and Regulation in Constrained Hypersonic Flight

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Acceleration-constrained hypersonic flight is examined in the context of optimal periodic control and regulation. Fuel-optimal periodic trajectories are found that yield an 11% improvement over the best static cruise solution, with a maximum vehicle acceleration of 5 *g*. These optimal periodic trajectories were implemented using a periodic regulator in feedback form, which minimized the second variation of the cost. This regulator was extended to account for a slowly varying system parameter (such as vehicle mass) and was shown to perform strikingly well in the constrained hypersonic cruise problem.

I. Introduction

THE field of optimal periodic control (OPC) is concerned with the study of autonomous dynamic systems that exhibit certain periodic trajectories possessing a lower average cost than its corresponding optimal steady-state solution. OPC has found important application in the fuel-optimal cruise problem for aircraft. The notion that periodic flight trajectories may be fuel-optimal was first introduced by Edelbaum in 1955 (Ref. 1). Subsequent work on aircraft cruise focused on establishing the local optimality of the steady-state cruise solution[2,3]; it was shown by Speyer[4] in 1973 that, for a point-mass aircraft model, an extremal path that satisfies first-order necessary conditions for local optimality, as well as the Legendre–Clebsch condition, generally fails another second-order necessary condition (the Jacobi test). This raised the question of whether a more general notion of cruise, periodic flight, could be optimal instead. From this point, the 1980s saw a strong interest in obtaining fuel-optimal periodic trajectories for aircraft models of varying complexity.

This study builds on these foundations of OPC theory and fuel-optimal periodic cruise in several important respects. First, fuel-optimal periodic paths are found for a highly nonlinear, realistic point-mass model and, second, exhibit dramatic fuel improvement and flight characteristics, which were not previously known. In fact, the particular structure of these resulting paths forced another extension to be made: the imposition of a mixed state/control inequality constraint taking the form of a maximum vehicle acceleration. Third, these constrained optimal periodic trajectories were synthesized into a usable controller by applying the optimal periodic regulator of Ref. 5, requiring new extensions and numerical aspects related to the constraint. Finally, a slowly varying regulator was derived for the general dynamic system exhibiting a slowly varying system parameter and was applied to the hypersonic vehicle case study by considering slow decrease in vehicle mass.

The results of this paper are presented in the following seven sections. Section II describes in detail the hypersonic vehicle model used in the study and the inequality constraint that is imposed on the model. Section III formally states the constrained OPC problem. Section IV states the first-order necessary conditions, which an extremal must satisfy for constrained OPC, and Sec. V describes the extremal periodic paths obtained for the model given in Sec. II. Section VI establishes second-order sufficient conditions for local optimality of the extremal solutions of Sec. V, and Sec. VII presents the main results on constrained optimal periodic regulation, again with application to the hypersonic vehicle case study.

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II. Hypersonic Vehicle Model

The problem considered involves a vehicle powered by airbreathing engines and capable of cruise at hypersonic speeds. To reduce the dimension of the problem, the motion of the vehicle is constrained to be in the vertical plane. The vehicle is modeled as a point mass moving relative to a nonrotating, spherical Earth. The forces acting on the vehicle during such planar flight are illustrated in the free-body diagram of Fig. 1. The equations of motion for such a system may be written as

$$\frac{\mathrm{d}\bar{h}}{\mathrm{d}\bar{r}} = \tan\gamma\left(1 + \frac{\bar{h}}{\bar{R}_0}\right) \tag{1}$$

$$\frac{\mathrm{d}M}{\mathrm{d}\bar{r}} = \left\{\frac{gF_r[T\cos(\alpha+\alpha_T) - D - W\sin\gamma]}{Ma^2W\cos\gamma}\right\}\left(1 + \frac{\bar{h}}{\bar{R}_0}\right) \tag{2}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\bar{r}} = \left\{\frac{gF_r[T\sin(\alpha+\alpha_T) + L - W\cos\gamma]}{M^2a^2W\cos\gamma} + \frac{1}{\bar{h}+\bar{R}_0}\right\}\left(1 + \frac{\bar{h}}{\bar{R}_0}\right) \tag{3}$$

In this formulation, actual altitude h and range r have been normalized by the scalar F_r to produce the dynamic state $\bar{h}$ and the independent variable $\bar{r}$. As is indicated in Fig. 1, the thrust force T is assumed to have a nonzero thrust offset angle α_T due to exhaust expansion on the vehicle afterbody. The speed of sound a is considered to be a constant normalization factor for velocity. The point-mass model given in Eqs. (1–3) is more complex than that used in previous studies of periodic hypersonic cruise, such as Ref. 6, because thrust is not assumed to be axial and no small angle approximation for angle of attack α is employed.

This model is implemented by specifying analytical relations between thrust, lift and drag, and the dynamic states. In the following expressions, the various constant parameters associated with the polynomials were computed based on a baseline hypersonic vehicle model.[7] The aerodynamic forces are written in terms of dynamic pressure and the aerodynamic coefficients as

$$T = ST_{\max} + (1-S)T_{\min}$$

$$L = \tfrac{1}{2}\rho M^2a^2S_bC_L, \qquad D = \tfrac{1}{2}\rho M^2a^2S_bC_D$$

$$T_{\max} = \tfrac{1}{2}\rho M^2a^2S_eC_{T_{\max}}, \qquad T_{\min} = -\tfrac{1}{2}\rho M^2a^2S_eC_{T_{\min}}$$

This thrust model assumes linear dependence on a throttle setting $S \in [0, 1]$ and includes an installation drag penalty term $T_{\min}$, where $C_{T_{\min}}$ is assumed a positive constant. The reference areas S_b and S_e

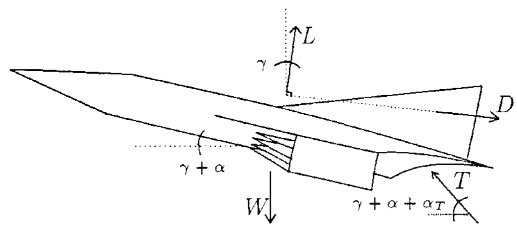

Fig. 1 Free-body diagram of hypersonic vehicle.

for aerodynamic and propulsive forces, respectively, are assumed constant. The lift and drag coefficients are assumed to be related to a Mach number dependent lift–drag polar

$$C_D = d_{00} + d_{01}M + (d_{10} + d_{11}M)\alpha + (d_{20} + d_{21}M)\alpha^2$$

and the lift coefficient is related to angle of attack as

$$C_L = b_{00} + b_{01}\alpha + (b_{10} + b_{11}\alpha)M$$

The maximum thrust coefficient is assumed to be related to M and α as

$$C_{T_{\max}} = C_{00} + C_{01}\alpha + (C_{10} + C_{11}\alpha)M + (C_{20} + C_{21}\alpha)M^2$$

In addition to a model of vehicle dynamics, a propulsion system model is necessary for the solution of fuel-optimal trajectories. In this study, a simple linear relationship between thrust-specific fuel consumption (TSFC) and Mach number is assumed:

$$\mathrm{TSFC} = b_0 + b_1 M$$

so that the average fuel consumed from $\bar{r} = 0$ to $\bar{r}_f$ is given by

$$J = \frac{1}{\bar{r}_f}\int_0^{\bar{r}_f} \frac{S(T_{\max} - T_{\min})\mathrm{TSFC}}{MaF_f\cos\gamma}\left(1 + \frac{\bar{h}}{\bar{R}_0}\right)\mathrm{d}\bar{r} \tag{4}$$

In Eq. (4), the installation drag penalty appears and F_f is a normalization factor for fuel.

This paper will be concerned with the study of a certain class of fuel-optimal flight trajectories that obey the dynamic equation (1–3) and which are optimal with respect to Eq. (4). This class of trajectories is restricted by requiring that the total acceleration of the vehicle at any instant due to acting aerodynamic forces be bounded from above. Thus, the following inequality constraint is imposed:

$$\begin{aligned} &C(\bar{h}, M, \gamma, C_L, S)\\ &\quad = \sqrt{[T\cos(\alpha + \alpha_T) - D]^2 + [T\sin(\alpha + \alpha_T) + L]^2}\\ &\quad - Wg_{\max} \le 0 \end{aligned} \tag{5}$$

where $g_{\max}$ is a (usually integer) constant that prescribes the maximum vehicle acceleration (load factor) as a multiple of the local acceleration of gravity.

The class of fuel-optimal trajectories of present interest is further constrained to be periodic in the aircraft dynamic states. That is, for some $\bar{r}_f \in \mathbb{R} > 0$, it is required that

$$\bar{h}(\bar{r}_f) = \bar{h}(0), \qquad M(\bar{r}_f) = M(0), \qquad \gamma(\bar{r}_f) = \gamma(0)$$

Clearly, any steady-state cruise point also satisfies these constraints for any $\bar{r}_f > 0$.

III. OPC Problem

In this section, the OPC problem is formally stated and solved in a more general setting, while still maintaining some of the particular features (such as the mixed state/control inequality constraint) exhibited by the hypersonic cruise problem of the preceding section.

Let the following autonomous, nonlinear dynamic system be considered:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), S(t)], \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{6}$$

with general associated average cost

$$J[\boldsymbol{x}_0, \boldsymbol{u}(\cdot), S(\cdot), \tau] = \frac{1}{\tau}\int_0^{\tau} L[\boldsymbol{x}(\sigma), \boldsymbol{u}(\sigma), S(\sigma)]\,\mathrm{d}\sigma \tag{7}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$ and $S(t) \in \mathbb{R}$ for all $t \in [0, \tau]$. To simplify the optimality conditions and to reflect the hypersonic cruise problem at hand, let the case of scalar control be taken, $m = 1$. This assumption has no bearing on the applicability of the theory for general m. Let the motion governed by Eq. (6) be required to satisfy the following scalar inequality constraints at every t:

$$C[\boldsymbol{x}(t), \boldsymbol{u}(t), S(t)] \le 0 \tag{8}$$

$$0 \le S(t) \le 1 \tag{9}$$

where the scalar function $S(t)$ is assumed to appear linearly in $\boldsymbol{f}$ and L but appears of higher order in C. The class of admissible controls for $\boldsymbol{u}$ and S, denoted henceforth as $\mathcal{U}$, is that of piecewise continuous functions on $\mathbb{R}$ such that Eqs. (8) and (9) are satisfied. The OPC problem is to find control functions $\boldsymbol{u}(t)$ and $S(t)$, initial state $\boldsymbol{x}_0$, and period τ such that $J[\boldsymbol{x}_0, \boldsymbol{u}(\cdot), S(\cdot), \tau]$ is minimized, subject to Eqs. (8) and (9) and to periodicity of dynamic state

$$\boldsymbol{x}(\tau) = \boldsymbol{x}_0 \tag{10}$$

Let $[\boldsymbol{x}_0^*, \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ be such a minimal solution to this OPC problem. Two assumptions regarding the system and its OPC solution are made at the outset.

Assumption 1. The functions $\boldsymbol{f}$: $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^n$, L: $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$, and C: $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are at least twice continuously differentiable in their arguments.

