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Improved Performance of Recursive Tracking Filters Using Batch Initialization and Process Noise Adaptation

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Performance of nonlinear recursive tracking filters may be improved using batch initialization and process-noise adaptation. These techniques are applied to improve radar tracking of a nonmaneuvering target vehicle during orbital flight. Using position measurements collected during a short interval following target acquisition, a batch maximum-likelihood algorithm generates an initial state estimate and covariance matrix for the recursive filter. The transient response of the recursive filter is improved, particularly in the convergence of velocity errors, because the batch covariance matrix has significant, nonzero correlations of position and velocity errors. Covariance propagation includes a process-noise matrix that compensates for errors in the nonlinear transition matrix that arise when the state estimate differs from the true state. The noise matrix is "tuned" with an adaptive procedure that uses the most recent state estimate, covariance matrix and gravity-gradient model for an inverse-square gravitational field. With process noise included, the recursive filter covariance shows improved agreement with the statistics of actual errors in the estimates.

I. Introduction

MANY methods have been developed for radar tracking of a nonmaneuvering vehicle during orbital flight.[1–5] State vector estimates and covariances may be generated from radar position measurements with a batch filter or a recursive filter. Recursive filters are usually preferred because batch filters impose a greater computational burden. However, radar tracking problems are inherently nonlinear, and performance of a recursive filter is encumbered by linear models of the dynamical system and measurements. Although nonlinearities can be accommodated in the update and propagation of the state estimate, covariance propagation is based on a linear model.

Linear filtering theory predicts that estimation accuracy should improve as more measurements are processed, and that the positive eigenvalues of the covariance matrix should decrease. In many nonlinear applications, the covariance matrix is usually not consistent with the statistics of actual errors in the state estimates obtained by Monte Carlo simulation. Without process noise, filter covariances are generally optimistic because of measurement linearization[2] and other modeling approximations. Consequently, a process-noise matrix is generally included to compensate for nonlinear effects, but the tuning procedure can be cumbersome.[4]

In this paper, the orbital tracking problem is revisited to improve performance of a nonlinear recursive filter. Two objectives are to accelerate the convergence of velocity errors during filter transient response and to demonstrate satisfactory agreement of the filter covariance with the statistics of actual errors in the estimates. It is shown that error convergence may be enhanced by proper initialization of the covariance matrix, and that covariance fidelity may be improved using a model-referenced technique for adaptation of the process-noise matrix. This work is an extension of earlier studies concerned with nonlinear estimation (and interception) of targets during re-entry[6] and boost.[7]

Batch and recursive tracking filters are formulated for a discrete, nonlinear dynamical system (Sec. II). Nonlinear measurements in radar-centered coordinates are transformed to linear measurements of the target position vector in inertial coordinates. Although radar measurement errors are independent in radar coordinates, the measurement noise matrix is not diagonal in inertial coordinates. Initial conditions for the recursive filter are generated by a maximum-likelihood batch filter. After initialization with the batch state estimate and covariance matrix, subsequent estimates and covariances are generated by the recursive filter (refer to Fig. 1).

Target orbital dynamics are represented by a nonlinear transition matrix. As the transition matrix depends on the state vector, errors in this matrix arise when the state estimate differs from the true state. These errors are compensated with a process-noise matrix in the equation for propagation of the covariance matrix (Sec. III). The algebraic structure of the noise matrix is determined from the gravity model, and its amplitude is modulated by the covariance matrix. In these respects, the adaptation technique may be characterized as a model-referenced method to distinguish it from other methods that rely instead on the statistical properties of the innovations sequence.[8]

Analysis of theoretical performance of a linear tracking filter demonstrates the importance of proper initialization of the covariance matrix (Sec. IV). Velocity errors can be corrected with the first position measurement when velocity errors are correlated to position errors. This situation occurs when the initial covariance matrix has nonzero off-diagonal terms, as compared to a different situation when the initial covariance is a diagonal matrix.

Performance of the nonlinear tracking filter is demonstrated by Monte Carlo simulation (Sec. V). The batch filter generates a covariance matrix with large correlations of position and velocity errors. This matrix is statistically consistent with the actual measurements. A comparison of probability distribution functions shows that the gravity-gradient noise matrix improves agreement of the recursive filter covariance with statistics of actual errors in the estimates.

