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$\mathcal{H}_\infty$ Bounded Fault Detection Filter

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A key issue in practical fault detection filter applications is sensitivity to system disturbances and sensor noise. In this paper, a stabilizing fault detection filter gain is found that bounds the $\mathcal{H}_\infty$ norm of the transfer matrix from system disturbances and sensor noise to the residual. For multidimensional faults, a residual direction is identified that enhances the fault signal-to-noise ratio while maintaining the $\mathcal{H}_\infty$ norm bound. By retaining the fault detection filter structure, the practical applicability of the proposed approach is the same as for fault detection filter designs found by conventional eigenvector assignment. In contrast to an unknown input observer approach wherein noise is explicitly decoupled from the fault signal, a noise bounding approach does not impose additional geometric constraints on the filter structure. This allows a given filter to isolate more faults, an important feature because in practical applications low-order filter dynamics are common. Further, the possibility of ill-conditioned filter eigenvectors, occasionally imposed by geometric constraints, may be reduced.

I. Introduction

ANALYTICAL redundancy methods for fault detection and identification use a modeled dynamic relationship between system inputs and measured system outputs to form a residual process. Nominally, faults are detected as the residual process is nonzero only when a fault has occurred and is zero at other times. An example of a residual process for an observable system when no disturbances or sensor noise are present is the innovations process of any stable linear observer. A detection filter is a linear observer with the gain constructed so that when a fault occurs, the residual responds in a known and fixed direction. Thus, when a nonzero residual is detected, a fault can be announced and identified at the same time. Because process disturbances and sensor noise also produce a nonzero residual, the ambiguity must be resolved with an appropriate threshold. Threshold selection schemes are not discussed here in any detail except to say that a residual generator with an improved signal-to-noise ratio should allow for a fault announcement algorithm with improved false-alarm and miss-alarm characteristics.

An objective of a detection filter design in the presence of disturbances is to reduce the component of the residual due to the disturbance without at the same time degrading the component of the residual due to the fault. This suggests as a cost function a ratio of transfer matrix norms.[1,2] In the numerator is the transfer matrix from the disturbance to the detection filter residual, and in the denominator is the transfer matrix from the fault to the detection filter residual. This formulation works well when only one fault is to be detected. Generalized eigenvector solutions are found using a parity equation approach in Ref. 1 and a quadratic optimization approach in Ref. 2. Unfortunately, for the detection filter structure where several faults are isolated simultaneously, no similar problem formulation is available. Edelmayer et al.[3] describe an $\mathcal{H}_\infty$ filtering approach that minimizes the $\mathcal{L}_2$ norm of the disturbance residual component. Chung and Speyer[4] develop in detail a related game-theoretic filter. In both filters, fault isolation is a feature to be developed. A multiobjective method proposed in Ref. 5 allows for either a μ synthesis or an $\mathcal{L}_1$ optimization. The filter design simultaneously bounds the error in an estimate of the fault signal, the residual component due to system disturbances and sensor noise, and the residual component due to structured system perturbations. Again, isolation of multiple faults remains a difficult issue.

The approach taken here is to retain the fault isolation structure of the detection filter. The design follows two steps. First, bound the $\mathcal{H}_\infty$ norm of the transfer matrix from the disturbance to the detection filter fault isolation residuals. Next, working within the noise bound constraint, enhance the residual component due to the associated isolated fault signal. This is done by maximizing the ratio of the residual component due to a fault to the residual component due to the noise. Note that by retaining the structure of a fault detection filter, no additional conditions are imposed on the problem. The practical applicability of the proposed approach is the same as for fault detection filter designs found by conventional eigenvector assignment.[6]

For one-dimensional faults, the primary effect of the first step is to reduce noise transmission through the complementary space, the state subspace independent of all detection spaces, and the second step is not usually needed. This is because, generically, a fault detection space is given by the fault direction itself, which means the detection space is spanned by a single fixed eigenvector. The associated eigenvalue is the only degree of freedom left, and so there is no way to increase the residual component due to a fault without at the same time increasing the residual component due to the noise. In practical applications, plant and actuator failures usually are modeled as one-dimensional faults. Sensor faults generically require a two-dimensional detection space, and so a design freedom exists where a residual component due to a fault could be enhanced.

This paper is organized as follows. Section II gives a very quick review of the Beard–Jones detection filter problem. Section III shows that the detection filter gain is not unique and, given a set of invariant subspaces that solve the detection filter problem, parameterizes the set of detection filter gains. Section IV defines a disturbance robust detection filter problem, and Sec. V provides a stabilizing and $\mathcal{H}_\infty$ bounding detection filter gain by solving a modified algebraic Riccati equation. Section VI enhances the residual component due to the associated isolated fault signal by solving a generalized eigenvalue problem. Section VII provides an application to a simplified aircraft accelerometer and elevon fault detection filter where wind and sensor noise is present. The example illustrates how a numerical integration approach can be applied to solve the modified Riccati equation. Section VIII contains a few concluding remarks.

II. Detection Filter Problem

A linear time-invariant system with q failure modes can be modeled[6–8] by

$$\dot{x} = Ax + B_u u + \sum_{i=1}^{q} F_i m_i \tag{1a}$$

$$y = Cx + \sum_{i=1}^{q} E_i m_i \tag{1b}$$

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All system variables belong to real vector spaces $x \in \mathcal{X}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$, and $m_i \in \mathcal{M}_i$ with $n = \dim \mathcal{X}$, $p = \dim \mathcal{U}$, $m = \dim \mathcal{Y}$, and $q_i = \dim \mathcal{M}_i$. The input u and the output y are known. The failure modes m_i are vectors that are unknown and arbitrary functions of time and are zero when there is no failure. The failure signatures $F_i : \mathcal{M}_i \mapsto \mathcal{F}_i \subseteq \mathcal{X}$ are maps that are known, fixed, and unique. A failure mode m_i models the time-varying amplitude of a failure, whereas a failure signature F_i models the directional characteristics of a failure. Assume the F_i are monic so that $m_i \neq 0$ implies $F_i m_i \neq 0$. Actuator and plant faults are modeled with F_i as the appropriate direction from A or B_u. For example, a stuck actuator is modeled with F_i as the column of A associated with the actuator dynamics and $m_i(t) = -u_i(t) + u_{ic}$ where u_{ic} is a constant.

Sensor faults are most naturally described as an additive term in the measurement equation (1b), that is, with $F_i = 0$ and E_i aligned with the faulty sensor. Thus, E_i, a sensor failure signature $E_i : \mathcal{M}_i \mapsto \mathcal{E}_i \subseteq \mathcal{Y}$, is a known, fixed, and monic map. A procedure described in detail in Refs. 6, 7, and 9 and by example in Sec. VII shows that, for the purpose of fault detection filter design, for every E_i there is an equivalent but higher-order F_i. Thus the following model is sufficient for design:

$$\dot{x} = Ax + B_u u + \sum_{i=1}^{q} F_i m_i \tag{2a}$$

$$y = Cx \tag{2b}$$

Note that no additional dynamics are needed. Also note that the models (1) and (2) are equivalent only in the sense that they generate the same detections spaces as defined later. The input/output properties of the two systems are quite different.

Consider a full-order observer of the form

$$\dot{\hat{x}} = (A + LC)\hat{x} + B_u u - Ly \tag{3a}$$

$$r = C\hat{x} - y \tag{3b}$$

The state estimation error $e = \hat{x} - x$ dynamics are

$$\dot{e} = (A + LC)e - \sum_{i=1}^{q} F_i m_i \tag{4}$$

If (C, A) is observable and L is chosen so that $A + LC$ is stable, then in steady state and in the absence of disturbances and modeling errors, the residual r is nonzero only if a failure mode m_i is nonzero and is almost always nonzero whenever m_i is nonzero. It follows that any stable observer can detect the occurrence of a fault. Simply monitor the residual, and when it is nonzero, a fault has occurred. A more difficult task is to determine which fault has occurred, and that is what a detection filter is designed to do.

