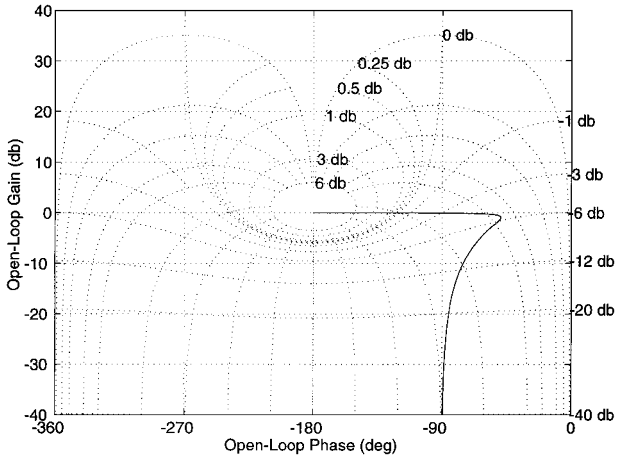

Fig. 4 Example 2, stabilizing an unstable system [$L(s) = K(s+1)/(s^2+9s-10)$], where $K = 10$.

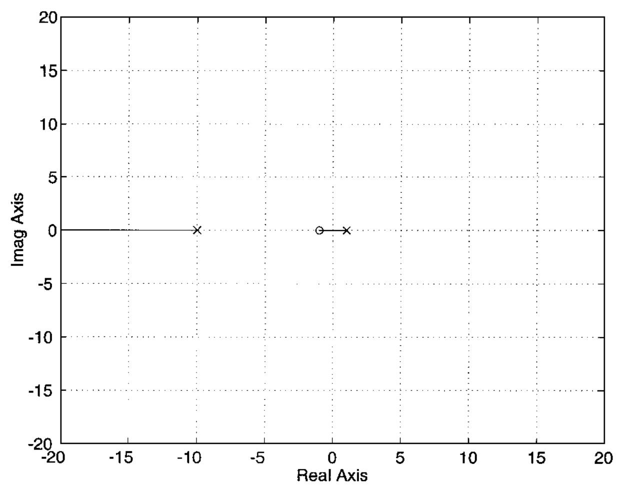

Fig. 5 Root locus of $L(s) = K(s+1)/(s^2+9s-10)$.

Consider a second example with one unstable pole. The objective here is to determine how to modify the open-loop transfer function so that the closed-loop transfer function will be stable. The open-loop transfer function is given by $L(s) = K(s+1)/(s^2+9s-10)$. Equation (2) states $\Phi = -(0.5P_\omega + P_{-1})180$ deg must be true for the system to be stable. The open-loop poles are -10 and 1. Because there are no poles on the imaginary axis and there is one unstable pole, $P_\omega = 0$ and $P_{-1} = 1$. Thus, for a stable closed-loop system, we require that $\Phi = -180$ deg. The Nichols plot of $L(s)$ is shown in Fig. 4 for a gain of $K = 10$. The initial phase at $\omega = 0$ is $\Phi_1 = -180$ deg. Of course, this is only true if the gain is raised slightly such that the initial magnitude is greater than 0 dB. If $K = 0$, the initial phase is not defined by this technique. If $K < 10$, then $\Phi_1 = -180$ deg because for $s < 0$ the 0 dB crossing is nearest 360 deg. The nearest multiple of 360 deg when the magnitude of $L(s)$ is 0 dB is 0 deg. Thus, if the gain is greater than 10, then $\Phi_1 - \Phi_3 = -180 + 0 = -180$ deg as required for stability. The root locus of Fig. 5 shows this to be the case.

Conclusions

A simple method for applying Nyquist's stability criterion on the Nichols Chart has been presented. The approach is applicable to open-loop systems with poles and/or zeros in the right-half plane. The result is a method for evaluating not only the stability of existing controlled systems, but also for determining control design objectives on the Nichols chart.

