Adaptive Two-Step Filter with Applications to Bearings-Only Measurement Problem

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Introduction

THE state estimation of a nonlinear system is still an important research subject. In some cases, the usual extended Kalman filter (EKF) produces large estimation errors and even diverges. In this paper, a two-step filter[1,2] for a class of nonlinear systems that consists of a linear dynamic model and a nonlinear measurement model is introduced. In many practical systems, the statistical properties of measurement noise are time varying and unknown a priori. For this case, a statistical estimator can be used to determine the mean and covariance of measurement noise online. A modified Sage–Husa time-varying measurement noise statistical estimator is integrated with the two-step filter to produce an adaptive two-step filter (ATSF). Finally, the ATSF is applied to the bearings-only measurement problem. Numerical comparisons of the ATSF with the EKF and the adaptive EKF (AEKF) are made using this application.

ATSF

Consider a nonlinear system

$$x_{k+1} = \Phi_k x_k + B_k u_k + \Gamma_k w_k \tag{1}$$

$$z_k = F(x_k, t_k) + v_k \tag{2}$$

where x_k is a state vector, u_k is a control vector, and w_k is a dynamic noise vector. The measurement z_k is expressed as a nonlinear function $F(\cdot,\cdot)$ of the system state and time, plus measurement noise v_k. The vectors w_k and v_k are assumed to be white, Gaussian processes with $E[w_k]=0$, $E[w_k w_k^T]=Q_k$, $E[v_k]=r_k$, $E[v_k v_k^T]=R_k$, and $E[w_k v_k^T]=0$.

The two-step filter[1,2] is summarized by the following set of equations: given the measurements

$$z_k = F(x_k, t_k) + v_k = H_k y_k + v_k, \qquad k = 1, 2, \ldots, N$$

and the nonlinearity $y_k = f_k(x_k, t_k)$, in general, choose

$$y_k = \begin{bmatrix} x_k \\ F(x_k, t_k) \end{bmatrix}$$

The first-step measurement update is

$$\hat{y}_k = \bar{y}_k + P_{y_k} H_k^T R_k^{-1}(z_k - H_k \bar{y}_k - r_k), \qquad k = 1, 2, \ldots, N \tag{3}$$

$$P_{y_k} = \left(M_{y_k}^{-1} + H_k^T R_k^{-1} H_k\right)^{-1} \tag{4}$$

where $M_{y_k} = E[(y_k - \bar{y}_k)(y_k - \bar{y}_k)^T]$, $P_{y_k} = E[(y_k - \hat{y}_k)(y_k - \hat{y}_k)^T]$, and $\bar{y}_k$ is the predicted vector of the state y_k.

The second-step measurement update (Newton–Raphson iterative algorithm) is

$$\hat{x}_{k,i+1} = \hat{x}_{k,i} - L_{G_{k,i}}^{-1} q_{k,i}^T \tag{5}$$

where i is the iteration number,

$$q_{k,i} = -[\hat{y}_k - f(\hat{x}_{k,i})]^T P_{y_k}^{-1} \left.\frac{\partial f}{\partial x_k}\right|_{x_k = \hat{x}_{k,i}} \tag{6}$$

$$L_{G_{k,i}} = \left.\frac{\partial f}{\partial x_k}\right|_{x_k = \hat{x}_{k,i}}^T P_{y_k}^{-1} \left.\frac{\partial f}{\partial x_k}\right|_{x_k = \hat{x}_{k,i}} \tag{7}$$