IV. First-Order Necessary Conditions

We are concerned with weak local optimality, by which we mean that $[\boldsymbol{x}_0^*, \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ minimizes J for small variations in initial state, control, and period. That is, a variation in the cost away from an extremal value J^*, which results from an extremal set $[\boldsymbol{x}_0^*, \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ will be examined, being generally written in the form

$$J = J^* + \delta J + \delta^2 J + \cdots \tag{11}$$

Conditions will be given that ensure that $\delta J = 0$ and $\delta^2 J > 0$ for small variations away from the extremal set. As will be seen, the linear structure of the problem with respect to S raises the possibility of either singular or bang-bang (discontinuous) control. The first-order necessary conditions will first be stated in general terms of the minimum principle without any preassumed control structure. The necessary conditions will then be specialized to the particular bang-bang structure encountered in the present application study. (Note here that the numerical optimization method employed in this research did not presuppose a control structure a priori.) In the second variational analysis, it will be assumed that the control variations are sufficiently small so as not to change the overall structure of the discontinuous extremal control S, that is, it will be assumed that no new instants of discontinuous control are added. Of course, this special discontinuous control structure will require special treatment in the second variational analysis to guarantee weak control variations.

The conditions for optimality of $[\boldsymbol{x}_0^*, \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ will be stated in terms of a scalar Hamiltonian function $\mathcal{H}$ defined as

$$\mathcal{H}(\boldsymbol{x}, \boldsymbol{u}, S, \lambda, \mu) = \tilde{\mathcal{H}}(\boldsymbol{x}, \boldsymbol{u}, S, \lambda) + \mu C(\boldsymbol{x}, \boldsymbol{u}, S) \tag{12}$$

$$\tilde{\mathcal{H}}(\boldsymbol{x}, \boldsymbol{u}, \lambda) = L(\boldsymbol{x}, \boldsymbol{u}, S) + \lambda^T \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, S) \tag{13}$$

where $\lambda(t) \in \mathbb{R}^n$ and $\mu(t) \in \mathbb{R}$ are Lagrange multipliers, with $\mu(t) \ge 0$ such that $\mu(t)C[\boldsymbol{x}(t), \boldsymbol{u}(t), S(t)] = 0$ (Ref. 8).

The first-order necessary conditions for a local solution to the OPC problems are as follows. If $[\boldsymbol{x}_0^*, \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ minimizes the cost $J[\boldsymbol{x}_0, \boldsymbol{u}(\cdot), \tau]$ subject to Eqs. (6–10), then

$$(\boldsymbol{u}^*, S^*) = \arg\min_{(\boldsymbol{u}, S)\in\mathcal{U}} \mathcal{H}(\boldsymbol{x}^*, \lambda^*, \boldsymbol{u}, S, \mu) \tag{14}$$

$$\dot{\lambda}^*(t) = -\mathcal{H}_x^c[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), S^*(t), \lambda^*(t), \mu^*(t)]$$

$$\lambda^*(\tau) = \lambda^*(0) \quad (15)$$

$$\mathcal{H}^c[\boldsymbol{x}^*(\tau^*), \boldsymbol{u}^*(\tau^*), S^*(\tau^*), \lambda^*(\tau^*), \mu^*(\tau^*)] - J^*\left[\boldsymbol{x}_0^*, \boldsymbol{u}(\cdot), S^*(\cdot)\right] = 0 \quad (16)$$

At any point t_i of discontinuous S^*, the Weierstrass–Erdmann conditions are satisfied:

$$\mathcal{H}^c\left(t_i^+\right) = \mathcal{H}^c\left(t_i^-\right) \quad (17)$$

$$\lambda^*\left(t_i^+\right) = \lambda^*\left(t_i^-\right) \quad (18)$$

Note that the notation $\mathcal{H}^c(t)$ means that $\mathcal{H}$ is evaluated at $[\boldsymbol{x}^*(t), \lambda^*(t), \boldsymbol{u}^*(t), S^*(t), \mu^*(t)]$.

These conditions are standard and are given elsewhere.[8,9] The minimum principle of Eq. (14) is a very general optimality condition; we wish to specialize this condition to the present problem structure. Because it is assumed that $\mathcal{H}$ is nonlinearly dependent on the control $\boldsymbol{u}$, this portion of the control satisfies a stationarity condition for every $t \in [0, \tau^*]$,

$$\mathcal{H}_u[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), S^*(t), \lambda^*(t), \mu^*(t)] = 0 \;\forall\; t \in [0, \tau] \quad (19)$$

Because of the affine dependence of $\tilde{\mathcal{H}}$ on S but general dependence of C on S, satisfaction of Eq. (14) results in a more complicated structure for $S^*(\cdot)$. Its structure is distinctly dependent on the activity of the constraint, as follows.

1) For the inactive constraint, $C < 0$, the extremal control $S^*(\cdot)$ may consist of arcs of purely bang-bang structure and/or arcs of singular control. Equation (14) satisfied along bang-bang arcs reduces to[8]

$$S^*(t) = \arg \min_{S \in \{0,1\}} \{\tilde{\mathcal{H}}[\boldsymbol{x}^*, \boldsymbol{u}(\boldsymbol{x}^*, \lambda^*, S), S]\} \quad (20)$$

Higher-order necessary optimality conditions may also be written for singular arcs; Ref. 10 investigates singular control in the context of optimal periodic aircraft cruise. If extremal control is computed numerically from condition (20) and it results that S^* approaches a high-frequency chattering solution, this indicates the onset of a singular arc[11] for which condition (20) is not valid. This condition was used in the numerical computation of extremal solutions of Sec. V, and no such chattering behavior was observed.

2) For the active constraint case $C = 0$, however, Eq. (14) admits intermediate values of S^* via the condition

$$\mathcal{H}_S[\boldsymbol{x}^*(t), \lambda^*(t), \boldsymbol{u}^*(t), S^*(t), \mu^*(t)] = 0 \quad (21)$$

Condition (21) determines S^* only when its solution is admissible for the bounds on S; such a case will be referred to as a constrained–unconstrained (CU) arc. Otherwise, condition (21) cannot be met, and $C = 0$ is satisfied solely by choice of $\boldsymbol{u}^*$, with S held at its upper or lower bound; such a case will be referred to as a constrained–constrained (CC) arc.

Finally, because the OPC problem is autonomous, condition (17) implies that $\mathcal{H}^c$ is a constant, even across instants of discontinuous control, for any extremal solution.

V. Periodic Extremal Cruise Trajectories

In this section, an indirect method used to solve the constrained OPC problem is outlined and the resulting periodic extremal path is presented. As these results indicate, potentially significant fuel savings may be realized by operating on these periodic cruise paths, even in the presence of a quite stringent constraint on maximum aerodynamic load.

To assess the benefit of the extremal trajectories, an optimal steady-state cruise point must be found. This problem is a simple static, nonlinear programming problem, because the right-hand sides of Eqs. (1–3) are set to zero and the scalar functional L is minimized subject to these equality constraints. Clearly, the constraint on load factor does not enter into this problem. Because L and f are linear in the throttle setting, S is necessarily on its upper or lower bound as determined by the sign of $\partial L/\partial S$. Of course, this problem may also admit a singular arc as a solution, resulting in a chattering cruise solution.[12] The presence of a chattering solution during the numerical solution is apparent if satisfaction of the maximum principal of Eq. (20) results in the coalescing of switch points in a small-time interval.[11] This phenomenon was not observed in this study. The static optimization problem was solved using a standard algorithm.[13] It was found that the optimal static cruise path yielded a nondimensional fuel consumption of $J \approx 0.9351$, with a fully open throttle, $S = 1$. The values of the states and lift coefficient corresponding to this static path are shown in Figs. 2–4.

The periodic boundary value problem was solved in two steps. First, an unconstrained Fletcher–Powell method (see Ref. 13) was used to find the state–costate initial condition that yielded periodicity of only dynamic state, by minimizing a quadratic form of state periodicity error. Second, a constraint manifold of periodic dynamic state is traversed toward the minimum of the original cost J using an accelerated gradient projection scheme.[14] Note that the flight-path angle initial condition was held at $\gamma(t_0) = 0$ for the periodic problem to be well defined. The numerical integration necessary to perform these steps was done with an adaptive-stepsize Runge–Kutta integration scheme.[13] The optimal periodic solution obtained is illustrated in Figs. 5–7 for maximum aerodynamic load factor of 5.0; the periodic fuel cost J of Eq. (7) for this solution was 0.8442, thus yielding an improvement over optimal static cruise of 10.8%. The optimal period was found to be approximately 830 miles. Figure 2 illustrates the extremal periodic orbit by plotting M against $\bar{h}$; traversal over time is in the clockwise sense. The large excursion in altitude causes the orbit to exhibit two distinct regions. At high altitude, aerodynamic forces are small in comparison to body

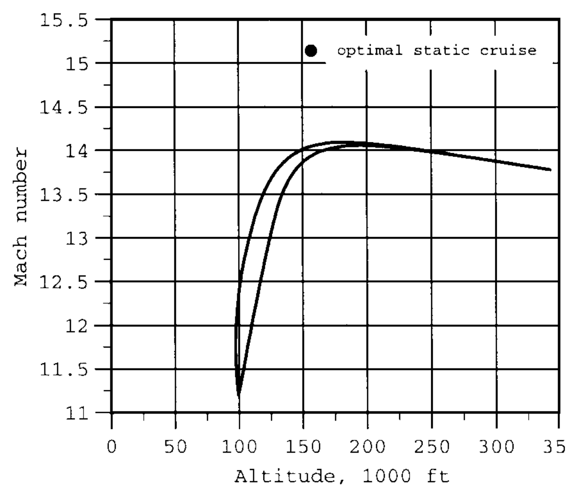

Fig. 2 Optimal periodic cruise: M vs h.

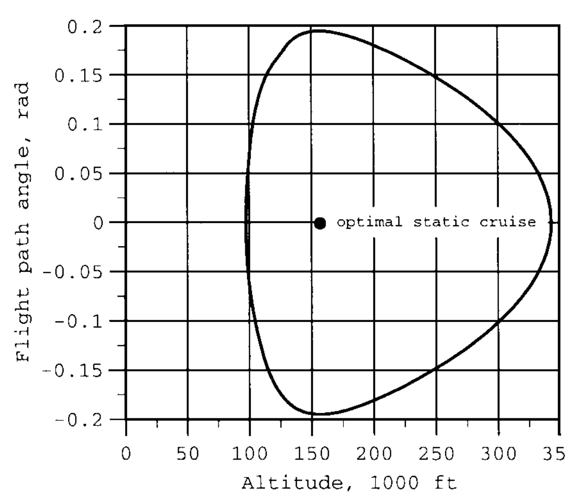

Fig. 3 Optimal periodic cruise: γ vs h.

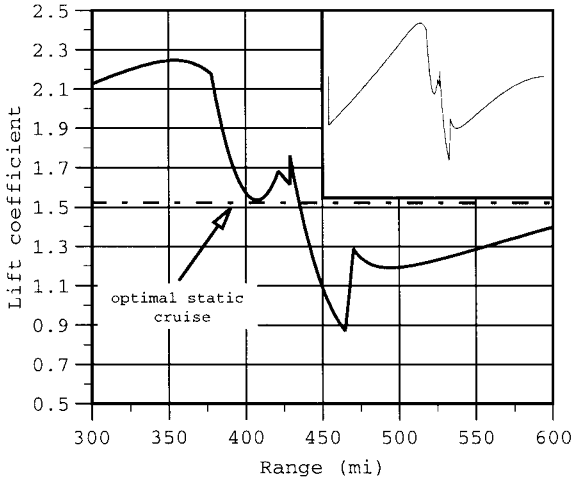

Fig. 4 Optimal periodic cruise: C_L vs *r*.