A detection filter is an observer with the property that when $m_i(t) \neq 0$, the error $e(t)$ remains in a (C, A)-invariant subspace $\mathcal{W}_i$ that contains the reachable subspace of $(A + LC, F_i)$. Thus, the residual remains in the output subspace $C\mathcal{W}_i$. Furthermore, the output subspaces $C\mathcal{W}_1, \ldots, C\mathcal{W}_q$ are independent so that

$$r \in \sum_{i=1}^{q} C\mathcal{W}_i$$

has a unique representation $r = z_1 + \cdots + z_q$ with $z_i \in C\mathcal{W}_i$. The fault is identified by projecting r onto each of the output subspaces $C\mathcal{W}_i$. The following statement of the detection filter problem is essentially the same as that found in Refs. 6 and 7 but is stated in the geometric language of Ref. 8.

Definition 2.1 (Detection Filter Problem). Given the system (2), the detection filter problem is to find a set of subspaces $\mathcal{W}_i \subseteq \mathcal{X}$, $i = 1, \ldots, q$, such that the following conditions are met:

Subspace invariance:

$$(A + LC)\mathcal{W}_i \subseteq \mathcal{W}_i$$

Fault inclusion:

$$\mathcal{F}_i \subseteq \mathcal{W}_i$$

Output separability:

$$C\mathcal{W}_i \cap \left(\sum_{j \neq i} C\mathcal{W}_j \right) = 0$$

for some map $L : \mathcal{Y} \mapsto \mathcal{X}$. □

When (C, A) is observable, Refs. 6 and 8 show that output separability implies $\mathcal{W}_1, \ldots, \mathcal{W}_q$ are independent.

To ensure stability, the invariant subspaces $\mathcal{W}_i$ are usually chosen as a set of mutually detectable, minimal unobservability subspaces or detection spaces.[7] An unobservability subspace (UOS) is a (C, A)-invariant subspace, $(A + LC)\mathcal{T} \subseteq \mathcal{T}$ for some L, with the property that, assuming (C, A) is observable, the spectrum of $(A + LC)$ may be placed arbitrarily. The detection space $\mathcal{T}^*$ is usually found as a minimal UOS because there is no known parameterization of all UOS and algorithms exist to compute the minimal UOS.[6,8] For example, Ref. 6 computes the detection space as the null space of a certain observability matrix. In Ref. 8 the detection space is found using transmission zero vectors. Finally, a mutually detectable set of detection spaces $\mathcal{T}_1^*, \ldots, \mathcal{T}_q^*$ is one that satisfies Definition 2.1 such that the spectrum of $(A + LC)$ may be placed arbitrarily.

Whereas the UOS are invariant subspaces, disturbances and sensor noise also may be transmitted through the complementary space, that is, an independent state subspace $\mathcal{X}_0 \neq 0$ formed when the sum of the independent detection spaces does not complete the state space, $\mathcal{T}_1^* \oplus \cdots \oplus \mathcal{T}_q^* \oplus \mathcal{X}_0 = \mathcal{X}$.

III. Detection Filter Gain Parameterization

Given a set of subspaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$ that solve the detection filter problem, the next problem is to characterize the set of maps $L : \mathcal{Y} \mapsto \mathcal{X}$ such that $L \in \cap_{i=1}^{q} \underline{L}(\mathcal{W}_i)$ where $\underline{L}(\mathcal{W}_i) = \{L \mid (A + LC)\mathcal{W}_i \subseteq \mathcal{W}_i\}$. A first step is to find a set $\underline{L}(\mathcal{W})$ for any one (C, A)-invariant subspace $\mathcal{W}$. Proposition 3.3 parameterizes $L \in \underline{L}(\mathcal{W})$ in two parameters $\alpha : C\mathcal{W} \mapsto \mathcal{W}$ and $\beta : \mathcal{Y} \mapsto \mathcal{X}$. Then, given a set of (C, A)-invariant subspaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$ that solve the detection filter problem, Proposition 3.4 parameterizes $L \in \cap_{i=1}^{q} \underline{L}(\mathcal{W}_i)$ in $q + 1$ parameters $\alpha_1, \ldots, \alpha_q$ and β. First, a lemma from Ref. 6 Lemma 1, except for the geometric language, is restated to provide a solution to a generalized inverse problem. Lemma 3.2 provides a few well-known properties of projections.

Lemma 3.1. Let $B : \mathcal{U} \mapsto \mathcal{X}$, $C : \mathcal{X} \mapsto \mathcal{Y}$, and $D : \mathcal{U} \mapsto \mathcal{Y}$ where B is monic. Then a general solution of $CB = D$ for C is given by

$$C = D\tilde{P}_B + K(I - P_B) \tag{5}$$

where $P_B : \mathcal{X} \mapsto \mathcal{X}$ is any projection such that $\operatorname{Im} P_B = \operatorname{Im} B$, $\tilde{P}_B : \mathcal{X} \mapsto \mathcal{U}$ is the natural projection where $B\tilde{P}_B = P_B$ and $K : \mathcal{X} \mapsto \mathcal{Y}$ is arbitrary.

Lemma 3.2. Let $C : \mathcal{X} \mapsto \mathcal{Y}$ and let $P : \mathcal{X} \mapsto \mathcal{X}$ be any projection. Then $\operatorname{Ker} P \subseteq \operatorname{Ker} C$ if and only if $C = CP$. Now let $\operatorname{Ker} P = \operatorname{Ker} C$ and let V decompose P as $VV^T = P$ and $V^T V = I$. Then CV is monic with $\operatorname{Im} CV = \operatorname{Im} C$.

An easy way to find a projector P that satisfies Lemma 3.2 is to find the singular value decomposition of C. For $C = U\Sigma V^T$ where Σ is a diagonal matrix of nonzero singular values, the V of the lemma are the right singular vectors of C. Thus $P = VV^T$ and $CV = U\Sigma V^T V = U\Sigma$ is monic with $\operatorname{Im} C = \operatorname{Im} U\Sigma$.

Proposition 3.3. Let $\mathcal{W} \subset \mathcal{X}$ be a (C, A)-invariant subspace with insertion map $W : \mathcal{W} \mapsto \mathcal{X}$. Let $P : \mathcal{W} \mapsto \mathcal{W}$ be any projection where $\operatorname{Ker} P = \operatorname{Ker} CW$ and let $\hat{F}$ decompose P as $\hat{F}\hat{F}^T = P$ and $\hat{F}^T \hat{F} = I$. Let $H : \mathcal{Y} \mapsto \mathcal{Y}$ be another projection where $\operatorname{Im} H = C\mathcal{W}$ and let $\tilde{H}$ be the associated natural projection that satisfies $CW\hat{F}\tilde{H} = H$ and $\tilde{H}CW\hat{F} = I$. Then $L : \mathcal{Y} \mapsto \mathcal{X}$ satisfies $(A + LC)W = WA_W$ for some $A_W : \mathcal{W} \mapsto \mathcal{W}$ if and only if

$$L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H) \tag{6}$$

for some $\alpha : C\mathcal{W} \mapsto \mathcal{W}$ and $\beta : \mathcal{Y} \mapsto \mathcal{X}$.

Proof. See Appendix A. □

The remark following Lemma 3.2 shows that $\hat{F}$ is the set of right singular vectors of CW.

Proposition 3.4. Let $\mathcal{W}_1, \ldots, \mathcal{W}_q \subset \mathcal{X}$ be a set of (C, A)-invariant subspaces that solve the detection filter problem and let the $W_i : \mathcal{W}_i \mapsto \mathcal{X}$ be the insertion maps. Let P_i, $\hat{F}_i$, H_i, and $\tilde{H}_i$ associated with $\mathcal{W}_i$ be as in Proposition 3.3 but partially specify the kernal of H_i and $\tilde{H}_i$ as

$$\sum_{j \neq i} C\mathcal{W}_j \subseteq \operatorname{Ker} H_i = \operatorname{Ker} \tilde{H}_i$$

Also, define the projection

$$H_0 = \left(I - \sum_{i=1}^{q} H_i \right)$$

and the associated natural projection $\tilde{H}_0$. Finally, define a set of maps

$$\underline{L}(\mathcal{W}_i) = \{ L : \mathcal{Y} \mapsto \mathcal{X} \mid (A + LC)\mathcal{W}_i \subseteq \mathcal{W}_i \}$$

Then $L \in \cap_{i=1}^{q} \underline{L}(\mathcal{W}_i)$ if and only if

$$L = \sum_{i=1}^{q} (-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\tilde{H}_0 \tag{7}$$

for some $\alpha_0 : \operatorname{Im} H_0 \mapsto \mathcal{X}$ and $\alpha_i : C\mathcal{W}_i \mapsto \mathcal{W}_i$ where $i = 1, \ldots, q$.