Equation (5) can be iterated until $\hat{x}_{k,i+1} - \hat{x}_{k,i} \to 0$. Define the covariance of the state x_k as

$$P_{x_k} = E\left[(\hat{x}_{k,i+1} - x_k)(\hat{x}_{k,i+1} - x_k)^T\right]$$

An approximate expression of P_{x_k} is

$$P_{x_k} = L_{G_{k,i}}^{-1} \tag{8}$$

The second-step time update is

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k + B_k u_k, \qquad k = 1, 2, \ldots, N-1 \tag{9}$$

$$M_{x_{k+1}} = \Phi_k P_{x_k} \Phi_k^T + \Gamma_k Q_k \Gamma_k^T \tag{10}$$

where $M_{x_{k+1}} = E[(x_{k+1} - \bar{x}_{k+1})(x_{k+1} - \bar{x}_{k+1})^T]$. The first-step time update is

$$\bar{y}_{k+1} \approx \hat{y}_k + f(\bar{x}_{k+1}) - f(\hat{x}_k) \tag{11}$$

$$M_{y_{k+1}} \approx P_{y_k} + \left.\frac{\partial f}{\partial x_{k+1}}\right|_{x_{k+1} = \bar{x}_{k+1}} M_{x_{k+1}} \left.\frac{\partial f}{\partial x_{k+1}}\right|_{x_{k+1} = \bar{x}_{k+1}}^T - \left.\frac{\partial f}{\partial x_k}\right|_{x_k = \hat{x}_k} P_{x_k} \left.\frac{\partial f}{\partial x_k}\right|_{x_k = \hat{x}_k}^T \tag{12}$$

where $M_{y_{k+1}} = E[(y_{k+1} - \bar{y}_{k+1})(y_{k+1} - \bar{y}_{k+1})^T]$.

In Eqs. (3) and (4), r_k and R_k are a priori unknown time-varying mean and covariance, respectively. For a linear state estimation problem, a well-known maximum a posteriori (MAP) measurement noise statistical estimator is the Sage–Husa estimator,

$$\hat{r}_k = \frac{1}{k}\sum_{i=1}^{k}[z_i - H_i \hat{y}_{i,k}]$$

$$\hat{R}_k = \frac{1}{k}\sum_{i=1}^{k}[z_i - H_i \hat{y}_{i,k} - r][z_i - H_i \hat{y}_{i,k} - r]^T$$

In adaptive filtering, Sage and Husa replace $\hat{y}_{i,k}$ approximately with $\hat{y}_{i,i-1}$ to obtain a recursive estimator,

$$\hat{r}_k = (1/k)[(k-1)\hat{r}_{k-1} + z_k - H_k \hat{y}_{k,k-1}] \tag{13}$$

$$\hat{R}_k = (1/k)\left[(k-1)\hat{R}_{k-1} + \varepsilon_k \varepsilon_k^T - H_k P_{y_{k,k-1}} H_k^T\right] \tag{14}$$

where $\varepsilon_k = z_k - H_k \hat{y}_k - \hat{r}_k$. In fact, replacing $\hat{y}_{i,k}$ with $\hat{y}_{i,i}$ can induce a modified recursive estimator,

$$\hat{r}_k = (1/k)[(k-1)\hat{r}_{k-1} + z_k - H_k \hat{y}_k] \tag{15}$$

$$\hat{R}_k = (1/k)\left[(k-1)\hat{R}_{k-1} + (I - H_k K_k)\varepsilon_k \varepsilon_k^T (I - H_k K_k)^T + H_k P_{y_k} H_k^T\right] \tag{16}$$

where $K_k = P_{y_k} H_k^T R_k^{-1}$. Clearly, the modified estimator is more accurate than the original one. Moreover, in Eq. (16), the positive

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definiteness of $\hat{\boldsymbol{R}}_k$ is naturally guaranteed because only plus manipulation occurs among all of the positive definite matrices on the right side. However, in Eq. (14), minus manipulation occurs among the positive definite matrices on the right side, so that the positive definiteness of $\hat{\boldsymbol{R}}_k$ may be destroyed by the large initial estimate error and other accidents. In short, robustness of the modified estimator against some uncertainties is better than that of the original one.