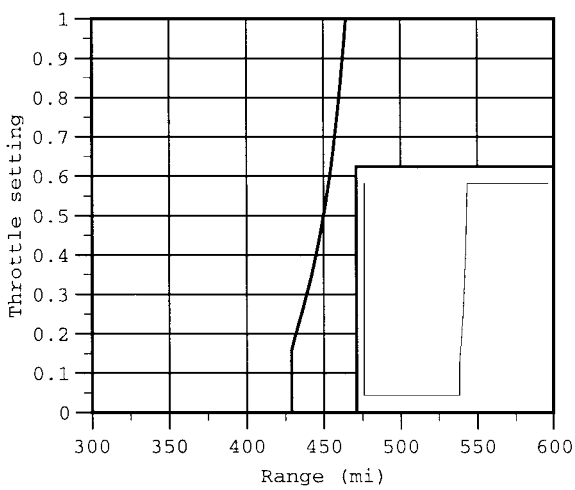

Fig. 5 Optimal periodic cruise: *S* vs *r*.

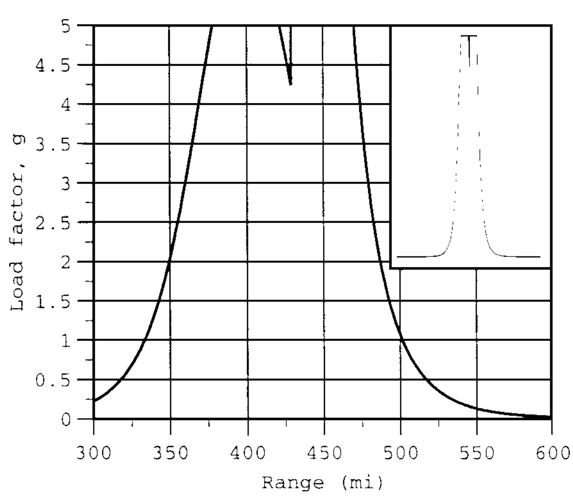

Fig. 6 Optimal periodic cruise: load factor vs *r*.

forces; the aircraft thus follows a trajectory of constant energy. At low altitude, aerodynamic forces are dominant. This separation of flight conditions is also apparent in Fig. 3, which gives flight-path angle and altitude, with traversal in time being in the clockwise sense.

The controls necessary to generate these periodic paths are shown in Figs. 4 and 5. Because their behavior is quite complex in the low-altitude regime, this interval is principally plotted, and the accompanying thumbnail image gives an overall view of the function. Figure 6 gives the load factor history, indicating the intervals for which the constraint is active. The initial phase of the periodic orbit is an unpowered glide from the maximum altitude. During this maneuver, aerodynamic lift and drag forces affect the maximum permissible acceleration on the vehicle; on this first constrained arc, throttle is still off, and the constraint is satisfied entirely by modulation of lift coefficient, as is indicated by the first drop in C_L of Fig. 4. The constraint becomes inactive in the region of minimum altitude. Shortly after the minimum altitude point, engine throttle jumps discontinuously to an intermediate value; during this phase, both throttle and C_L are jointly modulated to satisfy the load constraint and optimality conditions (19) and (21); this arc is an example of the CU arc described in the preceding section. Finally, throttle achieves its upper bound, forcing the constraint to be satisfied solely by C_L, yielding a second CC arc. Finally, the constraint becomes inactive as a result of higher altitude. Even though the optimality condition on S dictates that throttle remain open until a very high altitude, this thrusting cycle is effectively of much shorter duration, because the engines are assumed to be entirely airbreathing and, thus, cannot impart significant thrust to the vehicle for approximately $\bar{h} > 0.4$.

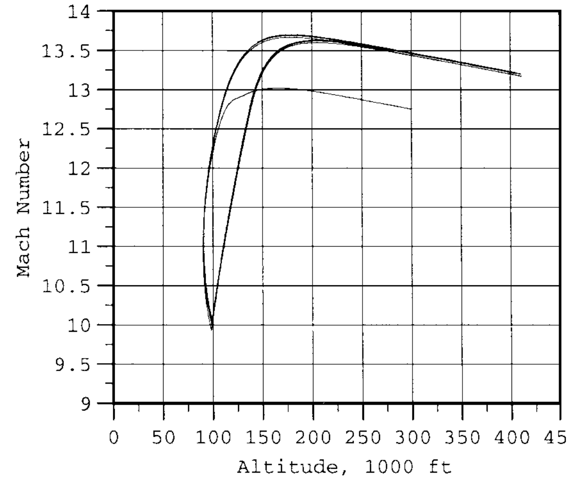

Fig. 7 Time-invariant regulator: *M* vs *h*.

VI. Second-Order Sufficient Conditions

In addition to the first-order necessary conditions, second-order conditions for optimality of the periodic optimal control problem without mixed state/control inequality constraint have been considered.[9,15] Zeidan and Zezza[15] considered a general periodic problem in the calculus of variations and developed necessary conditions based on the notion of coupled points for general boundary value problems. The necessary conditions were later extended to the optimal control problem in Ref. 16. Wang and Speyer[9] established necessary and sufficient conditions based on the existence of solutions to Riccati equations for both singly and infinitely repeated periodic problems, using an extension of conjugate point theory arising in fixed boundary value problems. Recently, second-order necessary and sufficient conditions have been studied for general control problems with mixed state/control inequality constraint[17,18]; these conditions both involve the existence of a solution to a modified Riccati equation involving the tangent space of active constraints. Inasmuch as sufficient conditions are required for the regulator results of Sec. VII, only second-order sufficient conditions for an infinitely repeated periodic process will be of concern here.

Sufficient optimality conditions will be determined by considering variations in the initial state, extremal control $\boldsymbol{u}^*(\cdot)$ along continuous subarcs, and period, which were determined in Sec. V. It is assumed here that these variations are sufficiently small that the qualitative structure of S^* does not change. That is, it is assumed that the number of points of discontinuous control is constant for all variations sufficiently small.

Small variations in $\boldsymbol{x}_0^*$ and $\boldsymbol{u}^*(\cdot)$ along continuous subarcs induce variations in instants of discontinuous control, $\mathrm{d}t_i$. In Sec. VI.A, it will be shown that these $\mathrm{d}t_i$ are of the same order as the continuous control variation, ensuring that they produce weak variations in

the state. Section VI.B will examine the continuous intervals using classical variational techniques. Finally, Sec. VI.C will bring these results together in a statement of weak, second-order sufficiency.

Before presenting these results, some assumptions necessary to this section will be given. These assumptions are made more transparent by defining a new control vector $\tilde{\boldsymbol{u}}$ in keeping with the special structure for S discussed earlier as

$$\tilde{\boldsymbol{u}} \in \mathbb{R}^{m(t)} \begin{cases} \tilde{\boldsymbol{u}} = \boldsymbol{u} = C_L \in \mathbb{R}^1 & \text{if } t \in U \text{ or } t \in \mathrm{CC} \Rightarrow m(t) = 1 \\ \tilde{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u} \\ S \end{bmatrix} \in \mathbb{R}^2 & \text{if } t \in \mathrm{CU} \Rightarrow m(t) = 2 \end{cases}$$

First, a regularity assumption on the constraint must be imposed.

Assumption 2. For each t for which $C^*(t) = 0$, it holds that $C^*_{\tilde{u}}(t) \neq 0$.

Let the set $\Im^* \triangleq [\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ be an extremal solution to the OPC problem; thus, $\Im^*$ satisfies the first-order necessary conditions. A normality condition must also be imposed on $\Im^*$, which is equivalent to a controllability condition of the linearized system about $(\boldsymbol{x}^*, \boldsymbol{u}^*, S^*)$ defined on a tangent space of active constraints.[17] Let $i(t)$ be an integer function such that $i(t) = 1$ if $C = 0$, and $i(t) = 0$ if $C < 0$. Let $C^{i(t)}(t)$ be defined as

$$C^{i(t)}(t) \triangleq \begin{cases} C & \text{if } \quad i(t) = 1 \\ 0 & \text{if } \quad i(t) = 0 \end{cases}$$

Define the subspace $T(t) \subseteq \mathbb{R}^{m(t)}$ as

$$T(t) \triangleq \left\{ \tilde{\boldsymbol{u}} \in \mathbb{R}^{m(t)} : C^{i(t)}_{\tilde{u}}(t)\tilde{\boldsymbol{u}} = 0 \right\} \tag{22}$$

Let the columns of the time-varying matrix $Y(t) \in \mathbb{R}^{m(t)} \times \mathbb{R}^{m(t)-i(t)}$ be an orthonormal basis for $T(t)$, with $Y(t) = 0$ if $T(t) = \emptyset$. Consider the linear autonomous system

$$\dot{\psi}(t) = \left[\boldsymbol{f}^*_{\boldsymbol{x}}(t) - \boldsymbol{f}^*_{\tilde{u}}(t)\beta(t)C^{i(t)}_{\boldsymbol{x}}(t) \right]\psi(t) \tag{23}$$

where $\beta(t)$ is given by

$$\beta(t) \triangleq \begin{cases} \left[C^{i(t)}_{\tilde{u}}(t)\right]^T \left\{ C^{i(t)}_{\tilde{u}}(t) \left[C^{i(t)}_{\tilde{u}}(t)\right]^T \right\}^{-1} & \text{if } \quad i(t) \neq 0 \\ 0 & \text{if } \quad i(t) = 0 \end{cases} \tag{24}$$

This definition for $\beta(t)$ is well defined by Assumption 2. Let $\hat{\Phi}$ denote the state transition matrix corresponding to Eq. (23). For some $t \in [0, \tau^*]$, define the following set in $\mathbb{R}^n$:

$$\mathcal{R}_t(\tau^*) \triangleq \left\{ \int_t^{t+\tau^*} \hat{\Phi}(t+\tau^*, s)\boldsymbol{f}^*_{\tilde{u}}(t)Y(s)\eta(s)\,\mathrm{d}s : \eta(\cdot) \in L^\infty\left[t, t+\tau^*; \mathbb{R}^{\,m(t)-i(t)}\right] \right\} \tag{25}$$

The following normality assumption is made.

Assumption 3. For every $t \in [0, \tau]$, it holds that $\mathcal{R}_t(\tau) = \mathbb{R}^n$.

In addition, satisfaction of a strong Legendre–Clebsch condition is assumed.

Assumption 4. The extremal solution $\Im^*$ is such that there exists a $\delta > 0$ such that

$$[Y(t)]^T \mathcal{H}^*_{\tilde{u}\tilde{u}}(t)Y(t) > \delta I_{m(t)-i(t)} \tag{26}$$

for all t for which $Y(t) \neq 0$, where $I_{m(t)-i(t)}$ is the identity matrix of dimension $m(t) - i(t) \times m(t) - i(t)$.