References

[1]Cohen, N., Chait, Y., Yaniv, O., and Borghesani, C., "Stability Analysis Using Nichols Charts," *International Journal of Robust and Nonlinear Control*, Vol. 4, No. 2, 1994, pp. 3–20.

[2]Kuo, B. C., *Automatic Control Systems*, 7th ed., Prentice–Hall, New York, 1991, Chap. 9.

Optimal Control and Filtering for Nonstandard Singularly Perturbed Linear Systems

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Introduction

A POWERFUL algorithm for the exact slow–fast decomposition of the algebraic Riccati equation of standard singularly perturbed systems is developed in Ref. 1, so that the optimal control and filtering tasks can be solved exactly and performed independently in slow and fast time scales.[2,3] In this Note, we show that the same algorithm, under the appropriate assumptions is applicable to the algebraic Riccati equation of nonstandard singularly perturbed control systems (having singular fast subsystem matrix). Nonstandard singularly perturbed systems are the modern research trend in control theory of singular perturbations.[4–9] The result obtained for the decomposition of the algebraic Riccati equation is used in this Note to obtain the exact pure-slow and pure-fast decomposition of optimal control and filtering tasks of nonstandard singularly perturbed linear systems. Note that in the control literature only approximate results for nonstandard singularly perturbed systems are available.

Before the results of Ref. 1 were available, control engineers were able to decompose exactly only linear singularly perturbed systems by using the celebrated Chang transformation.[10] In Ref. 11 the nonlinear algebraic Riccati equation was decomposed into slow and fast algebraic Riccati equations with the accuracy of $\mathcal{O}(\epsilon)$, where ϵ is a small positive singular perturbation parameter. Several real-world examples done in Refs. 2 and 12–14 indicate that very often an $\mathcal{O}(\epsilon)$ order of accuracy is not satisfactory. The results of Ref. 1 are, as a matter of fact, the extended and improved results of Ref. 11. It can be said that the results of Ref. 1 achieve the same goal as the results of Ref. 11, but with perfect accuracy.

The goal of this Note is to show that the results developed in Ref. 1 and the related work of Ref. 3 can be extended to nonstandard singularly perturbed systems. It should be pointed out that mechanical control systems in the modal coordinates[15] displaying slow and fast time scales are nonstandard singularly perturbed linear control systems—e.g., the linearized model of a flexible space structure.[16]

Conditions under which the first approximation of nonstandard singularly perturbed control systems can be studied are established in Ref. 8. It is important to point out that the results of Ref. 8 produce the $\mathcal{O}(\epsilon)$ accuracy only. In contrast, the results of this Note produce the exact solution and preserve the slow–fast decomposition features of Ref. 8.

Optimal Control of Nonstandard Singularly Perturbed Linear Systems

A nonstandard singularly perturbed control linear system is represented by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + B_1 u(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + B_2 u(t) \end{aligned} \tag{1}$$

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where $x_1(t) \in \Re^{n_1}$ are slow and $x_2(t) \in \Re^{n_2}$ are fast system state space variables, $u(t) \in \Re^m$ is a vector control input, and ϵ is a small positive singular perturbation parameter. Matrices $A_i (i = 1, \ldots, 4)$ and $B_j (j = 1, 2)$ are constant and of appropriate dimensions with A_4 being singular. Singularity of A_4 indicates the nonstandard singularly perturbed linear control system. In the case when the matrix A_4 is nonsingular, we have the so-called standard singularly perturbed control system.

With Eq. (1) a quadratic performance criterion to be minimized is associated:

$$J = \frac{1}{2}\int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]\,\mathrm{d}t$$

$$Q \geq 0, \qquad R > 0 \quad (2)$$

Note that the famous Chang transformation is not applicable to singularly perturbed systems having singular fast system matrix A_4. Also, the results of Ref. 11 are not applicable for the slow–fast decomposition of the corresponding algebraic Riccati equation because they require nonsingularity of A_4. However, in the following we show that the results of Ref. 1 can be applied under certain assumptions to both standard and nonstandard singularly perturbed control systems.