Extending Eqs. (15) and (16) to the time-varying case produces

$$\hat{\boldsymbol{r}}_k = (1 - d_k)\hat{\boldsymbol{r}}_{k-1} + d_k(\boldsymbol{z}_k - \boldsymbol{H}_k\hat{\boldsymbol{y}}_k) \tag{17}$$

$$\hat{\boldsymbol{R}}_k = (1 - d_k)\hat{\boldsymbol{R}}_{k-1} + d_k\Big[(\boldsymbol{I} - \boldsymbol{H}_k\boldsymbol{K}_k)\varepsilon_k\varepsilon_k^T(\boldsymbol{I} - \boldsymbol{H}_k\boldsymbol{K}_k)^T + \boldsymbol{H}_k\boldsymbol{P}_{y_k}\boldsymbol{H}_k^T\Big] \tag{18}$$

where $d_k = (1-b)/(1-b^k)$, $0 < b < 1$, and b is referred to as a forgetting factor.

Running Eqs. (3–12), Eq. (17), and Eq. (18) in the given sequence constitutes an ATSF.

Application to Bearings-Only Measurement Problem

In the Cartesian coordinate system, using the Singer model,[3–6] the three components of the target acceleration are assumed to be three independent Gauss–Markov stochastic processes. Then, the discrete dynamic model, which depicts the relative motion between the target and the missile,[3,6] is

$$\boldsymbol{x}_{k+1} = \boldsymbol{\Phi}_k\boldsymbol{x}_k + \boldsymbol{B}_k\boldsymbol{u}_k + \boldsymbol{\Gamma}_k\boldsymbol{w}_k \tag{19}$$

where

$$\boldsymbol{x}_k = [r_{xk} \quad r_{yk} \quad r_{zk} \quad v_{xk} \quad v_{yk} \quad v_{zk} \quad a_{Txk} \quad a_{Tyk} \quad a_{Tzk}]^T$$

with the first three variables representing relative positions, the second three variables representing relative velocities, and the last three variables representing target accelerations. The control vector is $\boldsymbol{u}_k = [a_{Mxk} \ a_{Myk} \ a_{Mzk}]^T$. The dynamic noise vector is $\boldsymbol{w}_k = [w_{Txk} \ w_{Tyk} \ w_{Tzk}]^T$ with $E[\boldsymbol{w}_k] = 0$ and $E[\boldsymbol{w}_k\boldsymbol{w}_k^T] = \boldsymbol{Q}_k = \sigma^2\boldsymbol{I}_3$, where $\boldsymbol{I}_3$ is a 3×3 identity matrix.

The dynamic coefficients for Eq. (19) are

$$\boldsymbol{\Phi}_k = \begin{bmatrix} \boldsymbol{I}_3 & \Delta t\boldsymbol{I}_3 & 1/\lambda^2(e^{-\lambda\Delta t} + \lambda\Delta t - 1)\boldsymbol{I}_3 \\ \boldsymbol{0}_3 & \boldsymbol{I}_3 & 1/\lambda(1 - e^{-\lambda\Delta t})\boldsymbol{I}_3 \\ \boldsymbol{0}_3 & \boldsymbol{0}_3 & e^{-\lambda\Delta t}\boldsymbol{I}_3 \end{bmatrix}$$

$$\boldsymbol{B}_k = \begin{bmatrix} -(\Delta t^2/2)\boldsymbol{I}_3 \\ -\Delta t\boldsymbol{I}_3 \\ \boldsymbol{0}_3 \end{bmatrix}, \qquad \boldsymbol{\Gamma}_k = \begin{bmatrix} \boldsymbol{0}_3 \\ \boldsymbol{0}_3 \\ \boldsymbol{I}_3 \end{bmatrix}$$

where Δt is the time interval between measurements and λ is determined from the bandwidth of the target acceleration.

The angle measurements for the three-dimensional bearings-only problem can be written as

$$\boldsymbol{z}_k = \begin{bmatrix} \tan^{-1}\dfrac{r_{yk}}{r_{xk}} & \tan^{-1}\dfrac{-r_{zk}}{\left(r_{xk}^2 + r_{yk}^2\right)^{\frac{1}{2}}} \end{bmatrix}^T + \boldsymbol{v}_k \tag{20}$$