A. Variations in Corner Times

Let t_i denote the instant in time at which the scalar control $S(\cdot)$ experiences a discontinuous change; this may occur for $t \in \mathrm{U}$ via switch condition (20), or at a corner between an unconstrained arc and a CU arc. The discontinuity in S results in a discontinuity of the control $\boldsymbol{u}$. A perturbation $\delta\boldsymbol{u}(\cdot)$ in the continuous control about its extremal $\boldsymbol{u}^*(\cdot)$ results in a perturbation $\mathrm{d}t_i$ of time t_i. In this section, it will be shown that $\mathrm{d}t_i$ is the same order as $\delta\boldsymbol{u}(t_i^-)$, ensuring that variations due to changes in t_i are weak with respect to the state. The present development is for discontinuous control on U; discontinuous control on a U–CU corner follows a similar development and yields the same result, with the constraint-augmented Hamiltonian appearing.

Consider a control variation of the form $\delta\boldsymbol{u}(t) = \epsilon\eta(t)$ for $t < t_i$, with $\eta \in L^\infty[0, \tau; \mathbb{R}^m]$ and ϵ a small number; this variation will satisfy other conditions in the section to follow. The neighboring extremal satisfies the Euler–Lagrange equations, written in the abbreviated form as

$$\dot{\boldsymbol{y}}(t) = \mathcal{F}[\boldsymbol{y}(t), \boldsymbol{u}(t), S] = K\tilde{\mathcal{H}}^T_{\boldsymbol{y}}[\boldsymbol{y}(t), \boldsymbol{u}(t), S] \tag{27}$$

where K is the fundamental symplectic matrix

$$K = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

From the theory of solutions to ordinary differential equations, the control variation produces a variation $\delta\boldsymbol{y}$ away from its extremal solution $\boldsymbol{y}^*(\cdot)$, which can be written to first order as $\delta\boldsymbol{y}(t) = \epsilon\boldsymbol{z}[t; \eta(\cdot)]$. The variation $\mathrm{d}t_i$ is a function of ϵ; the objective here is to show that the following order relation is satisfied: $\mathrm{d}t_i = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. From the definition of ordering,[19] we must show that

$$\lim_{\epsilon \to 0} \left\{ \frac{\mathrm{d}t_i}{\epsilon} \right\} = \lim_{\epsilon \to 0} \left\{ \frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} \right\} = \kappa < \infty \tag{28}$$

where the rule of de l'Hospital was employed in the evaluation of the limit. If such a κ exists, then $\mathrm{d}t_i$ can be written as an asymptotic series in ϵ as $\mathrm{d}t_i = \kappa\epsilon + \mathcal{O}(\epsilon^2)$. Let the following weak regularity assumption be imposed on the extremal solution $\mathcal{S}^*$.

Assumption 5. For every instant t_i of discontinuous control on the extremal path $\mathcal{S}^*$, the vector $\mathcal{H}^{*T}_{\boldsymbol{y}}(t_i^-)$ is not K orthogonal to $\mathcal{H}^{*T}_{\boldsymbol{y}}(t_i^+)$; that is, $\mathcal{H}^*_{\boldsymbol{y}}(t_i^-)K\mathcal{H}^{*T}_{\boldsymbol{y}}(t_i^+) \neq 0$.

It is shown in the Appendix that, if Assumption 5 holds, then κ is indeed bounded and is given by

$$\kappa = \frac{\mathcal{H}^*_{\boldsymbol{y}}\left(t_i^-\right) - \mathcal{H}^*_{\boldsymbol{y}}\left(t_i^+\right)}{\mathcal{H}^*_{\boldsymbol{y}}\left(t_i^+\right)K\mathcal{H}^{*T}_{\boldsymbol{y}}\left(t_i^-\right)} \cdot \boldsymbol{z}[t; \eta(\cdot)] \tag{29}$$

thus establishing that $\mathrm{d}t_i = \mathcal{O}(\epsilon)$. By expanding the variation $\delta\boldsymbol{y}(t_i + \mathrm{d}t_i) = \boldsymbol{y}(t_i + \mathrm{d}t_i) - \boldsymbol{y}^*(t_i + \mathrm{d}t_i)$, one obtains

$$\delta\boldsymbol{y}\left(t_i + \mathrm{d}t_i\right) = \delta\boldsymbol{y}(t_i) + \left[\mathcal{F}^*\left(t_i^-\right) - \mathcal{F}^*\left(t_i^+\right)\right]\mathrm{d}t_i + \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots \tag{30}$$

Upon substitution of Eqs. (29) and the expansion for $\mathrm{d}t_i$ into Eq. (30), the variations can be related as

$$\delta\boldsymbol{y}(t_i + \mathrm{d}t_i) = \Phi\left(t_i^+, t_i^-\right)\delta\boldsymbol{y}(t_i)$$

$$\Phi\left(t_i^+, t_i^-\right) = I + \left[\frac{1}{\mathcal{H}^*_{\boldsymbol{y}}\left(t_i^+\right)K\mathcal{H}^*_{\boldsymbol{y}}\left(t_i^-\right)}\right]K \cdot \left[\mathcal{H}^{*T}_{\boldsymbol{y}}\left(t_i^-\right) - \mathcal{H}^{*T}_{\boldsymbol{y}}\left(t_i^+\right)\right] \cdot \left[\mathcal{H}^*_{\boldsymbol{y}}\left(t_i^-\right) - \mathcal{H}^*_{\boldsymbol{y}}\left(t_i^+\right)\right] \tag{31}$$

A similar expansion procedure for the case of the U–CU arc yields an identical form but with the constraint-augmented Hamiltonian appearing.

The results of the Appendix also allow for an examination of the convexity of the cost with respect to changes in instants of discontinuous control. Using the established order relation for $\mathrm{d}t_i$, Eq. (A7) can be written as

$$\mathrm{d}\left(\frac{\partial J}{\partial t_i}\right) = \left\{\mathcal{H}^*_{\boldsymbol{y}}\left(t_i^+\right)\left[\mathcal{F}^*\left(t_i^-\right) - \mathcal{F}^*\left(t_i^+\right)\right]\right\}\mathrm{d}t_i + \left[\mathcal{H}^*_{\boldsymbol{y}}\left(t_i^+\right) - \mathcal{H}^*_{\boldsymbol{y}}\left(t_i^-\right)\right]\delta\boldsymbol{y}(t_i) + \mathcal{O}(\epsilon^2) = 0 \tag{32}$$

If $\delta\lambda(t)$ is assumed linearly related to $\delta\boldsymbol{x}(t)$ for $t \neq t_i$ by means of an $n \times n$ symmetric, time-varying matrix $P(t)$ as $\delta\lambda(t) = P(t)\delta\boldsymbol{x}(t)$, then Eq. (30) can be used to obtain

$$\begin{aligned}\delta\lambda(t_i) &= P(t_i + \mathrm{d}t_i)\delta\boldsymbol{x}(t_i + \mathrm{d}t_i) \\ &\quad + \left[\mathcal{H}_x^{*T}\left(t_i^-\right) - \mathcal{H}_x^{*T}\left(t_i^+\right)\right]\mathrm{d}t_i + \mathcal{O}(\epsilon^2) \\ &= P\left(t_i^+\right)\delta\boldsymbol{x}(t_i) + \left[\mathcal{H}_x^{*T}\left(t_i^-\right) - \mathcal{H}_x^{*T}\left(t_i^+\right)\right]\mathrm{d}t_i + \mathcal{O}(\epsilon^2) \end{aligned} \tag{33}$$

Substitution of Eq. (33) into Eq. (32) and collection of terms multiplicative in $\mathrm{d}t_i$ yields

$$\begin{aligned}\frac{\partial^2 J}{\partial t_i^2} &= \left\{\lambda^T\left[\boldsymbol{f}_x\left(t_i^-\right) - \boldsymbol{f}_x\left(t_i^+\right)\right] + L_x\left(t_i^-\right) - L_x\left(t_i^+\right)\right\}\boldsymbol{f}\left(t_i^-\right) \\ &\quad + \left[\boldsymbol{f}\left(t_i^-\right) - \boldsymbol{f}\left(t_i^+\right)\right]^T P\left(t_i^+\right)\left[\boldsymbol{f}\left(t_i^-\right) - \boldsymbol{f}\left(t_i^+\right)\right] \\ &\quad - \left[\lambda^T\boldsymbol{f}_x\left(t_i^-\right) + L_x\left(t_i^+\right)\right]\left[\boldsymbol{f}\left(t_i^-\right) - \boldsymbol{f}\left(t_i^+\right)\right] > 0 \end{aligned} \tag{34}$$

This expression was given in Ref. 20 for the case of pure bang-bang control.

B. Variations on Continuous Subarcs

Let the set of variations $\delta\Im \overset{\triangle}{=} (\delta\boldsymbol{x}, \delta\tilde{\boldsymbol{u}})$ represent small perturbations away from the extremal solution $\Im^*$. To develop sufficient conditions, we study the accessory minimum problem (AMP): minimize

$$J_2 = \int_0^{\tau^*} [\delta\boldsymbol{x}^T \quad \delta\tilde{\boldsymbol{u}}^T]\begin{bmatrix}\mathcal{H}_{xx}^* & \mathcal{H}_{x\tilde{u}}^* \\ \mathcal{H}_{\tilde{u}x}^* & \mathcal{H}_{\tilde{u}\tilde{u}}^*\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{x} \\ \delta\tilde{\boldsymbol{u}}\end{bmatrix}\mathrm{d}t \tag{35}$$

subject to linearized dynamics

$$\delta\dot{\boldsymbol{x}}(t) = \boldsymbol{f}_x^*(t)\delta\boldsymbol{x}(t) + \boldsymbol{f}_u^*(t)\delta\boldsymbol{u}(t), \qquad \delta\boldsymbol{x}(\tau) = \delta\boldsymbol{x}(0) \tag{36}$$

and subject to the linearized constraint

$$C_x^{i(t)}(t)\delta\boldsymbol{x}(t) + C_{\tilde{u}}^{i(t)}(t)\delta\tilde{\boldsymbol{u}}(t) = 0 \tag{37}$$

With Assumptions 3 and 4, the solution to AMP is given in terms of an absolutely continuous Lagrange multiplier vector $\boldsymbol{\delta\lambda}(t) \in \mathbb{R}^n$, and satisfies the periodic Hamiltonian system

$$\begin{bmatrix}\dot{\boldsymbol{x}}(t) \\ \boldsymbol{\delta}\dot{\boldsymbol{\lambda}}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}A(t) & -B(t) \\ -D(t) & -A^T(t)\end{bmatrix}}_{H(t)}\begin{bmatrix}\boldsymbol{x}(t) \\ \boldsymbol{\delta\lambda}(t)\end{bmatrix} \tag{38}$$

where the $n \times n$ matrices $A(t)$, $B(t) = B^T(t)$, and $D(t) = D^T(t)$ are functions of the system partial derivatives, with their functional form for $t \in$ U, $t \in$ CU or $t \in$ CC. For $t \in$ U, they are given by

$$A(t) = \boldsymbol{f}_x(t) - \boldsymbol{f}_u(t)\tilde{\mathcal{H}}_{uu}^{-1}(t)\tilde{\mathcal{H}}_{ux}(t) \tag{39}$$

$$B(t) = \boldsymbol{f}_u(t)\tilde{\mathcal{H}}_{uu}^{-1}(t)\boldsymbol{f}_u^T(t) \tag{40}$$

$$D(t) = \tilde{\mathcal{H}}_{xx}(t)\tilde{\mathcal{H}}_{xu}(t)\tilde{\mathcal{H}}_{uu}^{-1}(t)\tilde{\mathcal{H}}_{ux}(t) \tag{41}$$