Proof. See Appendix B. □

IV. Disturbance Robust Detection Filter Problem

Section III showed that a detection filter gain associated with a set of detection filter solution spaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$ is easy to find but generally is not unique. In this section, the $\mathcal{W}_1, \ldots, \mathcal{W}_q$ are found as for the deterministic case, but the nonuniqueness of the detection filter gain is treated as a degree of freedom in the detection filter design. This leads to the definition of a noise robust detection filter problem where the objective is to find a detection filter gain that minimizes or bounds a norm of the transfer matrix from the disturbance to the residual.

The linear time-invariant system of Eq. (2) with q failure modes is extended to include disturbances as

$$\dot{x} = Ax + B\omega + B_u u + \sum_{i=1}^{q} F_i m_i \tag{8a}$$

$$y = Cx + D\omega \tag{8b}$$

The input ω includes dynamic disturbances and sensor noise and is square integrable over $[0, \infty)$.

The error dynamics and residual of a full-order filter have the same form as the observer (3) and (4)

$$\dot{e} = (A + LC)e - (B + LD)\omega - \sum_{i=1}^{q} F_i m_i \tag{9a}$$

$$r = C\hat{x} - y = Ce - D\omega \tag{9b}$$

Because only forcing terms differentiate the residual process of the observer (3) and (4) from (9), the detection filter structure does not change with the introduction of disturbances and sensor noise. However, with the residual driven by an unknown signal, a nonzero residual does not necessarily mean a fault has occurred.

An objective of a detection filter design in the presence of disturbances is to reduce the component of the residual due to the disturbance without at the same time degrading the component of the residual due to the fault. This suggests as a cost function a ratio of transfer matrix norms.[1,2] The transfer matrix from the disturbance to the residual is in the numerator, and the transfer matrix from the fault to the residual is in the denominator. Unfortunately, this formulation requires some assumption about the functional form of the fault because a transfer matrix norm does not convey much information about the size of a transfer matrix output when nothing can be said about the input. Because it is a standard and reasonable assumption that process and sensor noise is white or nearly so, only the transfer matrix from the disturbance to the detection filter residual is retained in the definition of a noise robust detection filter problem.

Before continuing, it is necessary to carefully define what is meant by the component of the residual due to the fault. Define z_i as a projection of the observer residual (9) onto the output subspace $C\mathcal{W}_i$. Let $H_i : \mathcal{Y} \mapsto \mathcal{Y}$ be any projection onto $C\mathcal{W}_i$ and along the $C\mathcal{W}_{j \neq i}$ so that

$$C\mathcal{W}_i = \operatorname{Im} H_i, \qquad \sum_{j \neq i} C\mathcal{W}_j \subseteq \operatorname{Ker} H_i$$

Let $\tilde{H}_i$ be the associated natural projection and define $\tilde{z}_i$, a fault residual, as

$$\tilde{z}_i = \tilde{H}_i r \tag{10}$$

Using $\tilde{H}_i$ rather than H_i in Eq. (10) does not change any information given by the fault residual but is convenient later when certain matrix inverses are needed.

Now consider that for a system with q faults as in Eq. (8) there are q transfer matrices from the system disturbance to each of the fault residuals $\tilde{z}_i$, Eq. (10). There are several ways to proceed. One approach is to define a multiobjective problem where a detection filter gain L is found that in some way simultaneously bounds or makes small all of the transfer matrix norms $\|T_{\tilde{z}_i\omega}\|$, for example, a Pareto optimal solution. Another is to abandon the structure of the full-order detection filter for a system of q residual generators.[10] The q reduced-order filter gains are found independently of one another with the penalty that the order of the combined system usually is somewhat larger than the full-order detection filter. The approach taken here is to combine the fault residuals into a single detection filter output as follows.

Define a combined fault residual $z \in (C\mathcal{W}_1 \times \cdots \times C\mathcal{W}_q)$ by forming a map H from the $\tilde{H}_i$ in the expected way:

$$z = Hr, \qquad H^T = \left[\tilde{H}_1^T, \ldots, \tilde{H}_q^T\right] \tag{11}$$

The combined fault residual z provides the same information as the fault residuals, but it combines the $\tilde{z}_1, \ldots, \tilde{z}_q$ so that a single cost function can be defined for the detection filter. A noise robust detection filter problem is to find a set of subspaces $\mathcal{W}_i$ that solve the detection filter problem of Definition 2.1. Then, given the $\mathcal{W}_i$ and the associated filter gain sets

$$\underline{L}(\mathcal{W}_i) = \{ L_i \mid (A + L_i C)\mathcal{W}_i \subseteq \mathcal{W}_i \}$$

find a filter gain $L \in \cap\underline{L}(\mathcal{W}_i)$ that bounds or minimizes some norm $\|T_{z\omega}\|$ where $T_{z\omega}$ is the transfer matrix from the disturbance ω to the combined fault residual z of Eq. (11).

Note that L is found in a two-step process. First, a set of subspaces $\mathcal{W}_i$ is found that satisfies Definition 2.1. Then a map L is found from the set $\cap\underline{L}(\mathcal{W}_i)$. The alternative is to find L from the union of sets $\cap\underline{L}(\mathcal{W}_i)$, where the union is taken over all sets of subspaces $\mathcal{W}_i$ that satisfy Definition 2. Although the latter statement certainly is more general, it is impractical because there is no known parameterization of all (C, A)-invariant subspaces $\mathcal{W}_i$.

V. $\mathcal{H}_\infty$ Bounded Detection filter

The main result of this section is a proposition that provides an $\mathcal{H}_\infty$ norm bounding detection filter gain. Before this result is stated, a more general $\mathcal{H}_\infty$ norm bounding theorem is needed. Consider an observer with error dynamics and output

$$\dot{e} = (A + LC)e + (B + LD)\omega \tag{12a}$$

$$z = C_z e + D_z \omega \tag{12b}$$

The following theorem and corollary provide a filter gain L that stabilizes the filter and bounds the $\mathcal{H}_\infty$ norm of the transfer matrix from ω to z. This standard result is mainly from Lemma 1 of Ref. 11, and so no proof is provided here.

Theorem 5.1. Consider a system G with the form (12) and where $[A - BD^T(DD^T)^{-1}C]$ has no purely imaginary eigenvalues and

$(DD^T)^{-1}$ exists. Suppose there exists a scalar real constant $\gamma > 0$ and a symmetric positive definite real matrix $Y > 0$ that satisfies the following algebraic Riccati equation:

$$0 = (A + LC)Y + Y(A + LC)^T + (B + LD)(B + LD)^T + \gamma^{-2}\big(YC_z^T + BD_z^T\big)\big(YC_z^T + BD_z^T\big)^T \tag{13}$$

Then $(A + LC)$ is stable and $\|G\|_\infty \le [\gamma^2 + \sigma_{\max}^2(D_z)]^{1/2}$ where $\sigma_{\max}(D_z)$ is the largest singular value of D_z.

When the terms of Eq. (13) are manipulated to isolate L, a corollary that provides an L that stabilizes G and bounds $\|G\|_\infty$ follows immediately.

Corollary 5.2. Suppose a symmetric positive definite real matrix $Y > 0$ satisfies the following algebraic Riccati equation:

$$\begin{aligned}0 &= \big[A - BD^T(DD^T)^{-1}C + \gamma^{-2}BD_z^TC_z\big]Y \\ &\quad + Y\big[A - BD^T(DD^T)^{-1}C + \gamma^{-2}BD_z^TC_z\big]^T \\ &\quad + B\big[I - D^T(DD^T)^{-1}D + \gamma^{-2}D_z^TD_z\big]B^T \\ &\quad - Y\big[C^T(DD^T)^{-1}C - \gamma^{-2}C_z^TC_z\big]Y\end{aligned} \tag{14}$$

Then for

$$L = -(YC^T + BD^T)(DD^T)^{-1} \tag{15}$$

$(A+LC)$ is stable and $\|G\|_\infty \le [\gamma^2+\sigma_{\max}^2(D_z)]^{1/2}$ where $\sigma_{\max}(D_z)$ is the largest singular value of D_z.