II. Formulation

Several techniques for nonlinear estimation include minimum-variance estimation, estimation by statistical linearization, and batch least-squares estimation.[9–11] The minimum-variance technique is implemented because of its simplicity, from a computational standpoint, compared to the other two methods. The simplest form of the minimum-variance estimator is the extended Kalman filter (EKF). However, a batch filter is also recommended to generate an initial state estimate and covariance matrix for the recursive filter. Consequently, an algorithm suite is envisioned, consisting of a batch filter for initialization and a recursive filter for tracking and steering the radar beam (refer to Fig. 1).

Nonlinear Dynamics

Nonlinear batch and recursive filters are formulated in Earth-centered inertial (ECI) coordinates. For motion in a gravitational

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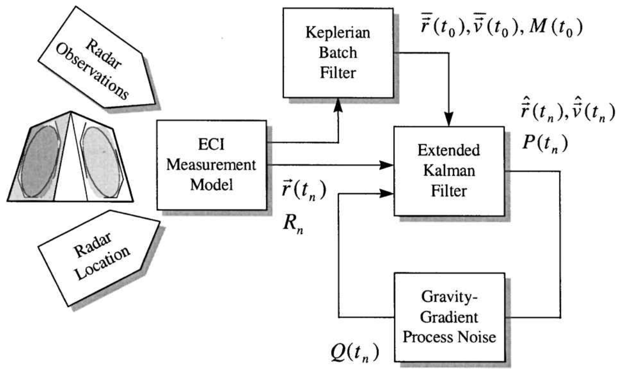

Fig. 1 Overview of the tracking process. A tracking filter for a phased-array radar is mechanized with a batch filter and a recursive filter. The batch filter generates an initial state estimate and covariance matrix for the recursive filter, at target acquisition. Subsequent estimates are updated sequentially by the recursive filter. The radar measurement model transforms both radar measurements and measurement noise matrix from radar-face coordinates to ECI coordinates. Recursive filter performance is stabilized with an adaptive gravity-gradient process-noise matrix.

field, a minimal representation for the target state vector includes the position and inertial velocity vectors in ECI coordinates:

$$\boldsymbol{x}_n = \begin{bmatrix} \boldsymbol{r}(t_n) \\ \boldsymbol{v}(t_n) \end{bmatrix}$$

Unlike earlier studies,[6,7] additional state variables are not required for the acceleration vector because the gravitational acceleration is completely specified by the position variables.

The time evolution of the state vector is described by a discrete, nonlinear model:

$$\boldsymbol{x}_{n+1} = \Phi(\boldsymbol{x}_n, t_{n+1} - t_n)\boldsymbol{x}_n$$

where the transition matrix depends on the state variable. In this paper, the effects of the gravitational field are approximated by a Keplerian transition matrix (see next section). Higher-order gravitational perturbations of the Earth, sun, and moon, and disturbances associated with atmospheric drag and solar radiation pressure, are neglected for simplicity.

Linear Measurements

Radar range and angle measurements are nonlinear functions of the target state vector in ECI coordinates:

$$z_n = \boldsymbol{h}(\boldsymbol{x}_n) + \zeta_n, \qquad D_n = E\left\{\zeta_n \zeta_n^T\right\}$$

In radar-centered coordinates, errors may be characterized as additive, independent, zero-mean Gaussian noise processes, having a diagonal noise matrix D_n. In the recursive update equation, this nonlinear measurement equation is adequately represented in the measurement residual term, but the filter gain is based on a linear measurement equation. This mismatch can adversely affect filter convergence because errors in the estimates tend to accumulate.[2]

Undesirable effects of measurement nonlinearities can be mitigated with a linear measurement equation in ECI coordinates:

$$\boldsymbol{y}_n = C\boldsymbol{x}_n + \boldsymbol{v}_n, \qquad C = [I_3 \quad O_3], \qquad R_n = E\left\{\boldsymbol{v}_n \boldsymbol{v}_n^T\right\}$$

(Another advantage is that this linearization technique is applicable to a sensor that measures a three-dimensional position vector in any coordinate system.) Measurement errors in ECI coordinates may be approximated by additive, zero-mean Gaussian noise processes, having a nondiagonal noise matrix R_n (see Appendix). Nonzero, off-diagonal terms are generated by the transformation of the diagonal noise matrix D_n from radar coordinates to ECI coordinates. To first-order of approximation in the error terms, R_n is consistent with the second-order moment of the ECI measurement errors. With this approach, the measurement residual term and the filter gains are consistent with a linear measurement equation.