For $t \in$ CU, the matrices of $H(t)$ are given by

$$\begin{aligned}A(t) &= \boldsymbol{f}_x(t) - \boldsymbol{f}_{\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)\left\{\mathcal{H}_{\tilde{u}x}(t) + \theta(t)C_{\tilde{u}}^T(t)\right. \\ &\quad \left.\times\left[C_x(t) - C_{\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)\mathcal{H}_{\tilde{u}x}(t)\right]\right\}\end{aligned} \tag{42}$$

$$B(t) = \boldsymbol{f}_{\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)\left[\boldsymbol{f}_{\tilde{u}}^T(t) - \theta(t)C_{\tilde{u}}^T(t)C_{\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)\boldsymbol{f}_{\tilde{u}}^T(t)\right] \tag{43}$$

$$\begin{aligned}D(t) &= \mathcal{H}_{xx}(t) - \mathcal{H}_{x\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)\left\{\mathcal{H}_{\tilde{u}x}(t) + \theta(t)C_{\tilde{u}}^T(t)\right. \\ &\quad \left.\times\left[C_x(t) - C_{\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)\mathcal{H}_{\tilde{u}x}(t)\right]\right\} \\ &\quad + \theta(t)C_x^T(t)\left[C_x(t) - C_{\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)\mathcal{H}_{\tilde{u}x}(t)\right]\end{aligned} \tag{44}$$

where the scalar function $\theta(t)$ is given by

$$\theta(t) = \frac{1}{C_{\tilde{u}}(t)\mathcal{H}_{\tilde{u}\tilde{u}}^{-1}(t)C_{\tilde{u}}^T(t)} \tag{45}$$

Finally, for $t \in$ CC, $H(t)$ reduces to the degenerate form

$$A(t) = \boldsymbol{f}_x(t) - \boldsymbol{f}_u(t)C_u^{-1}(t)C_x(t) \tag{46}$$

$$B(t) = 0 \tag{47}$$

$$\begin{aligned}D(t) &= \mathcal{H}_{xx}(t) - \mathcal{H}_{xu}(t)C_u^{-1}(t)C_x(t) \\ &\quad - C_x^T(t)C_u^{-1}(t)\left[\mathcal{H}_{ux}(t) + \mathcal{H}_{uu}(t)C_u^{-1}(t)C_x(t)\right]\end{aligned} \tag{48}$$

With the exception of the corner times $\{t_i\}$, the solution to Eq. (38) may be expressed for every $t \geq 0$ in terms of a state transition matrix $\Phi(t, 0)$, which satisfies the differential equation

$$\dot{\Phi}(t, 0) = H(t)\Phi(t, 0), \qquad \Phi(0, 0) = I_{2n} \tag{49}$$

where I_{2n} denoted the $2n$-dimensional identity matrix. If the augmented state/costate vector is defined as $\boldsymbol{y} \overset{\triangle}{=} \mathrm{col}\{\boldsymbol{x}, \lambda\} \in \mathbb{R}^{2n}$, then $\delta\Im$ can be described by $\delta\boldsymbol{y}$ from this state transition matrix as $\delta\boldsymbol{y}(t) = \Phi(t, 0)\delta\boldsymbol{y}(0)$. The state variation can be carried across t_i using Eq. (31). The regulators derived in Sec. VII will rely strongly on the spectral structure of $\Phi(t, 0)$. This structure derives from the fact that the monodromy matrix $\Phi(\tau, 0)$ [and in general any $\Phi(t, 0)$] is a symplectic matrix. Propagation of a symplectic initial condition across any given subarc is clearly symplectic, due to the Hamiltonian structure of Eq. (38) irrespective of the subarc. Moreover, simple matrix algebra (omitted here) shows that $\Phi(t_i^+, t_i^-)$ is a symplectic matrix for both possible discontinuity types examined in Sec. VI.A. The resulting symplectic property for $\Phi(\tau, 0)$ implies that its eigenvalues are symmetric about the unit circle.[21]

Of course, the time-varying matrix $P(\cdot)$ satisfies a generalized Riccati equation.[17] In terms of the Hamiltonian matrices appearing in Eq. (38), $P(\cdot)$ satisfies

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)P(t) - D(t) \tag{50}$$

In the case of the hypersonic cruise problem, differential equation (50) reduces to a Lyapunov differential equation for $t \in$ CC. In addition, $P(\cdot)$ is discontinuous at each t_i; this discontinuity can be written in terms of $\Phi(t_i^+, t_i^-)$ as

$$\begin{aligned}P\left(t_i^+\right) &= \left[\Phi_{21}\left(t_i^+, t_i^-\right) + \Phi_{22}\left(t_i^+, t_i^-\right)P\left(t_i^-\right)\right] \\ &\quad \times\left[\Phi_{11}\left(t_i^+, t_i^-\right) + \Phi_{12}\left(t_i^+, t_i^-\right)P\left(t_i^-\right)\right]^{-1}\end{aligned} \tag{51}$$

C. Sufficient Conditions

Having established this particular structure of the second variation, local sufficiency of the constrained OPC problem may now be addressed. As was stated earlier, attention is focused on the infinitely repeated periodic process. The second variation appearing in Eq. (11) may be written as

$$\begin{aligned}\delta^2 J &= \lim_{k\to\infty}\frac{1}{k\tau}\left\{\left[\delta\boldsymbol{x}_0^T \quad \mathrm{d}(k\tau)\right]\begin{pmatrix}0 & \mathcal{H}_x^{*T} \\ \mathcal{H}_x^* & -\mathcal{H}_x^*\boldsymbol{f}^*\end{pmatrix}_{t=0}\begin{bmatrix}\delta\boldsymbol{x}_0 \\ \mathrm{d}(k\tau)\end{bmatrix}\right. \\ &\quad + \int_0^{k\tau}(\delta\boldsymbol{x}^T \quad \delta\tilde{\boldsymbol{u}}^T)\begin{pmatrix}\mathcal{H}_{xx}^* & \mathcal{H}_{xu}^* \\ \mathcal{H}_{\tilde{u}x}^* & \mathcal{H}_{\tilde{u}\tilde{u}}^*\end{pmatrix}\begin{pmatrix}\delta\boldsymbol{x} \\ \delta\boldsymbol{u}\end{pmatrix}\mathrm{d}t \\ &\quad \left. + \sum_{j=1}^{k}\sum_{i=1}^{l}\left(\frac{\partial^2 J}{\partial t_i^2}\right)\mathrm{d}t_i^2(j\tau)\right\}\end{aligned} \tag{52}$$

where l is the number of instants of discontinuous control on $[0, \tau]$.

For second-order sufficient conditions, let the extremal solution $[\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ be an extremal solution to the constrained OPC problem that satisfies Assumptions 1–5. In addition, suppose the following.

1) There exists a real, symmetric, periodic solution $P(\cdot)$ propagated on $t \in$ U and $t \in$ CU by the mixed Riccati/Lyapunov differential equation (50).

2) If t_i is a point of discontinuous control, then the following convexity condition $\partial^2 J/\partial t_i^2 > 0$ is satisfied.

3) The monodromy matrix $\Phi(\tau, 0)$ possesses no unit magnitude eigenvalues beyond the coupled pair at unity.

Then $[\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), S^*(\cdot), \tau^*]$ is a weak local minimum of the infinitely repeated constrained OPC problem.

Remark. The proof of these conditions will not be given here, is given completely in Ref. 22, and parallels directly the results of Ref. 9 for the unconstrained case if the mathematical structure of Ref. 17 is utilized. □

VII. Optimal Periodic Regulation

For the development of the rest of this paper, it is assumed that the extremal periodic trajectory obtained satisfies the sufficient conditions for local optimality; this is required in order to compute the periodic feedback gains. A confusing aspect of regulation about periodic solutions is the difference a running time on the actual regulated system and an index time on the periodic path, which identifies the point about which state variations are defined. In the following sections, the running time will be denoted by σ and is, of course, monotonically increasing. The index time associated with the periodic solution will be denoted by t and is τ periodic.

A. Constant-Parameter Optimal Periodic Regulator

In Ref. 5, a regulator was derived using variational calculus in which $\boldsymbol{u}$ of Eq. (6) is given as a feedback control on a variation $\delta\boldsymbol{x}(t)$, providing exponential convergence to the periodic orbit $\Im^*$ to first order while minimizing the second variation of the performance index (7). The extension of these results to the present case of a mixed state/control inequality constraint is straightforward. The optimal periodic regulator will be briefly reviewed, its extension to the constrained case established, and its performance on the hypersonic vehicle studied.

The results of Ref. 5 for the unconstrained case will now be reviewed. Letting the variation in Lagrange multiplier $\delta\lambda(t)$ be related to $\delta\boldsymbol{x}(t)$ by $\delta\lambda(t) = P(t)\delta\boldsymbol{x}(t)$ and expanding Eq. (19) about the extremal periodic path $\Im^*$ yields

$$\delta\boldsymbol{u}(t) = \tilde{\mathcal{H}}_{\boldsymbol{uu}}^{-1}(t)\left[\tilde{\mathcal{H}}_{\boldsymbol{ux}}(t) + \boldsymbol{f}_{\boldsymbol{u}}^T(t)P(t)\right]\delta\boldsymbol{x}(t) \tag{53}$$

The particular choice of control variation $\delta\boldsymbol{u}(\cdot)$ in Eq. (53) ensures that the second variation $\delta^2 J$ of the periodic cost is extremized with respect to $\delta\boldsymbol{u}(\cdot)$. It remains to properly define the variation $\delta\boldsymbol{x}(t)$ because any point on a periodic path could be regarded as a possible point of reference (which is hereafter referred to as the index point). Define the following matrix concatenation of vectors:

$$\begin{bmatrix} V(0) \\ W(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1(0) & \cdots & \boldsymbol{v}_n(0) \\ \boldsymbol{w}_1(0) & \cdots & \boldsymbol{w}_n(0) \end{bmatrix} \tag{54}$$

where $[\boldsymbol{v}_1^T(0), \boldsymbol{w}_1^T(0)]^T = [\boldsymbol{f}^T(0), -\tilde{\mathcal{H}}_{\boldsymbol{x}}(0)]^T$ is the primary eigenvector of the unity eigenvalue of $\Phi(\tau, 0)$, and

$$\begin{bmatrix} \boldsymbol{v}_1(0) & \cdots & \boldsymbol{v}_n(0) \\ \boldsymbol{u}_1(0) & \cdots & \boldsymbol{w}_n(0) \end{bmatrix} = [\boldsymbol{e}_2(0) \quad \cdots \quad \boldsymbol{e}_n(0)] \cdot \Gamma \tag{55}$$

where $\{\boldsymbol{e}_2(0) \ \cdots \ \boldsymbol{e}_n(0)\}$ are the eigenvectors of $\Phi(\tau, 0)$ corresponding to those eigenvalues within the unit circle and Γ is such that the right-hand side of Eq. (55) is real. Then this matrix spans an invariant subspace of $\Phi(\tau, 0)$

$$\Phi(\tau, 0) \cdot \begin{bmatrix} V(0) \\ W(0) \end{bmatrix} = \begin{bmatrix} V(0) \\ W(0) \end{bmatrix} X \tag{56}$$

where X possesses one unity eigenvalue and $n-1$ eigenvalues whose magnitudes are strictly less than 1.