In Eq. (20), $\boldsymbol{v}_k$ is measurement noise with

$$E[\boldsymbol{v}_k] = 0, \qquad E\Big[\boldsymbol{v}_k\boldsymbol{v}_k^T\Big] = \boldsymbol{R}_k = 10^{-8}\boldsymbol{I}_2 + \boldsymbol{D}_k^{-1}\boldsymbol{\beta}_k\boldsymbol{D}_k^{-T} \tag{21}$$

where $\boldsymbol{\beta}_k = 0.1\boldsymbol{I}_2$, with $\boldsymbol{I}_2$ a 2×2 identity matrix, and

$$\boldsymbol{D}_k = \begin{bmatrix} \sqrt{r_{xk}^2 + r_{yk}^2 + r_{zk}^2} & 0 \\ 0 & \sqrt{r_{xk}^2 + r_{yk}^2 + r_{zk}^2} \end{bmatrix}$$

It is obvious that $\boldsymbol{R}_k$ is a time-varying matrix and unknown a priori. Both $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ are white Gaussian processes, and $E[\boldsymbol{w}_k\boldsymbol{v}_k^T] = 0$.

We apply the ATSF to the bearings-only measurement problem described by the system (19) and (20) and test its performance through numerical simulation.

Let

$$\boldsymbol{y}_k = \begin{bmatrix} \boldsymbol{x}_k \\ \boldsymbol{F}(\boldsymbol{x}_k, t_k) \end{bmatrix}, \qquad \boldsymbol{M}_{y0} = \begin{bmatrix} 10^5\boldsymbol{I}_3 & 0 & 0 \\ 0 & 10^4\boldsymbol{I}_3 & 0 \\ 0 & 0 & 10^2\boldsymbol{I}_5 \end{bmatrix}$$

$$\hat{\boldsymbol{R}}_0 = 10^{-8}\boldsymbol{I}_2$$

and $b = 0.98$, where $\boldsymbol{I}_5$ is a 5×5 identity matrix. Moreover, let $\sigma^2 = 1\ \mathrm{m^2/s^4}$, $\lambda = 1$, and $\Delta t = 10$ ms.

In both the elevation loop and the azimuth loop, a practical guidance law[7] associated with the augmented proportional guidance[7] is used to generate the missile's guidance command. For instance, in the elevation loop, the guidance law is

$$a_{\mathrm{Mel}} = NV_c\dot{q} + (N/2)a_{\mathrm{Tel}} + F_cr_cq \tag{22}$$

where a_{Mel} and a_{Tel} are, respectively, the missile and target accelerations perpendicular to the current line of sight (LOS); q and $\dot{q}$ are, respectively, the elevation angle and elevation angular rate; N is the effective navigation ratio; V_c is the closing velocity; r_c is the target–missile relative range; and F_c is a positive constant. By inducting the term F_cr_cq, this guidance law induces oscillatory motion of the LOS angle to enhance observability of the bearings-only tracking but without sacrificing terminal effectiveness.[7] In practical applications, the estimate values of a guidance filter are used to calculate r_c, V_c, and a_{Tel}.

For comparison, the EKF and an AEKF that uses the modified Sage–Husa time-varying measurement noise estimator [Eqs. (17) and (18)] to estimate $\boldsymbol{r}_k$ and $\boldsymbol{R}_k$ in real time are also applied to the bearings-only tracking problem under the same conditions as that on the ATSF. Because the linearization error in measurement model can be treated as an additional measurement noise, the AEKF is able to compensate for the bad influence of model error by estimating and adjusting $\boldsymbol{r}_k$ and $\boldsymbol{R}_k$ online. The EKF uses its estimate values to calculate $\boldsymbol{R}_k$ due to Eq. (21) and assumes that $\boldsymbol{r}_k$ is zero.

In the simulation, three typical launch scenarios are studied. The first one is given by $r_{x0} = 3500$ m, $r_{y0} = 15$ m, $r_{z0} = 10$ m, $v_{x0} = -1100$ m/s, $v_{y0} = -1.5$ m/s, $v_{z0} = -0.5$ m/s, and $a_{Tx0} = a_{Ty0} = a_{Tz0} = 100\ \mathrm{m/s^2}$. In this scenario, both the elevation angle and azimuth angle are small, and so the nonlinearity in the measurement model is weak.