Standard results strengthen Corollary 5.2 by replacing Eq. (14) with conditions on an associated Hamiltonian matrix and adding a system detectability requirement.[12] That is not done here because in the next proposition the Riccati equation (14) is modified to provide a detection filter gain and has no associated Hamiltonian matrix.

In the detection filter problem, L is constrained to generate a set of q invariant subspaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$. There is no reason to expect that L, at the same time, should satisfy Eq. (15). In the next proposition, Eq. (15) is modified so that L satisfies both constraints. When the modified relation is substituted for L in Eq. (13) and L is eliminated, the result is an algebraic Riccati equation with an extra term. The modified Riccati equation has no associated Hamiltonian, and conditions for the uniqueness or even the existence of a solution are unknown. However, Ref. 13 reports success in finding iterative numerical solutions to a similar relation arising from a decentralized control problem. An example in Sec. VII illustrates the application of a numerical integration approach.

Before stating the main proposition, it is convenient to rearrange the detection filter error dynamics by combining the error dynamics (9) with the detection filter gain (7). Then the problem of choosing the parameters α_0 and $\alpha_1, \ldots, \alpha_q$ has the same form as the problem of choosing a set of $q + 1$ constant feedback gains for the system

$$\dot{e} = \hat{A}e - \hat{B}\omega - \sum_{i=1}^{q} F_i m_i + W_1u_1 + \cdots + W_qu_q + u_0 \tag{16a}$$

$$y_1 = \tilde{H}_1Ce - \tilde{H}_1D\omega, \qquad u_1 = \alpha_1 y_1 \tag{16b}$$

$$\vdots \tag{16c}$$

$$y_q = \tilde{H}_qCe - \tilde{H}_qD\omega, \qquad u_q = \alpha_q y_q \tag{16d}$$

$$y_0 = \tilde{H}_0Ce - \tilde{H}_0D\omega, \qquad u_0 = \alpha_0 y_0 \tag{16e}$$

where

$$\hat{A} = A + \hat{L}C \tag{16f}$$

$$\hat{B} = B + \hat{L}D \tag{16g}$$

$$\hat{L} = -\sum_{i=1}^{q} AW_i\hat{F}_i\tilde{H}_i \tag{16h}$$

Proposition 5.3. Consider the system G with output given by Eq. (11):

$$G = \left[\begin{array}{c|cccccc} \hat{A} & -\hat{B} & W_1 & \cdots & W_q & I \\ \hline HC & -HD & 0 & \cdots & 0 & 0 \\ \tilde{H}_1C & -\tilde{H}_1D & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{H}_qC & -\tilde{H}_qD & 0 & \cdots & 0 & 0 \\ \tilde{H}_0C & -\tilde{H}_0D & 0 & \cdots & 0 & 0 \end{array}\right]$$

Define

$$C_2 = \begin{bmatrix} \tilde{H}_1C \\ \vdots \\ \tilde{H}_qC \\ \tilde{H}_0C \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} \tilde{H}_1D \\ \vdots \\ \tilde{H}_qD \\ \tilde{H}_0D \end{bmatrix}, \qquad V = D_{21}D_{21}^T$$

and the partitioning matrices $\Pi_1, \ldots, \Pi_q$ and Π_0

$$\Pi_1 = \begin{bmatrix} I \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \cdots \quad \Pi_q = \begin{bmatrix} 0 \\ \vdots \\ I \\ 0 \end{bmatrix} \qquad \Pi_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

such that

$$\Pi_i^T[C_2, D_{21}] = [\tilde{H}_iC, \tilde{H}_iD], \qquad \Pi_0^T[C_2, D_{21}] = [\tilde{H}_0C, \tilde{H}_0D]$$

Now define a set of projections $P_{W_1}, \ldots, P_{W_q}$ where $\operatorname{Im} P_{W_i} = \operatorname{Im} W_i$ and define a set of associated natural projections $\tilde{P}_{W_i}$ that satisfy $W_i\tilde{P}_{W_i} = P_{W_i}$. Assume $(\hat{A} - \hat{B}D_{21}^TV^{-1}C_2)$ has no eigenvalues on the imaginary axis. Let $\gamma > 0$ be a constant real scalar and suppose there exists $Y > 0$ such that

$$\begin{aligned}0 &= \big[\hat{A} - \hat{B}D_{21}^TV^{-1}C_2 + \gamma^{-2}\hat{B}D^TH^TC_2\big]Y \\ &\quad + Y\big[\hat{A} - \hat{B}D_{21}^TV^{-1}C_2 + \gamma^{-2}\hat{B}D^TH^TC_2\big]^T \\ &\quad + \hat{B}\big[I - D_{21}^TV^{-1}D_{21} + \gamma^{-2}D^TH^THD\big]\hat{B}^T \\ &\quad - Y\big[C_2^TV^{-1}C_2 - \gamma^{-2}C^TH^THC\big]Y \\ &\quad + \left[\sum_{i=1}^{q}(I - P_{W_i})\big(YC_2^T + \hat{B}D_{21}^T\big)V^{-1}\Pi_i\tilde{H}_iD\right] \\ &\quad \times \left[\sum_{i=1}^{q}(I - P_{W_i})\big(YC_2^T + \hat{B}D_{21}^T\big)V^{-1}\Pi_i\tilde{H}_iD\right]^T\end{aligned} \tag{17}$$

Then

$$\begin{aligned}\alpha_1 &= -\tilde{P}_{W_1}\big(YC_2^T + \hat{B}D_{21}^T\big)V^{-1}\Pi_1 \\ &\vdots \\ \alpha_q &= -\tilde{P}_{W_q}\big(YC_2^T + \hat{B}D_{21}^T\big)V^{-1}\Pi_q \\ \alpha_0 &= -\big(YC_2^T + \hat{B}D_{21}^T\big)V^{-1}\Pi_0\end{aligned}$$

stabilizes G and bounds the transfer matrix $T_{z\omega}$ as $\|T_{z\omega}\|_\infty \le [\gamma^2 + \sigma_{\max}^2(HD)]^{1/2}$ where $\sigma_{\max}(HD)$ is the largest singular value of HD.

Proof. The transfer matrix $T_{z\omega}$ is

$$T_{z\omega} = \left[\begin{array}{c|c} A_T & -B_T \\ \hline HC & -HD \end{array}\right]$$

where

$$A_T = \hat{A} + \sum_{i=1}^{q} W_i \alpha_i \tilde{H}_i C + \alpha_0 \tilde{H}_0 C$$

$$B_T = \hat{B} + \sum_{i=1}^{q} W_i \alpha_i \tilde{H}_i D + \alpha_0 \tilde{H}_0 D$$

By Theorem 5.1 and because $(\hat{A} - \hat{B} D_{21}^T V^{-1} C_2)$ has no eigenvalues on the imaginary axis, it is sufficient to show that $S = 0$ for some $Y > 0$ where

$$S = A_T Y + Y A_T^T + B_T B_T^T + \gamma^{-2}(YC^T + \hat{B}D^T) H^T H (YC^T + \hat{B}D^T)^T$$

The rest of the proof involves algebraic manipulations that put S in the form of the modified algebraic Riccati equation (17). □

VI. Fault Enhancement

As discussed in the Introduction, it is not enough to bound the residual component due to the process disturbances and sensor noise because this might, at the same time, make the fault residual component small. The approach taken here is to enhance each fault residual component while maintaining the disturbance and sensor noise bound.

One approach is to consider a cost function given as the fault signal to noise ratio

$$J_i = \frac{\|T_{z_i m_i}\|_\infty}{\|T_{z_i \omega}\|_\infty} \tag{18}$$

This is the cost function given in Ref. 1 for a set of parity equations. Results from Ref. 14 may be applied to maximize Eq. (18) with respect to a stable postfilter Q, that is,

$$J_i = \frac{\|QT_{z_i m_i}\|_\infty}{\|QT_{z_i \omega}\|_\infty} \tag{19}$$

As suggested in Ref. 14, the cost (19) is simple in that it uses only a single pair of frequencies. Furthermore, if the frequency content of a fault happens to be the same as for the disturbance, there can be no fault enhancement even if the ratio (19) is large. This does not mean that the cost is inappropriate, rather that a robust design requires careful attention to an even distribution of the singular values of $QT_{z_i m_i}$ and that they be greater than the maximum singular value of $QT_{z_i \omega}$.