This stable, invariant subspace for $\Phi(\tau, 0)$ can be propagated forward in time to be invariant for $\Phi(t + \tau, t)$ by invoking the Floquet theorem.[23] The state transition matrix of any linear system with periodic coefficients may be written in terms of a matrix exponential as

$$\Phi(t, 0) = R(t)e^{\bar{G}t} \tag{57}$$

where $R(t)$ is a $2n \times 2n$ periodic matrix function and $\bar{G}$ is a $2n \times 2n$ constant matrix. It can be shown that the columns of the time-varying matrix

$$\begin{bmatrix} V(t) \\ W(t) \end{bmatrix} = R(t)\begin{bmatrix} V(0) \\ W(0) \end{bmatrix} \tag{58}$$

form an invariant subspace of $\Phi(t + \tau, t)$ and produce the same invariant form X as in Eq. (56). The existence of this stable, invariant subspace for every $t \in [0, \tau]$ suggests a method for choosing the index point t such that $\delta\boldsymbol{x}(t) = \boldsymbol{x}(\sigma) - \boldsymbol{x}^*(t)$ [as propagated by the Hamiltonian system (38)] converges to zero as $\sigma \to \infty$. Let $\delta\boldsymbol{x}(t)$ be written in terms of the column vectors of $V(t)$ as $\delta\boldsymbol{x}(t) = V(t)\delta\alpha(t)$, where $\delta\alpha^T(t) = [\delta\alpha_1(t), \delta\alpha_{n-1}^T(t)]^T \in \mathbb{R}^n$. It is shown in Ref. 5 that, if t is chosen such that the resulting $\delta\boldsymbol{x}(t)$ has the property that $\delta\alpha_1(t) = 0$, then $\boldsymbol{x}(\sigma)$ is exponentially convergent to $\boldsymbol{x}^*(\cdot)$ as $\sigma \to \infty$. The following convergent computational algorithm was also given to compute this index point:

$$\mathrm{d}t_k \triangleq \delta\alpha_1(t_k) = \Gamma_1^T(t_k)[\boldsymbol{x} - \boldsymbol{x}^*(t_k)], \qquad t_{k+1} = t_k + \mathrm{d}t_k \tag{59}$$

where $\Gamma_1(t)$ is the first column of the inverse of $V^T(t)$.

The control variation $\delta\tilde{\boldsymbol{u}}(t)$, thus, can be written as a feedback on $\delta\boldsymbol{x}(t)$ of the form

$$\delta\tilde{\boldsymbol{u}}(t) = G_{\boldsymbol{ux}}(t)\delta\boldsymbol{x}(t) \tag{60}$$

The gain vector $\boldsymbol{G}_{\boldsymbol{ux}}(\cdot)$ is a function of $\Im^*$, and its form depends on whether $t \in$ U, $t \in$ CC, or $t \in$ CU. For $t \in$ U,

$$\boldsymbol{G}_{\boldsymbol{ux}}(t) = -\mathcal{H}_{\boldsymbol{uu}}^{\ast-1}(t)\left[\mathcal{H}_{\boldsymbol{ux}}^{\ast}(t) + \boldsymbol{f}_{\boldsymbol{u}}^{*T}(t)P(t)\right] \tag{61}$$

For $t \in$ CC, feedback is not generally applicable because C_L is chosen entirely by satisfaction of constraint. For $t \in$ CU,

$$\begin{aligned} \boldsymbol{G}_{\boldsymbol{ux}}(t) = {} & -\mathcal{H}_{\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}}^{\ast-1}(t)\Big\{\mathcal{H}_{\tilde{\boldsymbol{u}}\boldsymbol{x}}^{\ast}(t) + \theta(t)C_{\tilde{\boldsymbol{u}}}^T(t) \\ & \times \left[C_{\boldsymbol{x}}(t) - C_{\tilde{\boldsymbol{u}}}(t)\mathcal{H}_{\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}}^{-1}(t)\mathcal{H}_{\tilde{\boldsymbol{u}}\boldsymbol{x}}(t)\right]\delta\boldsymbol{x}(t) \\ & + \left[\boldsymbol{f}_{\tilde{\boldsymbol{u}}}^T(t) - \theta(t)C_{\tilde{\boldsymbol{u}}}^T(t)C_{\tilde{\boldsymbol{u}}}(t)\mathcal{H}_{\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}}^{\ast-1}(t)\boldsymbol{f}_{\tilde{\boldsymbol{u}}}^T(t)\right]P(t)\Big\} \end{aligned} \tag{62}$$

This expression gives the entire control vector $\tilde{\boldsymbol{u}}(\cdot)$, which solves the accessory minimum problem and which satisfies the equality constraint to first order. To satisfy the constraint exactly, only a subset of this control may be used for optimization, with the remaining elements chosen to satisfy the constraint. This approach was taken in the present study, with C_L computed based on feedback and S determined by the constraint.

To implement the regulator for the hypersonic cruise problem, a means of predicting on-line the variation in throttle switch time is necessary. Using the semigroup property of transition matrices, $\mathrm{d}t_i$ may be rewritten (to first order) as

$$\mathrm{d}t_i = \left\{\left[\frac{\mathcal{H}_{\boldsymbol{y}}^{\ast}\left(t_i^-\right) - \mathcal{H}_{\boldsymbol{y}}^{\ast}\left(t_i^+\right)}{\mathcal{H}_{\boldsymbol{y}}^{\ast}\left(t_i^+\right)K\mathcal{H}_{\boldsymbol{y}}^{\ast T}\left(t_i^-\right)} \cdot \Phi\left(t_i^-, 0\right)\right] \cdot \Phi^{-1}(t, 0)\right\}\delta\boldsymbol{y}(t) \tag{63}$$

The row vector inside the braces of Eq. (63) is a constant for a given extremal solution; thus, knowledge of $\Phi(t, 0)$ for $t < t_i$ allows for a prediction for the variation in switch time.

With these additional considerations, the autonomous optimal periodic regulator was implemented on the hypersonic vehicle case study. The regulator was implemented on an extremal solution computed for $g_{\max} = 6$; thus, the nominal trajectory possessed a larger maximum altitude than Figs. 2–6 but was otherwise structurally identical. The Riccati solution $P(\cdot)$ and $\Phi(\tau, 0)$ were computed for this solution and satisfied the sufficient optimality conditions. The performance of the regulator is shown in Figs. 7 and 8. The regulator was simulated through about four and a half cycles. The results show a rapid initial convergence rate, even with the presence of load constraint and with a rather large initial state perturbation.

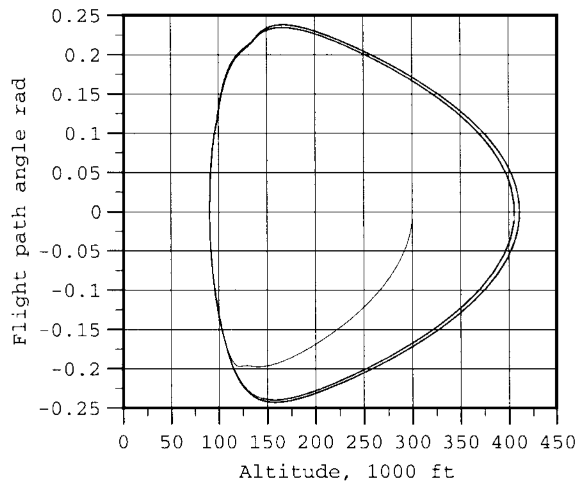

Fig. 8 Time-invariant regulator: γ vs h.

B. Slowly Varying Regulator

The regulator of the preceding section possesses the additional advantage over other possible regulation schemes by also minimizing the second variation of the cost away from its extremal value, while also providing rapid convergence, even for relatively large initial state deviation. As with all aspects of OPC theory, however, this regulator assumes that system parameters are time invariant. This assumption is never generally true, but in problems of aircraft cruise, this assumption is logically baseless, as vehicle mass decreases monotonically with fuel consumption, a model of which is included in the problem through the formulation of cost. Therefore, it is desirable to consider how OPC theory may be applied to slowly varying systems. Of course, extremal solutions of periodic state and control do not exist for time-varying systems; in the case of regulation, however, the problem of driving a slowly varying system to a quasiperiodic solution may be posed, where this target trajectory traverses a family of extremal periodic OPC solutions on the slow time scale. In Ref. 5, an optimal periodic regulator was derived, which provided convergence to an extremal periodic path corresponding to a time-invariant parameter variation. It will turn out that the form of the feedback controller for this slowly varying problem will be identical to this constant–parameter perturbation regulator, but with the parameter varying slowly. A marked difference from Ref. 5, however, will be the presence of the constraints and a prediction of a proper variation in corner times as a function of both state variation and parameter change.

We concern ourselves with regulation of a slowly varying system of the form

$$\dot{\boldsymbol{x}}(\sigma) = \boldsymbol{f}[\boldsymbol{x}(\sigma), \boldsymbol{u}(\sigma), S(\sigma), m(\zeta)], \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{64}$$

where the scalar parameter m is assumed to vary on the slow time scale defined by $\zeta = \epsilon\sigma$, with $0 < \epsilon \ll 1$. In the implementation of this regulator, it is assumed that the OPC problem has been solved to obtain the pair $\{\boldsymbol{y}^*(\cdot), \boldsymbol{u}^*(\cdot)\}$ for a nominal value of the parameter, e.g., m_0. It is further assumed that a periodic direction vector $\Delta\hat{\boldsymbol{y}}(\cdot)$ is known that describes changes on the extremal periodic orbit for small constant variations in m, by the equation $\boldsymbol{y}^*(\sigma; m_0 + \mathrm{d}m) = \boldsymbol{y}^*(t, m_0) + \Delta\hat{\boldsymbol{y}}(t)\,\mathrm{d}m$. This family direction can be computed as in Ref. 5 by solving two linear differential equations on the nominal periodic path. The accessory minimum problem is now formulated with the extremal solution about which variations are made changing on the slow scale according to $m(\zeta)$. The solution to this problem solves a Hamiltonian system analogous to Eq. (38) but depending on two time scales, as

$$\delta\dot{\boldsymbol{y}}(t, \zeta) = H(t, \zeta)\delta\boldsymbol{y}(t, \zeta) \tag{65}$$

where the overdot represents the total derivative. The solution to Eq. (65) may be written in terms of a state transition matrix $\Phi(t, 0; \zeta)$, which is dependent on the slow time scale. This matrix can also be written in terms of the Floquet theorem as

$$\Phi(t, 0; \zeta) = R(t, \zeta)e^{\bar{G}(\zeta)t} \tag{66}$$

where, for a fixed ζ, the matrix $R(\cdot, \zeta)$ is periodic of period $\tau(\zeta)$. Introducing the transformation $\delta\boldsymbol{y}(t, \zeta) = R(t, \zeta)\delta z(t, \zeta)$, one obtains upon substitution the following differential equation in δz:

$$\delta\dot{z}(t, \zeta) = [\bar{G}(\zeta) - \epsilon\hat{G}(t, \zeta)]\delta z(t, \zeta) \tag{67}$$

where $\hat{G}(t, \zeta) = R^{-1}(t, \zeta)R_\zeta(t, \zeta)$. This transformed system may now be solved using a multiple time scale procedure.[19] Let the solution to Eq. (67) be written as an asymptotic expansion of the form $\delta z(t, \zeta) = \boldsymbol{f}_0(t, \zeta) + \epsilon\boldsymbol{f}_1(t, \zeta) + \cdots$, where each $\boldsymbol{f}_i$ is a bounded function of t and ζ in $\mathbb{R}^{2n}$. Substitution of this form into Eq. (67) and equating like powers of t yields

$$\mathcal{O}(1): \frac{\partial \boldsymbol{f}_0}{\partial t} - \bar{G}(\zeta)\boldsymbol{f}_0(t, \zeta) = 0 \tag{68}$$

$$\mathcal{O}(\epsilon): \frac{\partial \boldsymbol{f}_1}{\partial t} - \bar{G}(\zeta)\boldsymbol{f}_1(t, \zeta) = -\hat{G}(t, \zeta)\boldsymbol{f}_0(t, \zeta) - \frac{\partial \boldsymbol{f}_0}{\partial \zeta} \tag{69}$$