Let P be the positive semidefinite stabilizing solution of the algebraic Riccati equation corresponding to the optimization problem defined in Eqs. (1–2), i.e.,

$$A^T P + PA + Q - PSP = 0, \qquad P = \begin{bmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & \epsilon P_3 \end{bmatrix} \quad (3)$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ (1/\epsilon)A_3 & (1/\epsilon)A_4 \end{bmatrix}$$

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix} \quad (4)$$

$$S = \begin{bmatrix} S_1 & (1/\epsilon)Z \\ (1/\epsilon)Z^T & (1/\epsilon^2)S_2 \end{bmatrix} = BR^{-1}B^T, \qquad B = \begin{bmatrix} B_1 \\ (1/\epsilon)B_2 \end{bmatrix}$$

The optimal control for Eq. (1) is given in terms of P as

$$u(t) = -R^{-1}B^T Px(t) = -F_1 x_1(t) - F_2 x_2(t)$$

$$x^T(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}, \qquad F_1 = R^{-1}\left(B_1^T P_1 + B_2^T P_2^T\right) \quad (5)$$

$$F_2 = R^{-1}\left(\epsilon B_1^T P_2 + B_2^T P_3\right)$$

The decomposition results of Ref. 1 are summarized in the following lemma. We have also simplified and algorithmically organized the main steps on the exact pure-slow and pure-fast decomposition of singularly perturbed linear control systems originally obtained in Ref. 1.

Lemma 1. Consider the closed-loop system

$$\begin{bmatrix} \dot{x}_1(t) \\ \epsilon\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 - B_1F_1 & A_2 - B_1F_2 \\ A_3 - B_2F_1 & A_4 - B_2F_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (6)$$

There exists a nonsingular transformation $\boldsymbol{T}$ such that

$$\begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = \boldsymbol{T}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Rightarrow \begin{matrix} \dot{x}_s(t) = (a_1 + a_2P_s)x_s(t) \\ \epsilon\dot{x}_f(t) = (b_1 + b_2P_f)x_f(t) \end{matrix} \quad (7)$$

where P_s and P_f are the unique solutions of the exact pure-slow and pure-fast algebraic Riccati equations given by

$$\begin{aligned} P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s &= 0 \\ P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f &= 0 \end{aligned} \quad (8)$$

where matrices $a_i, b_i (i = 1, \ldots, 4)$ are obtained from

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_1 - T_2 L, \qquad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = T_4 + \epsilon L T_2 \quad (9)$$

with

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \qquad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \qquad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix} \quad (10)$$

The matrix L satisfies the Chang transformation equations

$$\begin{aligned} T_4 L - T_3 - \epsilon L(T_1 - T_2 L) &= 0 \\ -H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H &= 0 \end{aligned} \quad (11)$$

The solution of the original global algebraic Riccati equation [Eq. (3)] can be obtained from

$$P = \left\{\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right\} \left\{\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 \\ 0 & P_f \end{bmatrix}\right\}^{-1} \quad (12)$$

where

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \boldsymbol{\Omega} = E_1 \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon L H \end{bmatrix} E_2 \quad (13)$$

with

$$E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \epsilon I_{n_2} \end{bmatrix}, \qquad E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \quad (14)$$

The decomposition transformation $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = (\Pi_1 + \Pi_2 P) \quad (15)$$

with

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = \Omega^{-1} \quad (16)$$

For standard singularly perturbed systems, the preceding lemma is valid under the assumption that the slow and fast subsystems are stabilizable-detectable.[11] Let

$$A_0 = A_1 - A_2 A_4^{-1} A_3, \qquad B_0 = B_1 - A_2 A_4^{-1} B_2$$

$$q_0 = q_1 - q_2 A_4^{-1} A_3$$

then, the required assumption follows.

Assumption 1. The triples (A_0, B_0, q_0) and (A_4, B_2, q_2) are stabilizable-detectable. For nonstandard singularly perturbed systems we need the following assumption.