Although an $\mathcal{H}_\infty$ norm might be chosen for its tractability and immediate application to robustness, it is suggested in Ref. 5 that an $\mathcal{L}_1$ norm is an appropriate measure for the fault transmission signal. The choice of the signal or transfer matrix induced norm is determined, ideally, by a description of the physical system and its operating environment and, pragmatically, by the available synthesis tools and whether the resulting filter design is feasible to implement. The approach developed here is based on the signal $\mathcal{L}_2$ norm and provides a simple constant gain mapping found by solving a set of eigenvalue problems.

First, consider the fault detection filter transfer matrix for the fault isolation residual z_i. By the filter unobservability subspace structure, only the fault m_i influences residual z_i, and so a reduced-order realization is written. The subscript i is dropped for notational convenience:

$$\dot{\bar{e}}(t) = \bar{A}\bar{e}(t) + \bar{F}m(t) + \bar{B}\omega(t)$$

$$z(t) = \bar{C}\bar{e}(t) + D_m m(t) + D_\omega \omega(t)$$

The error $\bar{e}$ lies in the factor space

$$\bar{e} \in \mathcal{X} \Big/ \sum_{j \neq i} \mathcal{T}_j$$

the observable factor space with respect to z. All maps are taken as induced on this factor space. Now consider signals $m(t)$ and $\omega(t)$ as elements of $\mathcal{L}_2(-\infty, 0]$ spaces of appropriate dimensions and define the controllability operators

$$\psi_m : \mathcal{L}_2(-\infty, 0] \mapsto \mathcal{R}^n \triangleq \int_{-\infty}^{0} e^{-\bar{A}\tau} \bar{F} m(\tau)\, d\tau$$

$$\psi_\omega : \mathcal{L}_2(-\infty, 0] \mapsto \mathcal{R}^n \triangleq \int_{-\infty}^{0} e^{-\bar{A}\tau} \bar{B} \omega(\tau)\, d\tau$$

Then $z = z_m + z_\omega$ where z_m and z_ω are residual components due to $m(t)$ and $\omega(t)$ given by

$$z_m(t) = \bar{C} e^{\bar{A}t} \bar{e}_{0_m} = \bar{C} e^{\bar{A}t} \psi_m m$$

$$z_\omega(t) = \bar{C} e^{\bar{A}t} \bar{e}_{0_\omega} = \bar{C} e^{\bar{A}t} \psi_\omega \omega$$

A detection filter fault enhancement problem may be stated as follows. Consider the residual components z_m and z_ω as elements of $\mathcal{L}_2[0, T]$ spaces where T is an observation window. Find a constant mapping q that maximizes a cost defined with respect to the worst case fault isolation and disturbance transmission signals

$$J = \frac{\|q^T z_m^*\|^2_{\mathcal{L}_2[0,T]}}{\|q^T z_\omega^*\|^2_{\mathcal{L}_2[0,T]}} \tag{20}$$

The worst case signals z_ω^* and z_m^* are given by optimization problems to be defined shortly. Note that $q^T z_m^*$ and $q^T z_\omega^*$ are scalar functions of time. When maximized with respect to q^T, the cost J penalizes large residual components due to a disturbance ω and small residual components due to a fault m.

The choice of the observation window T and the fault detection threshold is a design decision based on the functional form of the expected faults and disturbances. A detailed discussion is found in Ref. 15. However, it is worthwhile to point out that a window of zero length, $T = 0$, is not practical. First, because faults and disturbances enter the residual directly through D_m and D_ω, it is not possible to distinguish a fault from a disturbance at any one point in time. Second, the operators that map signals $m(t)$ and $\omega(t) \in \mathcal{L}_2(-\infty, 0]$ to the respective residual components at time $t = 0$ are given by

$$\bar{\psi}_m : \mathcal{L}_2(-\infty, 0] \mapsto \mathcal{R}^m \triangleq \bar{C}\psi_m m(t) + D_m m(0)$$

$$\bar{\psi}_\omega : \mathcal{L}_2(-\infty, 0] \mapsto \mathcal{R}^m \triangleq \bar{C}\psi_\omega \omega(t) + D_\omega \omega(0)$$

These operators are not bounded. For example, let

$$m_h(t) = \begin{cases} 1/\sqrt{h} & -h \le t \le 0 \\ 0 & t < -h \end{cases} \tag{21}$$

Then $m_h(t) \in \mathcal{L}_2(-\infty, 0]$ and $\|m_h\| = 1$ for all h but $\bar{\psi}_m m_h \to \infty$ as $h \to 0$. Hence, further restrictions on m and ω need to be made before a cost function such as the following could be used:

$$\frac{\|\bar{C}\psi_m m + D_m m(0)\|_{\mathcal{R}^m}}{\|\bar{C}\psi_\omega \omega + D_\omega \omega(0)\|_{\mathcal{R}^m}}$$

A well-known result[16] is that for a given initial state $\bar{e}_{0_\omega}$ the smallest signal $\omega \in \mathcal{L}_2(-\infty, 0]$ that produces $\bar{e}_{0_\omega}$ has a norm given by

$$\inf_{\omega \in \mathcal{L}_2(-\infty,0]} \left\{ \|\omega\|^2 \mid \bar{e}(0) = \bar{e}_{0_\omega} \right\} = \bar{e}_{0_\omega}^T X_\omega^{-1} \bar{e}_{0_\omega} \tag{22}$$

where X_ω is the controllability gramian given as the solution to the steady-state Lyapunov equation

$$0 = \bar{A}X_\omega + X_\omega \bar{A}^T + \bar{B}\bar{B}^T$$

A worst case initial state $\bar{e}_{0_\omega}$ is found by maximizing the ratio

$$J_\omega = \sup_{\omega \in \mathcal{L}_2(-\infty,0]} \frac{\|z_\omega\|^2_{\mathcal{L}_2[0,T]}}{\|\omega\|^2_{\mathcal{L}_2(-\infty,0]}} = \max_{\bar{e}_{0_\omega} \neq 0} \frac{\bar{e}_{0_\omega}^T \left(\int_0^T e^{\bar{A}^T \tau} \bar{C}^T \bar{C} e^{\bar{A}\tau}\, d\tau \right) \bar{e}_{0_\omega}}{\bar{e}_{0_\omega}^T X_\omega^{-1} \bar{e}_{0_\omega}}$$

This is solved as an eigenvalue problem

$$\left(X_\omega \int_0^T e^{\bar{A}^T \tau} \bar{C}^T \bar{C} e^{\bar{A}\tau}\, d\tau \right) \bar{e}_{0_\omega} = \lambda_{\omega\max} \bar{e}_{0_\omega} \tag{23}$$

with $J_\omega = \lambda_{\omega\max}$. The worst-case signal is given by $z_\omega^*(t) = \bar{C} e^{\bar{A}t} \bar{e}_{0_\omega}$. Note that in the case where $T = \infty$, J_ω is the Hankel norm of the transfer matrix $T_{z_i\omega}$.