Let $s > t$. From Eq. (68), $\boldsymbol{f}_0(s, \zeta)$ may be written to first order in ϵ as

$$\boldsymbol{f}_0(s, \zeta) = e^{\bar{G}(\zeta)(s-t)}\delta z(t, \zeta) = e^{\bar{G}(s-t)}V(\zeta)\delta\boldsymbol{k}(t, \zeta) \tag{70}$$

where $\delta z(t, \zeta)$ was written in terms of the slowly varying basis set $V(\zeta)$. By examining the solution to Eq. (69) using the solution to Eq. (68), a condition on $\delta\boldsymbol{k}(t, \zeta)$ can be found such that the asymptotic expansion for $\boldsymbol{f}$ is uniformly valid. Though the manipulations are laborious (they are omitted here but are contained in Ref. 22), the result is analogous to the time-invariant regulator: $\delta\boldsymbol{k}(t, \zeta)$ must be such that

$$\delta\boldsymbol{k}(t, \zeta) = \begin{bmatrix} \delta\boldsymbol{k}_s(t, \zeta) \\ 0 \end{bmatrix} = \left\{ \begin{matrix} \begin{bmatrix} \delta\boldsymbol{k}_s^{n-1}(t, \zeta) \\ \delta\boldsymbol{k}_s^{n}(t, \zeta) \end{bmatrix} \\ 0 \end{matrix} \right\} \tag{71}$$

Reflecting the solution for $\delta z(t, \zeta)$ through the transformation matrix $R(t, \zeta)$ with this structure for $\delta\boldsymbol{k}(t, \zeta)$ yields the following:

$$\begin{bmatrix} \delta\boldsymbol{x}(t, \zeta) \\ \delta\lambda(t, \zeta) \end{bmatrix} = R(t, \zeta) \begin{bmatrix} \boldsymbol{v}_{11}^{(1)}(\zeta) & \cdots & \boldsymbol{v}_{11}^{(n-1)}(\zeta) \\ \boldsymbol{v}_{21}^{(1)}(\zeta) & \cdots & \boldsymbol{v}_{21}^{(n-1)}(\zeta) \end{bmatrix} \delta\boldsymbol{k}_s^{n-1}(t, \zeta)$$

$$+ \nu(t, \zeta) \begin{bmatrix} \boldsymbol{f}^*(t, \zeta) \\ -\mathcal{H}_{\boldsymbol{x}}^*(t, \zeta) \end{bmatrix} \delta\boldsymbol{k}_s^{n}(t, \zeta) \tag{72}$$

where $\nu(t, \zeta)$ is a general nonzero scalar function. Convergence to the slowly varying family $\Im^*(\zeta)$ of extremal periodic solutions to order $\mathcal{O}(\epsilon)$ thus requires that $\delta\boldsymbol{x}(t, \zeta)$ have no component in the direction of $\boldsymbol{f}^*(t, \zeta)$.

This condition, familiar from Sec. VII.A, is still not implementable in a feedback control setting because only the nominal periodic path $\Im^*(0)$ is available to form the state variation via the index point algorithm given in Sec. V. Let $t_{\mathrm{d}m}$ denote the index point associated with orbit $\Im^*(m + \mathrm{d}m)$ for a given state vector $\boldsymbol{x}$, and let t denote the index point associated with $\Im^*(m)$ for the same state $\boldsymbol{x}$. Through an order analysis, it is easily shown that $t_{\mathrm{d}m}$ and t differ at most by $\mathcal{O}(\mathrm{d}m)$. This can be shown by expanding the first-order approximation for $t_{\mathrm{d}m}$ appearing in Eq. (59):

$$t_{\mathrm{d}m} = t + \Gamma_1^T(t_{\mathrm{d}m}; m + \mathrm{d}m)\big[\boldsymbol{x} - \boldsymbol{x}^*(t_{\mathrm{d}m}; m + \mathrm{d}m)\big] \tag{73}$$

in terms of m. This fact implies that, to first order in state perturbation and $\mathrm{d}m$,

$$\delta\boldsymbol{x}(t_{\mathrm{d}m}; m + \mathrm{d}m) = \delta\boldsymbol{x}(t; m) - \Delta\hat{\boldsymbol{x}}(t; m)\,\mathrm{d}m \tag{74}$$

Expanding the stationarity condition $\mathcal{H}_{\boldsymbol{u}}(\boldsymbol{y}, \boldsymbol{u}, m + \mathrm{d}m) = 0$ with Eq. (74) and $t_{\mathrm{d}m} = t + \mathcal{O}(\mathrm{d}m)$ and retaining principal terms yields a feedback control for the slowly varying regulation problem of the form

$$\boldsymbol{u} = \boldsymbol{u}^*(t; m) + \boldsymbol{G}_{\boldsymbol{ux}}(t; m)\delta\boldsymbol{x}(t; m) + \boldsymbol{G}_{\boldsymbol{um}}(t; m)\,\mathrm{d}m(\zeta) \tag{75}$$

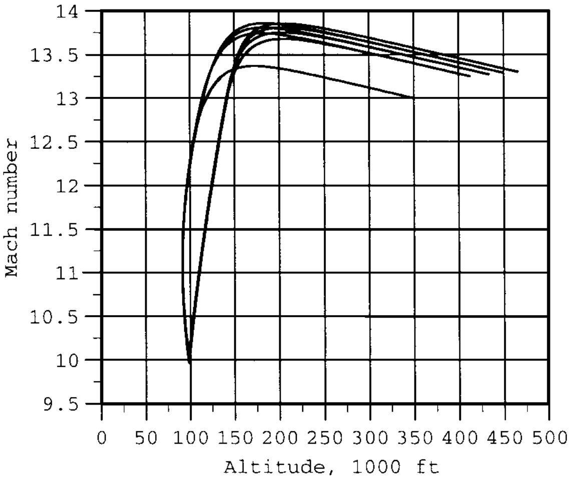

Fig. 9 Slowly varying regulator: *M* vs *h*.

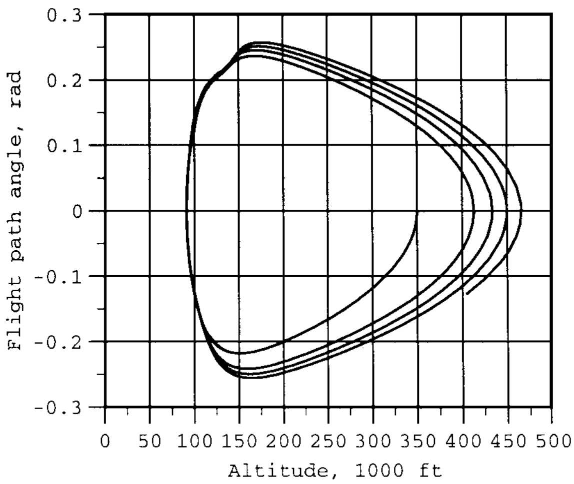

Fig. 10 Slowly varying regulator: γ vs *h*.

where the form of $\boldsymbol{G}_{ux}$ is identical to that given in Sec. VII.A and the vector gain $\boldsymbol{G}_{um}$ is given on the unconstrained arcs by

$$\boldsymbol{G}_{um}(t;m) = -\mathcal{H}^{f\,-1}_{uu}(t)\left\{\boldsymbol{f}_u^T(t)[\Delta\hat{\lambda}(t) - P(t)\Delta\hat{\boldsymbol{x}}(t)] + \mathcal{H}^f_{um}(t)\right\} \tag{76}$$

and on the CU constrained arc by

$$\boldsymbol{G}_{um}(t;m) = -\mathcal{H}^{f\,-1}_{uu}(t)\left\{\left[\boldsymbol{f}_u^T(t) - \theta(t)C_u^T(t)C_u(t)\mathcal{H}^{f\,-1}_{uu}(t)\boldsymbol{f}_u^T(t)\right] \times [\Delta\hat{\lambda}(t) - P(t)\Delta\hat{\boldsymbol{x}}(t)] + \mathcal{H}^f_{um}(t) - \theta(t)C_u^T(t)C_u(t)\mathcal{H}^{f\,-1}_{uu}(t) \times \mathcal{H}^f_{um}(t) + \theta(t)C_u(t)C_m(t)\right\} \tag{77}$$

A full implementation of the regulator in the presence of discontinuous control requires a first-order prediction of the change in the corner time from $t_i(m)$ to $t_i(m + \mathrm{d}m)$. The form for this prediction may be again derived by an expansion of the Weierstrass–Erdmann condition as in the Appendix, here accounting for the parameter perturbation. The prediction $\mathrm{d}t_i$ takes the following form:

$$\mathrm{d}t_i = \left[\frac{\mathcal{H}^f_y(t_i^-) - \mathcal{H}^f_y(t_i^+)}{\mathcal{H}^f_y(t_i^+)K\mathcal{H}^f_y(t_i^-)}\right][\delta\boldsymbol{y}(t_i) + \Delta\hat{\boldsymbol{y}}(t_i)\,\mathrm{d}m] + \left[\frac{\mathcal{H}^f_m(t_i^-) - \mathcal{H}^f_m(t_i^+)}{\mathcal{H}^f_y(t_i^+)K\mathcal{H}^f_y(t_i^-)}\right]\mathrm{d}m \tag{78}$$

This regulator was implemented on the present hypersonic vehicle case study by considering the vehicle weight (or mass) fraction as the slowly varying parameter, that is, $\mathrm{d}W/W_0$. The simulation presented was performed for $\epsilon = -3.0 \times 10^{-3}$; this choice assumes that the vehicle mass relative to gross takeoff mass decreases by 3% across every periodic cycle of the extremal solutions of Sec. V. The results of the simulation are shown in Figs. 9 and 10. After an initial strong regulation phase to the neighborhood of the nominal solution, the regulated system begins to traverse a family of periodic extremal trajectories of increasing energy. Note that the orbit is almost invariant at low altitude; this is because, in regions of high atmospheric density, aerodynamic forces greatly dominate the vehicle body forces, suggesting that the orbit is rather insensitive to mass changes at low altitude.