Keplerian Batch Filter

The recursive filter must be initialized at target acquisition with prior estimates of the target state vector and covariance matrix. In the absence of prior information (e.g., from nonradar sensors such as satellites), these quantities must be generated by processing radar observations over a specified time interval. A Keplerian batch filter (KBF) generates a maximum-likelihood state estimate and covariance matrix at target acquisition:

$$M_0 = (H^T W^{-1} H)^{-1}, \qquad \hat{\boldsymbol{x}}_0 = M_0 H^T W^{-1} \boldsymbol{Y}$$

$$H = \begin{bmatrix} C\Phi(\bar{\boldsymbol{x}}_0, 0) \\ C\Phi(\bar{\boldsymbol{x}}_0, t_1 - t_0) \\ \vdots \\ C\Phi(\bar{\boldsymbol{x}}_0, t_N - t_0) \end{bmatrix}, \qquad W^{-1} = \begin{bmatrix} R_0^{-1} & O_3 & \cdots & O_3 \\ O_3 & R_1^{-1} & \cdots & O_3 \\ \vdots & \vdots & \vdots & \vdots \\ O_3 & O_3 & \cdots & R_N^{-1} \end{bmatrix}$$

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{y}_0 \\ \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_N \end{bmatrix}$$

Radar observations are processed during a certain time interval from acquisition at t_0 to some later time t_N. Elements of the H matrix reflect the propagation of the initial state estimate from acquisition to each of the observation times at $t_0, t_1, t_2, \ldots, t_N$. Although short time intervals are desirable, the duration $t_N - t_0$ must be large enough to generate a properly conditioned covariance matrix with significant correlations of position and velocity errors. [For example, at least 10 observations should be collected during a 10–20 s interval. However, much longer intervals (of 60 s or more) will result in smaller correlations of errors.]

The KBF is an iterative process because H contains the transition matrix, which depends on the state estimate. Consequently, a prior state estimate (but not a covariance matrix) is needed to start the KBF process. A prior state estimate may be generated with a batch-polynomial filter. Successive approximations of the KBF state estimate may be refined by repeating the KBF update several times, using the same measurements. For each iteration step, H is evaluated with the state estimate from the previous iteration step. KBF performance depends on the sample size $N + 1$ and the number of iterations for the batch estimate.

Extended Kalman Filter

The EKF is initialized with the KBF state estimate and covariance matrix at acquisition t_0. At later times, the EKF state estimate and covariance matrix are updated sequentially to reflect the effect of the measurement at time t_n:

$$K_n = M_n C^T \left(C M_n C^T + R_n\right)^{-1}, \qquad P_n = (I_6 - K_n C)M_n$$

$$\hat{\boldsymbol{x}}_n = \bar{\boldsymbol{x}}_n + K_n(\boldsymbol{y}_n - C\bar{\boldsymbol{x}}_n)$$

The prior state estimate is corrected with a weighted residual, or the difference between the measurement and the estimated measurement based on the prior state estimate. (An important limitation of this algorithm is that the prior covariance matrix is not updated using the actual statistics of the measurement residuals. Instead, these statistics are inferred from the filter's own models of the state-error covariance matrix and the measurement noise matrix. Although this is not an issue for a linear process, nonlinear effects can cause differences between actual and inferred statistics of the measurement residuals. This, in turn, can result in differences between the statistics of actual errors in the estimates and the filter covariance matrix, i.e., the latter is not a true representation of the former.) As noted, the measurement residual term and the filter gain K_n are consistent with a linear measurement equation. The state estimate and covariance matrix are propagated to the next update at t_{n+1}:

$$\bar{\boldsymbol{x}}_{n+1} = \hat{\Phi}_n \hat{\boldsymbol{x}}_n, \qquad M_{n+1} = \hat{\Phi}_n P_n \hat{\Phi}_n^T + Q_n$$

These calculations generate a prior state estimate and prior covariance matrix for the next update, and the recursive process is repeated.