Similarly, find a worst-case fault by maximizing the inverse ratio

$$J_m = \max_{\bar{e}_{0m} \neq 0} \frac{\bar{e}_{0_m}^T X_m^{-1} \bar{e}_{0_m}}{\bar{e}_{0_m}^T \left(\int_0^T e^{\bar{A}^T \tau} \bar{C}^T \bar{C} e^{\bar{A}\tau}\, d\tau \right) \bar{e}_{0_m}}$$

so that $\bar{e}_{0_m}$ is the eigenvector associated with

$$\left(X_m \int_0^T e^{\bar{A}^T \tau} \bar{C}^T \bar{C} e^{\bar{A}\tau}\, d\tau \right)^{-1} \bar{e}_{0_m} = \lambda_{m\max} \bar{e}_{0_m} \tag{24}$$

Now maximize Eq. (20) with respect to q using $\bar{e}_{0_\omega}$ and $\bar{e}_{0_m}$ from Eqs. (23) and (24). This is solved as another eigenvalue problem

$$J = \max_{q \neq 0} \frac{\left\| q^T z_m \right\|^2_{\mathcal{L}_2[0,T]}}{\left\| q^T z_\omega \right\|^2_{\mathcal{L}_2[0,T]}} = \lambda_{\max} \tag{25}$$

where

$$\left(\bar{C} \int_0^T e^{\bar{A}\tau} \bar{e}_{0_m} \bar{e}_{0_m}^T e^{\bar{A}^T \tau}\, d\tau \bar{C}^T \right)^T q = \lambda_{\max} \left(\bar{C} \int_0^T e^{\bar{A}\tau} \bar{e}_{0_\omega} \bar{e}_{0_\omega}^T e^{\bar{A}^T \tau}\, d\tau \bar{C}^T \right)^T q \tag{26}$$

Finally, the controllability gramians in Eq. (26) for the case $T = \infty$ may be found as solutions to a pair of steady-state Lyapunov equations. Let

$$X_{0_m} = \int_0^T e^{\bar{A}\tau} \bar{e}_{0_m} \bar{e}_{0_m}^T e^{\bar{A}^T \tau}\, d\tau, \qquad X_{0_\omega} = \int_0^T \bar{e}^{\bar{A}\tau} e_{0_\omega} \bar{e}_{0_\omega}^T e^{\bar{A}^T \tau}\, d\tau$$

Then

$$0 = \bar{A} X_{0_m} + X_{0_m} \bar{A}^T + \bar{e}_{0_m} \bar{e}_{0_m}^T, \qquad 0 = \bar{A} X_{0_\omega} + X_{0_\omega} \bar{A}^T + \bar{e}_{0_\omega} \bar{e}_{0_\omega}^T$$

VII. Application to an Aircraft Fault Detection System

This example considers a simplified aircraft fault detection filter that monitors an elevon actuator and normal accelerometer in the presence of wind and sensor noise. The design approach includes bounding the $\mathcal{H}_\infty$ norm of the transfer matrix from the wind and sensor noise to the filter residual. In this example, other approaches such as the unknown input observer[1] are difficult to apply. An unobservability subspace formed with respect to the wind, thereby decoupling the wind from the fault isolation residuals, happens to be nonmutually detectable with respect to the faults. The faults and the wind direction combine to place a system transmission zero at 0.002, forcing any fault detection filter design with these fault directions to have an unstable closed-loop pole.

The dynamics of an F16XL are linearized about a trimmed level flight condition at 10,000 ft altitude and Mach 0.9. The five-state model includes longitudinal dynamics only, no lateral dynamics and no actuator dynamics. A first-order Dryden wind gust model is included:

$$\dot{x} = Ax + B_\omega \omega + B_\delta \delta, \qquad y = Cx + D\nu$$

The states are

u = longitudinal body axis velocity, ft/s
w = normal body axis velocity, ft/s
q = pitch rate, deg/s
θ = pitch angle, deg
w_g = wind gust, ft/s

The measurements are

q = pitch rate, deg/s
α = angle of attack, deg
A_z = normal acceleration, ft/s^2
A_x = longitudinal acceleration, ft/s^2

The disturbances are

ω = wind gust, ft/s
ν_q = pitch rate sensor noise
ν_α = angle-of-attack sensor noise
ν_{Az} = normal accelerometer sensor noise
ν_{Ax} = longitudinal accelerometer sensor noise

The input is

δ = elevon deflection angle, deg

All disturbances are zero-mean uncorrelated white noise processes with unit spectral density. The port and starboard elevons are modeled as a slaved system because only longitudinal dynamics are considered for this simple example. The elevon actuator dynamics are not included. The system matrices are

$$A = \begin{bmatrix} -0.0674 & 0.0430 & -0.8886 & -0.5587 & 0.0430 \\ 0.0205 & -1.4666 & 16.5800 & -0.0299 & -1.4666 \\ 0.1377 & -1.6788 & -0.6819 & 0 & -1.6788 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.1948 \end{bmatrix}$$

$$B_\omega = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.57 \end{bmatrix}, \qquad B_\delta = \begin{bmatrix} -0.1672 \\ -1.5179 \\ -9.7842 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0.0591 & 0 & 0 & 0.0591 \\ 0.0139 & 1.0517 & 0.1485 & -0.0299 & 0 \\ -0.0677 & 0.0431 & 0.0171 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.143 & 0 & 0 \\ 0 & 0 & 0.245 & 0 \\ 0 & 0 & 0 & 0.245 \end{bmatrix}$$

Now consider a fault detection system with two faults: a normal accelerometer sensor fault and an elevon fault. The normal accelerometer fault can be modeled as an additive term in the measurement equation

$$y = Cx + E_{Az}\mu_{Az} \qquad \text{where} \qquad E_{Az} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{27}$$

and where μ_{Az} is an arbitrary time-varying real scalar. For the purpose of determining an associated detection space, the fault E_{Az} in Eq. (27) is equivalent to a two-dimensional fault F_{Az} (Refs. 6, 7, and 9)

$$\dot{x} = Ax + F_{Az} m_{Az} \qquad \text{with} \qquad F_{Az} = \left[F_{Az}^1, F_{Az}^2\right]$$

where the directions F_{Az}^1 and F_{Az}^2 are given by

$$E_{Az} = CF_{Az}^1, \qquad F_{Az}^2 = AF_{Az}^1$$

so that

$$F_{Az} = \begin{bmatrix} 0 & 0.9986 \\ 0 & 0.0534 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The elevon fault is given simply as $F_\delta = B_\delta$. Because CF_{Az}^1, CF_{Az}^2, and CF_δ are all nonzero and because none of the triples (C, A, F_{Az}^1), (C, A, F_{Az}^2), and (C, A, F_δ) have invariant zeros, the minimal unobservability subspaces for the faults are given by the fault directions themselves, that is, $\mathcal{T}_{Az}^{1*} = \operatorname{Span} F_{Az}^1$, $\mathcal{T}_{Az}^{2*} = \operatorname{Span} F_{Az}^2$, and $\mathcal{T}_\delta^* = \operatorname{Span} F_\delta$. The faults are mutually detectable, and so there are no constraints on the spectrum of the detection filter.

The first step toward finding a fault detection filter gain is to find $\hat{L}$ as in Eq. (16h). This gain forms an observer with the correct detection space structure but without regard to stability or any performance considerations:

$$\hat{L} = -\sum_{i=1}^{q} AW_i \hat{F}_i \tilde{H}_i$$

Considering the two-dimensional normal accelerometer sensor fault as a pair of output separable faults, the $\hat{F}_i$ are identity matrices and the W_i are just the fault directions themselves. To find the $\tilde{H}_i$, let $F = [F_{Az}, F_\delta]$ and form the left inverse of CF as $(CF)^{-\ell} = (F^T C^T CF)^{-1} F^T C^T$. Now take $\tilde{H}_{Az}$ as the first two rows of $(CF)^{-\ell}$ and $\tilde{H}_\delta$ as the third row. Finally $\hat{L} = -AF_{Az}\tilde{H}_{Az} - AF_\delta \tilde{H}_\delta$ and all components needed to apply Proposition 5.3 are now given.

Application of Proposition 5.3 involves solving a modified algebraic Riccati equation. One approach that has achieved practical success is to form a modified differential Riccati equation and to numerically integrate until a steady state is reached. An initial condition for the integration is chosen by solving the algebraic Riccati equation found by truncating the modifying quadratic term. Choosing an $\mathcal{H}_\infty$ bounding parameter $\gamma = 1.35$ results in a filter with eigenvalues -27.1035, -2.0181, -0.3846, -0.0041, and -0.8413.

Figure 1 shows the maximum singular values in decibels of two fault detection filter transfer matrices. One is from the wind disturbance and sensor noise to the residual that isolates a normal accelerometer fault. The other is from the normal accelerometer sensor to the same residual. A third transfer matrix, one from the elevon deflection, is zero, as it should be, and is not shown. Figure 2 shows the maximum singular values of transfer matrices to the elevon fault residual. Here the transfer matrix from the normal accelerometer sensor is zero and is not shown. In both figures, the residual is scaled so that the dc gain of the disturbance component is 0 db. Both faults have been scaled by 2 to emphasize that fault detection in the presence of disturbances resolves to a threshold selection problem.