VIII. Conclusions

The theory of optimal periodic control and regulation was applied to realistic, vertical-plane hypersonic flight under an acceleration constraint. An optimal periodic cruise trajectory was found using a complex, nonlinear point-mass model, which yielded more than 10% improvement over optimal static cruise, with maximum vehicle acceleration constrained to be below 5 times that of gravity. The nature of the periodic cruise trajectories offers other advantages beyond their fuel optimality, including lower surface heating, due to a high-altitude quenching phase. An optimal periodic regulator was implemented for this case study, demonstrating remarkable convergence to the extremal periodic path, in spite of the active acceleration constraint during regulation. Finally, the regulator was extended to accommodate the slow decay in vehicle mass, which is ignored under classical OPC theory.

Appendix: Order Relation for Corner Times

To establish the order relation, we examine the Weierstrass–Erdmann optimality condition for time instant $t_i + \mathrm{d}t_i$ of discontinuous control resulting from a control variation of the form $\delta\boldsymbol{u}(t) = \epsilon\eta(t)$. This is given by

$$\tilde{\mathcal{H}}^f\left\{\boldsymbol{y}(t_i + \mathrm{d}t_i), \boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^+\right], S = 1\right\} - \tilde{\mathcal{H}}^f\left\{\boldsymbol{y}(t_i + \mathrm{d}t_i), \boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^-\right], S = 0\right\} = 0 \tag{A1}$$

Consider first an appropriate expansion of the first term in Eq. (A1). We first expand as

$$\tilde{\mathcal{H}}^f\left\{\boldsymbol{y}(t_i + \mathrm{d}t_i), \boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^+\right], S = 1\right\} = \tilde{\mathcal{H}}^f\left[(t_i + \mathrm{d}t_i)^+\right] + \tilde{\mathcal{H}}^f_y\left[(t_i + \mathrm{d}t_i)^+\right]\delta\boldsymbol{y}(t_i + \mathrm{d}t_i) + \mathcal{O}\left\{\|\delta\boldsymbol{y}(t_i + \mathrm{d}t_i)\|^2, \left|\delta\boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^+\right]\right|^2\right\} \tag{A2}$$

where $\tilde{\mathcal{H}}^f_u[(t_i + \mathrm{d}t_i)^+] = 0$ was used. We wish to expand further in terms of $\mathrm{d}t_i$. Using the fundamental theorem of calculus, one can write

$$\tilde{\mathcal{H}}^f_y\left[(t_i + \mathrm{d}t_i)^+\right] = \tilde{\mathcal{H}}^f_y\left(t_i^+\right) + \int_{t_i^+}^{(t_i + \mathrm{d}t_i)^+} \dot{\tilde{\mathcal{H}}}^f_y[\boldsymbol{y}^*(t), \boldsymbol{u}^*(t), S = 1]\,\mathrm{d}t = \tilde{\mathcal{H}}^f_y\left(t_i^+\right) + a_0^*\,\mathrm{d}t_i + a_1^*\,\mathrm{d}t_i^2 + \cdots$$

where the integral has been written as an asymptotic expansion in powers of $\mathrm{d}t_i$. A superscript asterisk appears on the expansion coefficients to indicate that they are a function only of $\boldsymbol{y}^*$ and $\boldsymbol{u}^*$ and not $\mathrm{d}t_i$. For example, a_0^* can easily be written as

$$a_0\left(t_i^+\right) = \left[\tilde{\mathcal{H}}^f_{yy}\left(t_i^+\right) - \tilde{\mathcal{H}}^f_{yu}\left(t_i^+\right)\tilde{\mathcal{H}}^{f\,-1}_{uu}\left(t_i^+\right)\tilde{\mathcal{H}}^f_{uy}\left(t_i^+\right)\right]\mathcal{F}^*\left(t_i^+\right)$$

With this expansion and noting that $\tilde{\mathcal{H}}^f$ is a constant, Eq. (A2) can be written as

$$\tilde{\mathcal{H}}^f\left\{\boldsymbol{y}(t_i + \mathrm{d}t_i), \boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^+\right], S = 1\right\} = \tilde{\mathcal{H}}^f\left(t_i^+\right) + \left[\tilde{\mathcal{H}}^f_y\left(t_i^+\right) + a_0^*\,\mathrm{d}t_i + a_1^*\,\mathrm{d}t_i^2 + \cdots\right]\delta\boldsymbol{y}(t_i + \mathrm{d}t_i) + \mathcal{O}\left\{\|\delta\boldsymbol{y}(t_i + \mathrm{d}t_i)\|^2, \left|\delta\boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^+\right]\right|^2\right\} \tag{A3}$$

Turning our attention to the second term of Eq. (A1), this can also be expanded as

$$\tilde{\mathcal{H}}\left\{\boldsymbol{y}(t_i + \mathrm{d}t_i), \boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^-\right], S = 0\right\}$$

$$= \tilde{\mathcal{H}}\left(t_i^-\right) + \dot{\tilde{\mathcal{H}}}\left(t_i^-\right)\mathrm{d}t_i + b_1\,\mathrm{d}t_i^2 + \cdots$$

In this expansion, b_1 is not evaluated on the extremal path and is at most $\mathcal{O}(1)$ with respect to ϵ as $\epsilon \to 0$. Expanding these terms further, we have

$$\tilde{\mathcal{H}}\left(t_i^-\right) = \tilde{\mathcal{H}}^*\left(t_i^-\right) + \tilde{\mathcal{H}}_y^*\left(t_i^-\right)\delta\boldsymbol{y}(t_i) + \mathcal{O}(\epsilon^2)$$

where $\tilde{\mathcal{H}}_u^*(t_i^-) = 0$ was used. Noting that $\dot{\tilde{\mathcal{H}}}(t_i^-)$ is at most $\mathcal{O}(\epsilon)$, we obtain an expansion of the second term as

$$\tilde{\mathcal{H}}\left\{\boldsymbol{y}(t_i + \mathrm{d}t_i), \boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^-\right], S = 0\right\} = \tilde{\mathcal{H}}^*\left(t_i^-\right)$$

$$+ \tilde{\mathcal{H}}_y^*\left(t_i^-\right)\delta\boldsymbol{y}(t_i) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots \quad \text{(A4)}$$

We have now expanded both terms of Eq. (A1) such that $\mathrm{d}t_i$ appears explicitly in integer powers and ordering is performed with gauge functions of ϵ. Before substituting into Eq. (A1), we note that $\delta\boldsymbol{y}(t_i + \mathrm{d}t_i)$ can be expanded as

$$\delta\boldsymbol{y}(t_i + \mathrm{d}t_i) = \delta\boldsymbol{y}(t_i) + \left[\mathcal{F}^*\left(t_i^-\right) - \mathcal{F}^*\left(t_i^+\right)\right]\mathrm{d}t_i$$

$$+ \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots \quad \text{(A5)}$$

This equation is easily verified by using the definition $\delta\boldsymbol{y}(t_i + \mathrm{d}t_i) = \boldsymbol{y}(t_i + \mathrm{d}t_i) - \boldsymbol{y}^*(t_i + \mathrm{d}t_i)$ and expanding. Equation (A5) implies that

$$\mathcal{O}\left\{\|\delta\boldsymbol{y}(t_i + \mathrm{d}t_i)\|^2, \left|\delta\boldsymbol{u}\left[(t_i + \mathrm{d}t_i)^+\right]\right|^2\right\}$$

$$= \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots \quad \text{(A6)}$$

We are now in a position to substitute Eqs. (A3) [with the order relation (A6)], (A4), and (A5) into Eq. (A1) to obtain

$$\left[\tilde{\mathcal{H}}_y^*\left(t_i^+\right) + a_0^*\left(t_i^+\right)\mathrm{d}t_i + a_1^*\left(t_i^+\right)\mathrm{d}t_i^2 + \cdots\right]\cdot\left\{\epsilon z[t_i;\eta(\cdot)]\right.$$

$$\left. + \left[\mathcal{F}^*\left(t_i^-\right) - \mathcal{F}^*\left(t_i^+\right)\right]\mathrm{d}t_i + \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots\right\}$$

$$+ \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots - \epsilon\tilde{\mathcal{H}}_y^*\left(t_i^-\right)z[t_i;\eta(\cdot)]$$

$$+ \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots = 0 \quad \text{(A7)}$$

Operating on both sides of this equation by $\mathrm{d}/\mathrm{d}\epsilon$ gives

$$\left[a_0^*\left(t_i^+\right)\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} + 2a_1^*\left(t_i^+\right)\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} + \cdots\right]\cdot\left\{\epsilon z[t_i;\eta(\cdot)]\right.$$

$$\left. + \left[\mathcal{F}^*\left(t_i^-\right) - \mathcal{F}^*\left(t_i^+\right)\right]\mathrm{d}t_i + \mathcal{O}(\epsilon)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + \cdots\right\}$$

$$+ \left[\tilde{\mathcal{H}}_y^*\left(t_i^+\right) + a_0^*\left(t_i^+\right)\mathrm{d}t_i + a_1^*\left(t_i^+\right)\mathrm{d}t_i^2 + \cdots\right]$$

$$\times\left\{z[t_i;\eta(\cdot)] + \left[\mathcal{F}^*\left(t_i^-\right) - \mathcal{F}^*\left(t_i^+\right)\right]\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} + \mathcal{O}(1)\mathrm{d}t_i\right.$$

$$\left. + \mathcal{O}(\epsilon)\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} + \mathcal{O}(1)\,\mathrm{d}t_i^2 + 2\mathcal{O}(1)\,\mathrm{d}t_i\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} + \cdots\right\} + \mathcal{O}(\epsilon)$$

$$+ \mathcal{O}(\epsilon)\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} + \mathcal{O}(1)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + 2\mathcal{O}(1)\,\mathrm{d}t_i\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon}$$

$$+ \cdots - \tilde{\mathcal{H}}_y^*\left(t_i^-\right)z[t_i;\eta(\cdot)] + \mathcal{O}(\epsilon) + \mathcal{O}(\epsilon)\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon}$$

$$+ \mathcal{O}(1)\,\mathrm{d}t_i + \mathcal{O}(1)\,\mathrm{d}t_i^2 + 2\mathcal{O}(1)\,\mathrm{d}t_i\frac{\mathrm{d}(\mathrm{d}t_i)}{\mathrm{d}\epsilon} + \cdots = 0 \quad \text{(A8)}$$

If Eq. (A8) is multiplied out, a linear equation in $\mathrm{d}(\mathrm{d}t_i)/\mathrm{d}\epsilon$ is obtained, e.g., $p(\epsilon)[\mathrm{d}(\mathrm{d}t_i)/\mathrm{d}\epsilon] + q(\epsilon) = 0$. The single real root to this equation tends to infinity if and only if $p(\epsilon) \to 0$ as $\epsilon \to 0$. However, inspection of Eq. (A8) yields that

$$\lim_{\epsilon\to 0} p(\epsilon) = \tilde{\mathcal{H}}_y^*\left(t_i^+\right)\mathcal{F}^*\left(t_i^-\right) \quad \text{(A9)}$$

$$\lim_{\epsilon\to 0} q(\epsilon) = \mathcal{H}_y^*\left(t_i^+\right) - \mathcal{H}_y^*\left(t_i^-\right) \quad \text{(A10)}$$

Thus, Assumption 5 guarantees that $\mathrm{d}t_i = \kappa\epsilon + \mathcal{O}(\epsilon^2)$, with κ given by Eq. (29).

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