The second launch scenario is given by $r_{x0} = 3500$ m, $r_{y0} = 1500$ m, $r_{z0} = 1000$ m, $v_{x0} = -1100$ m/s, $v_{y0} = -150$ m/s, $v_{z0} = -50$ m/s, and $a_{Tx0} = a_{Ty0} = a_{Tz0} = 100\ \mathrm{m/s^2}$, where the elevation angle and azimuth angle are medium, which means that the nonlinearity in the measurement model is medium.

The third launch scenario is given by $r_{x0} = 3500$ m, $r_{y0} = 3000$ m, $r_{z0} = 2000$ m, $v_{x0} = -1100$ m/s, $v_{y0} = -300$ m/s, $v_{z0} = -100$ m/s, and $a_{Tx0} = a_{Ty0} = a_{Tz0} = 100\ \mathrm{m/s^2}$, where the elevation angle and azimuth angle are large, which means that the nonlinearity in the measurement model is strong.

In all of the scenarios, assume that the initial estimate of the filters is $\hat{r}_{x0} = 3000$ m, $\hat{r}_{y0} = 1200$ m, $\hat{r}_{z0} = 800$ m, $\hat{v}_{x0} = -950$ m/s, $\hat{v}_{y0} = -100$ m/s, $\hat{v}_{z0} = -100$ m/s, and $\hat{a}_{Tx0} = \hat{a}_{Ty0} = \hat{a}_{Tz0} = 0\ \mathrm{m/s^2}$.

Apparently, for scenarios 1 and 3, the initial estimated error is large, and for scenario 2, the initial estimated error is relatively small. For each scenario, let the guidance ratios in Eq. (22) be $N = 4$ and $F_c = 0.4$ and run a Monte Carlo simulation to calculate the averaged miss distances while feeding back the estimates of ATSF, AEKF, and EKF to the guidance law. The results are listed in Table 1.

Table 1 Miss distances while feeding back filters' estimates, m

Filter	Scenario 1	Scenario 2	Scenario 3
ATSF	0.026	0.022	0.015
AEKF	0.026	0.032	0.042
EKF	5.554	0.496	7.706

Analyze the results in Table 1. In scenario 1, although the nonlinearity in the measurement model is weak, the large initial estimated error still produces a large initial linearization error. The EKF is not able to deal with this error, and so it performs badly and the miss distance by using EKF is large. The AEKF is able to compensate for the linearization error well in this weak nonlinear case, and so its performance is good. Note that the observability of the state components in the x axis is weak in this case, where the elevation angle and azimuth angle are small. Because of this weak observability and the weak nonlinearity of the measurement model, the advantage of the ATSF over the AEKF is not reflected by the simulation results in as much as the miss distances indicate that the AEKF does as well as the ATSF in this case.

In scenario 2, where the nonlinearity in the measurement model is medium, the EKF is still inferior to the ATSF and AEKF, although its performance is improved over that in scenario 1 for the relative small initial estimated error. Because the nonlinearity in the measurement model becomes stronger, the advantage of the ATSF over the AEKF is indicated by the miss distances.

Scenario 3 is the worst case because of the strong measurement nonlinearity and large initial estimated error. The EKF has no way to deal with the linearization error, and so its performance is quite bad in this worst case. By compensating for the linearization error by adjusting the mean and covariance of measurement noise online, the AEKF performs better than the EKF. Unfortunately, this compensation is limited for improving the filtering performance while meeting this serious nonlinear case. In the first step filtering of the ATSF, the measurement model is linear, and the noise statistical estimator is able to estimate the mean and covariance of measurement noise, so that no model error exists. In the second step filtering of the ATSF, the Newton–Raphson iterative algorithm is used to deal with the nonlinearity in the measurement model, which is more effective in alleviating the bad influence of the nonlinearity. Moreover, the iteration is apt to eliminate the influence of large initial error. Therefore, the ATSF performs significantly better than the AEKF and the EKF in this strong nonlinear case.