Assumption 2. The triples $(A_s, \sqrt{S_s}, \sqrt{Q_s})$ and (A_4, B_2, q_2) are stabilizable-detectable. The matrices A_s, S_s, Q_s are obtained in Ref. 8 as

$$T_1 - T_2 T_4^{-1} T_3 = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} \tag{17}$$

All steps in the preceding lemma can be easily computed by using MATLAB. The pure-slow and pure-fast algebraic Riccati equations [Eqs. (8–9)] can be solved in terms of Lyapunov iterations, which is in fact the Newton method for solving Eqs. (8–9) as demonstrated in Ref. 1. The initial conditions for the Newton method are obtained from the $\mathcal{O}(\epsilon)$-approximate slow and fast algebraic Riccati equations derived in Ref. 8, i.e.,

$$\begin{aligned} A_s^T P_s^{(0)} + P_s^{(0)} A_s + Q_s - P_s^{(0)} S_s P_s^{(0)} &= 0 \\ A_4^T P_f^{(0)} + P_f^{(0)} A_4 + Q_3 - P_f^{(0)} S_2 P_f^{(0)} &= 0 \end{aligned} \tag{18}$$

The unique positive semidefinite stabilizing solutions of the preceding algebraic Riccati equations exist under Assumption 2. By the implicit function theorem the unique solutions of the pure-slow and pure-fast algebraic Riccati equations [Eqs. (8)] exist for sufficiently small values of the small perturbation parameter ϵ.

The Chang transformation equations [Eqs. (11)] can be solved as linear equations by using either the fixed point iterations or by the Newton method. In addition, they can be solved by using the Taylor series as demonstrated in Ref. 17 and by the eigenvector method of Ref. 18.

Solvability of Eqs. (11) requires invertibility of the matrix T_4. In addition, this matrix has to be nonsingular to preserve the slow–fast decomposition of the corresponding state–costate variables, i.e., to keep slow variables slow and fast variables fast.

In Ref. 8 the linear quadratic control problem of nonstandard singularly perturbed systems is solved with the accuracy of $\mathcal{O}(\epsilon)$ by requiring nonsingularity of T_4. The following lemma is established in Ref. 8.

Lemma 2. The matrix T_4 is invertible if and only if

$$\operatorname{rank}[A_4 \quad B_2] = n_2, \qquad \operatorname{rank}\begin{bmatrix} A_4^T & D_2^T \end{bmatrix} = n_2 \tag{19}$$

where $D_2^T D_2 = Q_3$.

Lemma 2 produces the required conditions that assures invertibility of T_4 and applicability of results of Ref. 1 for solving the algebraic Riccati equation of nonstandard singularly perturbed systems in terms of reduced-order pure-slow and pure-fast algebraic Riccati equations. However, the stabilizability-detectability of the triple (A_4, B_2, q_2) also guarantees the invertibility of the matrix T_4. Thus, the lemma established in Ref. 8 states only another set of conditions under which the matrix T_4 is invertible.

Example 1. To compare the results of this Note and that of Ref. 8 and to demonstrate an improvement over the results of Ref. 8, we have considered the following example:

$$A_1 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0.345 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0.524 \\ 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0.262 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$R = 1, \qquad x(0) = [1 \quad 0 \quad 1 \quad 0]^T$$

In Table 1 we compare, for different values of the small singular perturbation parameter ϵ, the values for the approximate optimal criterion J_{WF92} obtained by using the methodology of Ref. 8 and the optimal criterion values obtained by using the technique presented in this Note. It can be seen from Table 1 that for very small values of ϵ the satisfactory results are obtained by both methods. However, for relatively bigger values of ϵ the results of Ref. 8 are not accurate. In such cases, the reduced-order slow–fast decomposition technique proposed in this Note has to be used.