Note that in the case of the elevon fault both the residual and the detection space are one dimensional, and so the associated filter eigenvector is fixed. There is no way to increase the residual component due to the fault without at the same time increasing the component due to the noise.

This is not the case for the normal accelerometer residual because it is two dimensional. A fault enhancing residual direction is found from Eq. (26) as $q_i^T = [-0.57, -0.82]$. The singular value frequency responses for the improved residual are also shown in Fig. 1. Disturbance reduction is seen mainly at frequencies above 1 rad/s. A modest increase in the fault signal is seen at all frequencies.

Figures 3 and 4 show residual histories where white noise is applied to the wind gust model and the sensors. Figure 3 shows the normal accelerometer residual history when a 2 ft/s^2 bias is added to the accelerometer signal after 1 s. Figure 4 shows the elevon residual history when a 2-deg bias is added to the elevon deflection after 1 s. Clearly, in both cases, a hard fault is detectable with an appropriate threshold.[15]

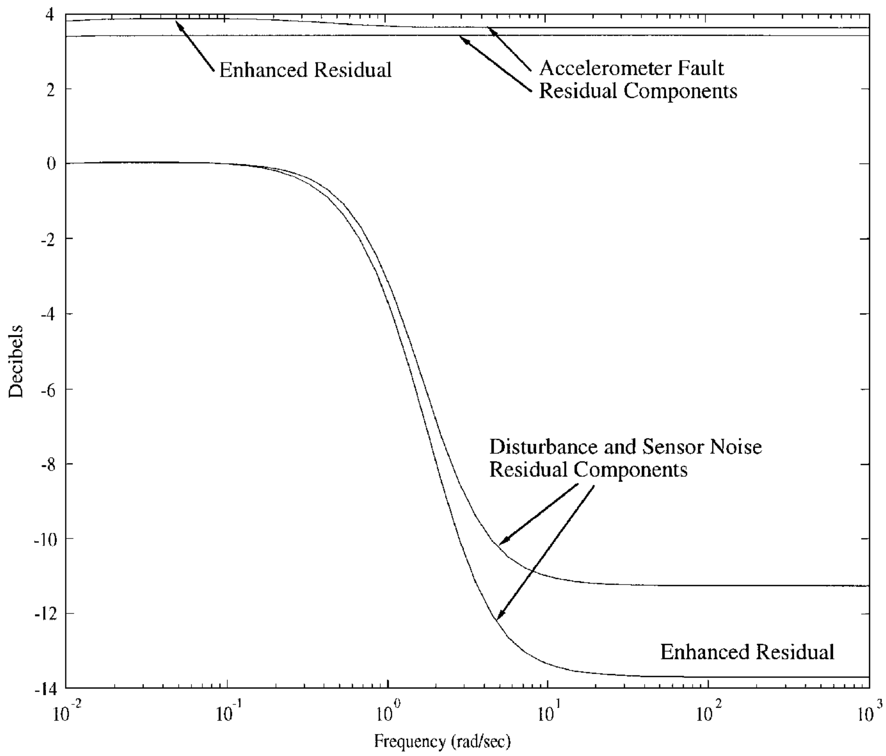

Fig. 1 Magnitude of transfer functions to the normal accelerometer fault isolation residual.

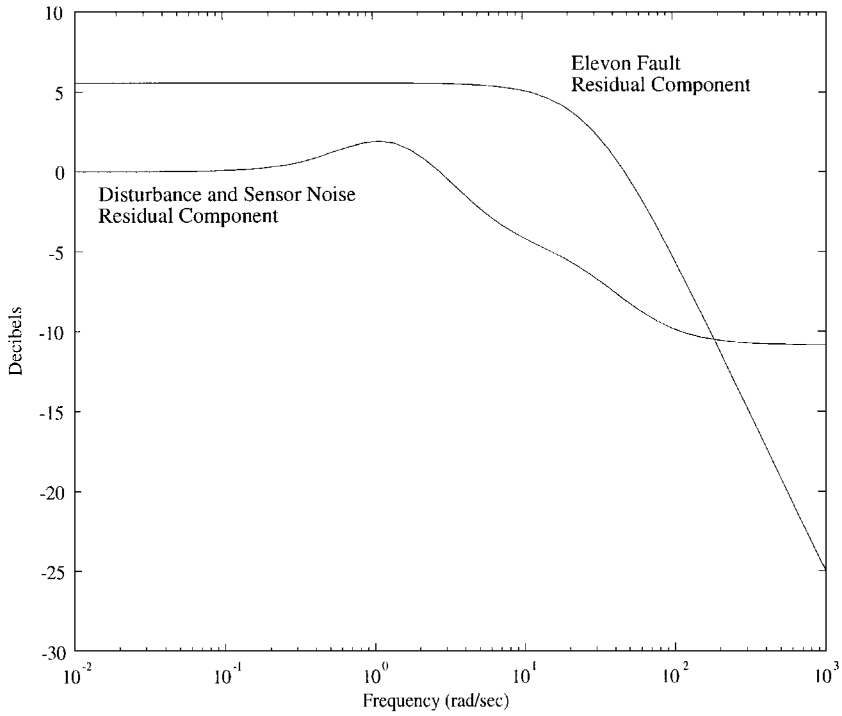

Fig. 2 Magnitude of transfer functions to the elevon fault isolation residual.

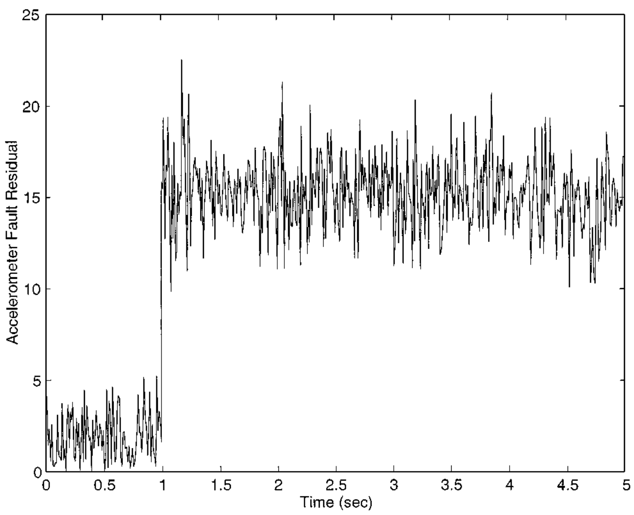

Fig. 3 Normal accelerometer fault isolation residual; 2 ft/s^2 accelerometer fault occurs at t = 1 s.

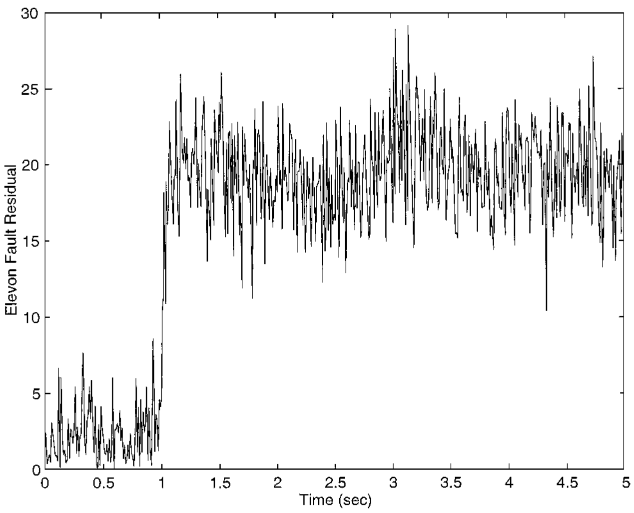

Fig. 4 Elevon fault isolation residual; 2-deg elevon fault occurs at t = 1 s.

VIII. Conclusions

Several approaches to disturbance robust fault detection are available. The noise bounding approach presented here is most applicable when disturbances cannot be decoupled from faults because of low-order filter dynamics or separability concerns or when decoupling the noise imposes an ill-conditioned filter eigenstructure. The geometric structure that allows for fault isolation in an $\mathcal{H}_\infty$ bounded fault detection filter is preserved by modifying an algebraic Riccati equation. A theoretical concern is that this equation has no associated Hamiltonian and no analytically derived solution. However, in practice, convergent numerical schemes have been successfully applied.