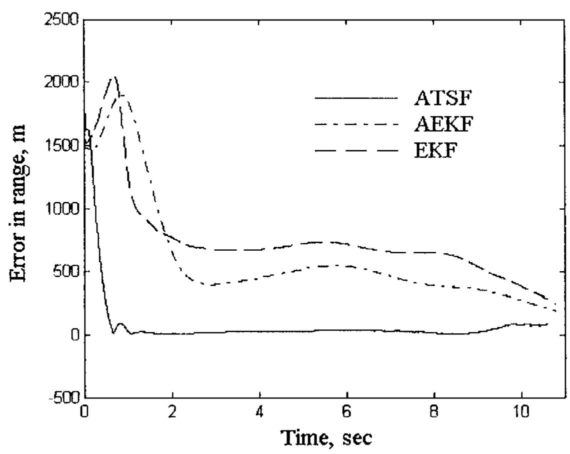

Fig. 1 Errors in range estimates of the ATSF, AEKF, and EKF for scenario 3.

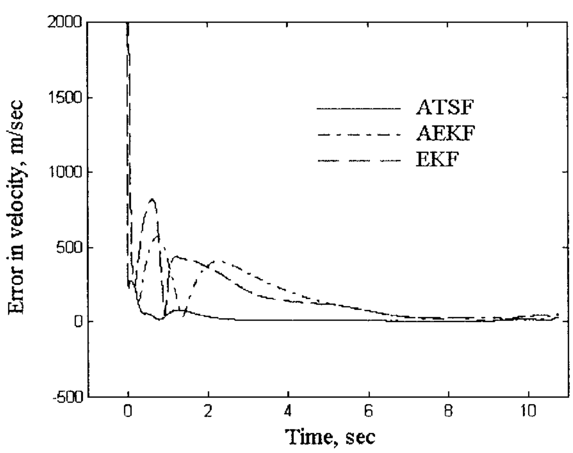

Fig. 2 Errors in velocity estimates of the ATSF, AEKF, and EKF for scenario 3.

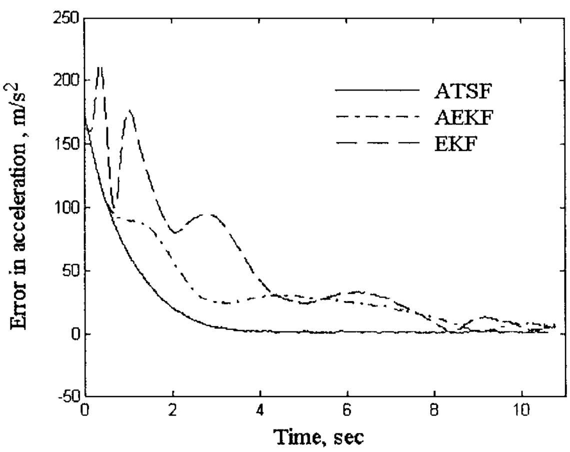

Fig. 3 Errors in acceleration estimates of the ATSF, AEKF, and EKF for scenario 3.

The results of 100 runs of Monte Carlo simulation for scenario 3 are presented in Figs. 1–3, which compare the filters' performance more obviously. The error in the range estimate at time step k in Fig. 1, for example, is plotted by using the rms-type value[6] $\sqrt{(E[e_{r_{xk}}]^2 + E[e_{r_{yk}}]^2 + E[e_{r_{zk}}]^2)}$, where $E[e_{r_{xk}}]$ is the averaged value of the error in the estimate of the r_{xk} over 100 runs of Monte Carlo simulation. Similar rms-type quantities are plotted in Figs. 2 and 3. Figures 1–3 show that the estimated errors of the ATSF are apparently smaller than those of the EKF and AEKF.

Conclusions

The ATSF for a class of nonlinear systems consisting of a linear dynamic model and a nonlinear measurement model is able to determine the mean and covariance of measurement noise online, so that it performs well in the cases where the statistical properties are unknown a priori. The ATSF is applied to the bearings-only measurement problem and performs dramatically better than the EKF and the AEKF, especially while meeting strong measurement nonlinearity and large initial estimated error.

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