Table 1 Comparison of two algorithms

ϵ	J_{opt}	J_{WF92}	$J_{\text{opt}} - J_{WF92}$, %
0	3.1423	3.1423	0
0.01	3.2530	3.2548	0.06
0.05	3.7246	3.7780	1.43
0.1	4.3813	4.6392	5.89
0.25	6.8166	10.1217	48.5

Kalman Filtering for Nonstandard Singularly Perturbed Systems

A linear stochastic nonstandard singularly perturbed system is represented by

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_2 x_2(t) + G_1 w(t) \\ \epsilon \dot{x}_2(t) &= A_3 x_1(t) + A_4 x_2(t) + G_2 w(t) \end{aligned} \tag{20}$$

where $w(t)$ represents an r-dimensional Gaussian zero-mean stationary white noise stochastic process with intensity matrix $W > 0$. Matrices G_1 and G_2 are constant and of appropriate dimensions. It should be emphasized that the matrix A_4 is singular, in contrast to the Kalman filtering problem of standard singularly perturbed systems where this matrix is nonsingular.[3] With system (20) a measurement equation is associated:

$$y(t) = C_1 x_1(t) + C_2 x_2(t) + v(t) \tag{21}$$

where $y(t)$ is a p-dimensional measurement vector, $v(t)$ is a p-dimensional measurement zero-mean stationary Gaussian white noise stochastic process with intensity matrix $V > 0$, and C_1, C_2 are constant matrices of appropriate dimensions.

The Kalman filtering problem can be studied by using duality with the optimal control problem. In this case, the duality is achieved by replacing matrices T_1, T_2, T_3, T_4, defined in Eqs. (10), respectively, by the following matrices:

$$\begin{aligned} T_{1_F} &= \begin{bmatrix} A_1^T & -C_1^T V^{-1} C_1 \\ -G_1 W G_1^T & -A_1 \end{bmatrix} \\ T_{2_F} &= \begin{bmatrix} A_3^T & -C_1^T V^{-1} C_2 \\ -G_1 W G_2^T & -A_2 \end{bmatrix} \\ T_{3_F} &= \begin{bmatrix} A_2^T & -C_2^T V^{-1} C_1 \\ -G_2 W G_1^T & -A_3 \end{bmatrix} \\ T_{4_F} &= \begin{bmatrix} A_4^T & -C_2^T V^{-1} C_2 \\ -G_2 W G_2^T & -A_4 \end{bmatrix} \end{aligned} \tag{22}$$

Using Eqs. (22) we form the matrix dual to the matrix defined in Eqs. (18) as

$$T_{1_F} - T_{2_F} T_{4_F}^{-1} T_{3_F} = \begin{bmatrix} A_{s_F}^T & -C_s^T V_s C_s \\ -G_s W_s G_s^T & -A_{s_F} \end{bmatrix} \tag{23}$$

The following assumption is dual to Assumption 2.

Assumption 3. The triples (A_{s_F}, C_s, G_s), (A_4, C_2, G_2) are stabilizable-detectable.

Lemma 2 of Ref. 8 in the case of Kalman filtering of nonstandard singularly perturbed linear systems should be replaced by the following dual lemma.

Lemma 3. The matrix T_{4_F}, defined by Eq. (23), is invertible if and only if

$$\operatorname{rank}\begin{bmatrix} A_4^T & C_2^T \end{bmatrix} = n_2, \qquad \operatorname{rank}[A_4 \quad G_2] = n_2 \tag{24}$$

However, as pointed out for the regulator problem, the stabilizability-detectability conditions imposed on the fast subsystem in Assumption 3 also guarantee nonsingularity of the matrix T_{4_F}.