Appendix A: Proof of Proposition 3.3

Proof. ($\Rightarrow$) Assume L satisfies $(A + LC)W = WA_W$ for some map A_W. Then $LCW = -AW + WA_W$ and

$$LCW\hat{F} = -AW\hat{F} + WA_W\hat{F} \tag{A1}$$

Now $\hat{F}$ is defined so that $\hat{F}\hat{F}^T$ is a projection with $\operatorname{Ker} CW = \operatorname{Ker} \hat{F}\hat{F}^T$ and $\hat{F}^T\hat{F} = I$. Therefore, by Lemma 3.2, $CW\hat{F}$ is monic and by Eq. (A1) and Lemma 3.1

$$L = (-AW\hat{F} + WA_W\hat{F})\tilde{H} + \beta(I - H)$$

Therefore $(A + LC)W = WA_W$

$$\Rightarrow L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H)$$

where $\alpha = A_W\hat{F}$ and β is anything.

($\Leftarrow$) Suppose $L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H)$. Now $HCW\hat{F} = CW\hat{F}$ and $\tilde{H}CW\hat{F} = I$, and so $LCW\hat{F} = (-AW\hat{F} + W\alpha)$ and

$$(A + LC)W\hat{F} = W\alpha \tag{A2}$$

The term $\hat{F}$ is defined so that $\hat{F}\hat{F}^T$ is a projector with $\operatorname{Ker} CW = \operatorname{Ker} \hat{F}\hat{F}^T$ and $\hat{F}^T\hat{F} = I$. Therefore, by Lemma 3.2, $CW = CW\hat{F}\hat{F}^T$, and it follows that $CW(I - \hat{F}\hat{F}^T) = 0$ and

$$\operatorname{Im}[W(I - \hat{F}\hat{F}^T)] \subseteq \mathcal{W} \cap \operatorname{Ker} C \tag{A3}$$

Because for any (C, A)-invariant subspace $\mathcal{W}$ it is true that $A(\mathcal{W} \cap \operatorname{Ker} C) \subseteq \mathcal{W}$, it follows from Eq. (A3) that for some $\tilde{A}_W$

$$AW(I - \hat{F}\hat{F}^T) = W\tilde{A}_W \tag{A4}$$

and

$$(A + LC)W(I - \hat{F}\hat{F}^T) = W\tilde{A}_W$$

By Eq. (A2), $(A + LC)W\hat{F}\hat{F}^T = W\alpha\hat{F}^T$. Therefore

$$(A + LC)W = W\left(\alpha\hat{F}^T + \tilde{A}_W\right)$$

and $L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H)$

$$\Rightarrow (A + LC)W = WA_W$$

where $A_W = \alpha\hat{F}^T + \tilde{A}_W$ and where $\tilde{A}_W$ satisfies Eq. (A4). Note that $\tilde{A}_W = \tilde{A}_W(I - \hat{F}\hat{F}^T)$, and so

$$A_W = \alpha\hat{F}^T + \tilde{A}_W(I - \hat{F}\hat{F}^T)$$

By Lemma 3.1, A_W is a particular solution to $\alpha = A_W\hat{F}$. □

Appendix B: Proof of Proposition 3.4

Proof. ($\Rightarrow$) Assume $L \in \underline{L}(\mathcal{W}_i)$. Then L satisfies $(A + LC)W_i = W_iA_{W_i}$ for some $A_{W_i} : \mathcal{W} \mapsto \mathcal{W}$ for $i = 1, \ldots, q$. Therefore $LCW_i = -AW_i + W_iA_{W_i}$ and $LCW_i\hat{F}_i = -AW_i\hat{F}_i + W_iA_{W_i}\hat{F}_i$ and

$$\begin{aligned} &L[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q] \\ &\quad = \left[\left(-AW_1\hat{F}_1 + W_1A_{W_1}\hat{F}_1\right), \ldots, (-AW_q\hat{F}_q + W_qA_{W_q}\hat{F}_q)\right] \end{aligned} \tag{B1}$$

The $\hat{F}_i$ are defined so that $\hat{F}_i\hat{F}_i^T$ is a projector with $\operatorname{Ker} CW_i = \operatorname{Ker}\hat{F}_i\hat{F}_i^T$ and $\hat{F}_i^T\hat{F}_i = I$. Therefore, Lemma 3.2 shows that $\operatorname{Im} CW_i = \operatorname{Im} CW_i\hat{F}_i$ and $CW_i\hat{F}_i$ is monic. Because the $\mathcal{W}_1, \ldots, \mathcal{W}_q$ solve the detection filter problem, they are output separable, which means the output subspaces $C\mathcal{W}_1, \ldots, C\mathcal{W}_q$ are independent. Therefore, $[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q]$ is monic.

In Proposition 3.3 $\operatorname{Ker} H$ is not specified and is not important. Here, however, $H_iCW_j = 0$, and so if H is the projection

$$H = \sum_{i=1}^{q} H_i$$

then

$$H[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q] = [CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q]$$

A natural projection $\tilde{H}$ associated with H is

$$\tilde{H} = \begin{bmatrix} \tilde{H}_1 \\ \vdots \\ \tilde{H}_q \end{bmatrix}$$

because

$$[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q]\tilde{H} = \sum_{i=1}^{q} CW_i\hat{F}_i\tilde{H}_i = \sum_{i=1}^{q} H_i = H$$

and

$$\tilde{H}[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q] = \operatorname{diag}(\tilde{H}_iCW_i\hat{F}_i) = I$$

Because $[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q]$ is monic and H and $\tilde{H}$ meet the requirements of Lemma 3.1, the general solution of Eq. (B1) for L is

$$\begin{aligned} L &= \left[\left(-AW_1\hat{F}_1 + W_1A_{W_1}\hat{F}_1\right), \ldots, \right. \\ &\qquad \left.\left(-AW_q\hat{F}_q + W_qA_{W_q}\hat{F}_q\right)\right]\tilde{H} + \hat{\beta}(I - H) \\ &= \sum_{i=1}^{q}\left(-AW_i\hat{F}_i + W_iA_{W_i}\hat{F}_i\right)\tilde{H}_i + \hat{\beta}(I - H) \\ &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \hat{\beta}(I - H) \end{aligned}$$

where $\alpha_i = A_{W_i}\hat{F}_i$ and $\hat{\beta}$ is anything. Finally, it follows directly from the definitions of H and $\tilde{H}_0$ that for any $\hat{\beta}$ there exists β such that $\hat{\beta}(I - H) = \beta\tilde{H}_0$. Therefore,

$$L = \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\tilde{H}_0$$

($\Leftarrow$) Assume

$$\begin{aligned} L &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\tilde{H}_0 \\ &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \hat{\beta}(I - H) \end{aligned}$$

where the equality follows from the definitions of H and $\tilde{H}_0$. Because $H_iH_j = 0$,

$$(I - H) = \left(I - \sum_{i=1}^{q} H_i\right) = \left(I - \sum_{j \neq i} H_j\right)(I - H_i)$$

Then

$$\begin{aligned} L &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\left(I - \sum_{i=1}^{q} H_i\right) \\ &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\left(I - \sum_{j \neq i} H_j\right)(I - H_i) \\ &= (-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i \\ &\quad + \left[\sum_{j \neq i}\left(-AW_j\hat{F}_j + W_j\alpha_j\right)\tilde{H}_j + \beta\left(I - \sum_{j \neq i} H_j\right)\right](I - H_i) \end{aligned}$$

Therefore, L has the form

$$L = (-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta_i(I - H_i)$$

where

$$\beta_i = \sum_{j \neq i}\left(-AW_j\hat{F}_j + W_j\alpha_j\right)\tilde{H}_j + \beta\left(I - \sum_{j \neq i} H_j\right)$$

By Proposition 3.3, $L \in \underline{L}(\mathcal{W}_i)$ for each $\mathcal{W}_i$, which means $L \in \cap_{i=1}^{q}\underline{L}(\mathcal{W}_i)$. □

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