According to the results of Ref. 3, the exact reduced-order, independent, pure-slow and pure-fast, Kalman filters driven by the system measurements are given by

$$\begin{aligned} \dot{\hat{x}}_s(t) &= \left(a_{1_F} + a_{2_F} P_{s_F}\right)^T \hat{x}_s(t) + K_s y(t) \\ \epsilon \dot{\hat{x}}_f(t) &= \left(b_{1_F} + b_{2_F} P_{f_F}\right)^T \hat{x}_f(t) + K_f y(t) \end{aligned} \tag{25}$$

where the newly defined matrices are

$$\begin{aligned} \begin{bmatrix} a_{1_F} & a_{2_F} \\ a_{3_F} & a_{4_F} \end{bmatrix} &= \left(T_{1_F} - T_{2_F} L_F\right) \\ \begin{bmatrix} b_{1_F} & b_{2_F} \\ b_{3_F} & b_{4_F} \end{bmatrix} &= \left(T_{4_F} + \epsilon L_F T_{2_F}\right) \end{aligned} \tag{26}$$

$$\begin{bmatrix} K_s \\ (1/\epsilon) K_f \end{bmatrix} = \left(\Pi_{1_F} + \Pi_{2_F} P_F\right)^{-T} P_F \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} V^{-1} \tag{27}$$

The matrix P_F is obtained by using formula (12) with the solutions of the pure-slow and pure-fast algebraic filter Riccati equations obtained from

$$\begin{aligned} P_{s_F} a_{1_F} - a_{4_F} P_{s_F} - a_{3_F} + P_{s_F} a_{2_F} P_{s_F} &= 0 \\ P_{f_F} b_{1_F} - b_{4_F} P_{f_F} - b_{3_F} + P_{f_F} b_{2_F} P_{f_F} &= 0 \end{aligned} \tag{28}$$

and with

$$\begin{aligned} \Omega_F^{-1} &= \begin{bmatrix} \Omega_{1_F} & \Omega_{2_F} \\ \Omega_{3_F} & \Omega_{4_F} \end{bmatrix}^{-1} = \begin{bmatrix} \Pi_{1_F} & \Pi_{2_F} \\ \Pi_{3_F} & \Pi_{4_F} \end{bmatrix} \\ &= E_2^T \begin{bmatrix} I - \epsilon H_F L_F & -\epsilon H_F \\ L_F & I \end{bmatrix} E_3 \end{aligned} \tag{29}$$

where

$$E_3 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & (1/\epsilon) I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{30}$$

The Chang filter decoupling algebraic equations satisfy

$$\begin{aligned} T_{4_F} L_F - T_{3_F} - \epsilon L_F \left(T_{1_F} - T_{2_F} L_F\right) &= 0 \\ -H_F\left(T_{4_F} + \epsilon L_F T_{2_F}\right) + T_{2_F} + \epsilon\left(T_{1_F} - T_{2_F} L_F\right) H_F &= 0 \end{aligned} \tag{31}$$

The initial conditions for the Newton method for solving the pure-slow and pure-fast algebraic Riccati equations are obtained, respectively, from

$$\begin{aligned} P_{s_F}^{(0)} A_{s_F}^T + A_{s_F} P_{s_F}^{(0)} + W_s - P_{s_F}^{(0)} V_s P_{s_F}^{(0)} &= 0 \\ P_{f_F}^{(0)} A_4^T + A_4 P_{f_F}^{(0)} + G_2 W G_2^T - P_{f_F}^{(0)} C_2^T V^{-1} C_2 P_{f_F}^{(0)} &= 0 \end{aligned} \tag{32}$$

It can be easily shown that initial conditions are $\mathcal{O}(\epsilon)$ approximations of the exact solutions.

The optimal estimates obtained from Eqs. (25) are related to the optimal estimates of the state variables of Eqs. (20) by the following nonsingular transformation:

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \left(\Pi_{1_F} + \Pi_{2_F} P_F\right)^T \begin{bmatrix} \hat{x}_s(t) \\ \hat{x}_f(t) \end{bmatrix} \tag{33}$$

Conclusions

We have shown that the standard exact slow–fast decomposition method is applicable under certain assumptions to nonstandard singularly perturbed linear systems. In contrast to the $\mathcal{O}(\epsilon)$ accurate results presently available in the literature, the presented methodology produces the exact solution to the linear quadratic optimal control and filtering problems for this class of systems.

References

[1] Su, W., Gajic, Z., and Shen, X., "The Exact Slow-Fast Decomposition of the Algebraic Riccati Equation of Singularly Perturbed Systems," *IEEE Transactions on Automatic Control*, Vol. AC-37, Sept. 1992, pp. 1456–1459.

[2] Gajic, Z., and Shen, X., *Parallel Algorithms for Optimal Control of Large Scale Linear Systems*, Springer–Verlag, London, 1993.

[3] Gajic, Z., and Lim, M., "A New Filtering Method for Linear Singularly Perturbed Systems," *IEEE Transactions on Automatic Control*, Vol. AC-39, Sept. 1994, pp. 1952–1955.

[4] Chow, J., *Time-Scale Modeling of Dynamic Networks with Applications to Power Systems,* Springer–Verlag, New York, 1982.

[5] Khalil, H., "Time Scale Decomposition of Linear Implicit Singularly Perturbed Systems," *IEEE Transactions on Automatic Control*, Vol. AC-29, Nov. 1984, pp. 1054–1056.

[6] Khalil, H., "Feedback Control of Nonstandard Singularly Perturbed Systems," *IEEE Transactions on Automatic Control*, Vol. AC-34, Oct. 1989, pp. 1052–1060.

[7] Kokotovic, P., Khalil, H., and O'Reilly, J., *Singular Perturbation Methods in Control: Analysis and Design*, Academic Press, Orlando, FL, 1986.

[8] Wang, Y., and Frank, P., "Complete Decomposition of Sub-Optimal Regulators for Singularly Perturbed Systems," *International Journal of Control*, Vol. 55, Jan. 1992, pp. 49–60.

[9] Wang, Y., Frank, P., and Wu, N., "Near-Optimal Control of Nonstandard Singularly Perturbed Systems," *Automatica*, Vol. 30, No. 2, 1994, pp. 277–292.

[10] Chang, K., "Singular Perturbations of a General Boundary Value Problem," *SIAM Journal of Mathematical Analysis*, Vol. 3, Aug. 1972, pp. 520–526.

[11] Chow, J., and Kokotovic, P., "A Decomposition of Near-Optimum Regulators for Systems with Slow and Fast Modes," *IEEE Transactions on Automatic Control*, Vol. AC-21, Oct. 1976, pp. 701–705.

[12] Gajic, Z., Petkovski, D., and Harkara, N., "The Recursive Algorithm for Optimal Static Output Feedback Control Problem of Linear Singularly Perturbed Systems," *IEEE Transactions on Automatic Control*, Vol. AC-34, April 1989, pp. 465–468.

[13] Skataric, D., and Gajic, Z., "Linear Control of Nearly Singularly Perturbed Hydro Power Plants," *Automatica* , Vol. 28, Feb. 1992, pp. 159–163.

[14] Mizukami, K., and Suzumura, F., "Closed-Loop Stackelberg Strategies for Singularly Perturbed Systems: The Recursive Approach," *International Journal of Systems Science*, Vol. 24, No. 5, 1993, pp. 887–900.

[15] Baruh, H., and Choe, K., "Sensor Placement in Structural Control," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 3, 1990, pp. 524–533.

[16] Moerder, D. D., and Calise, A. J., "Two-Time Scale Stabilization of Systems with Output Feedback," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 6, 1985, pp. 731–736.

[17] Derbel, N., Kamoun, M., and Poloujadoff, M., "New Approach to Block-Diagonalization of Singularly Perturbed Systems by Taylor Expansion," *IEEE Transactions on Automatic Control* , Vol. AC-39, April 1994, pp. 1429–1431.

[18] Kecman, V., Bingulac, S., and Gajic, Z., "Eigenvector Method for Exact Decomposition of Singularly Perturbed and Regular Optimal Control Problems," *Automatica*, Vol. 35, Jan. 1999, in press.