The nonlinear transition matrix is evaluated with the most recent state estimate:

$$\hat{\Phi}_n \equiv \Phi(\hat{\boldsymbol{x}}_n, t_{n+1} - t_n)$$

An error in the transition matrix arises when the state estimate differs from the true state. This error must be properly compensated with a process-noise matrix Q_n in the equation for covariance propagation. The error in the state estimate may be expressed by

$$\delta\hat{\boldsymbol{x}}_n = \hat{\boldsymbol{x}}_n - \boldsymbol{x}_n$$

The propagation of the error state may be expressed by

$$\delta\hat{\boldsymbol{x}}_{n+1} = \hat{\boldsymbol{x}}_{n+1} - \boldsymbol{x}_{n+1} = \hat{\Phi}_n\hat{\boldsymbol{x}}_n - \Phi_n\boldsymbol{x}_n$$

After rearrangement of terms, the effect of the error in the transition matrix is clearly shown:

$$\delta\hat{\boldsymbol{x}}_{n+1} = \hat{\Phi}_n(\hat{\boldsymbol{x}}_n - \boldsymbol{x}_n) + (\hat{\Phi}_n - \Phi_n)\boldsymbol{x}_n = \hat{\Phi}_n\delta\hat{\boldsymbol{x}}_n + \delta\hat{\Phi}_n\boldsymbol{x}_n$$

To first order in the error terms, the true state (which is not known) may be replaced by the estimated state in the last term:

$$\delta\hat{\boldsymbol{x}}_{n+1} \cong \hat{\Phi}_n\delta\hat{\boldsymbol{x}}_n + \delta\hat{\Phi}_n\hat{\boldsymbol{x}}_n + \text{hot}$$

where the neglected higher-order terms are second order in the error terms. A matrix equation for the propagation of the covariance matrix may be determined by forming the expectation:

$$E\left\{\delta\hat{\boldsymbol{x}}_{n+1}\delta\hat{\boldsymbol{x}}_{n+1}^T\right\} = \hat{\Phi}_n E\left\{\delta\hat{\boldsymbol{x}}_n\delta\hat{\boldsymbol{x}}_n^T\right\}\hat{\Phi}_n^T + \hat{\Phi}_n E\left\{\delta\hat{\boldsymbol{x}}_n\hat{\boldsymbol{x}}_n^T\delta\hat{\Phi}_n^T\right\}$$
$$+ E\left\{\delta\hat{\Phi}_n\hat{\boldsymbol{x}}_n\delta\hat{\boldsymbol{x}}_n^T\right\}\hat{\Phi}_n^T + E\left\{\delta\hat{\Phi}_n\hat{\boldsymbol{x}}_n\hat{\boldsymbol{x}}_n^T\delta\hat{\Phi}_n^T\right\}$$

The first term corresponds to the linear model for the propagation of the covariance matrix, and the last three terms are included in the process-noise matrix:

$$Q_n = E\left\{\delta\hat{\Phi}_n\hat{\boldsymbol{x}}_n\hat{\boldsymbol{x}}_n^T\delta\hat{\Phi}_n^T\right\} + \hat{\Phi}_n E\left\{\delta\hat{\boldsymbol{x}}_n\hat{\boldsymbol{x}}_n^T\delta\hat{\Phi}_n^T\right\} + E\left\{\delta\hat{\Phi}_n\hat{\boldsymbol{x}}_n\delta\hat{\boldsymbol{x}}_n^T\right\}\hat{\Phi}_n^T$$

The several terms in this expression will next be determined analytically from the Keplerian transition matrix.

III. Gravity-Gradient Noise Matrix

In this formulation, orbital dynamics of the target are approximated by a Keplerian transition matrix. A process-noise matrix is included in the equation for the propagation of the covariance matrix to compensate for errors associated with the gravity-gradient effect, which arise when the position estimates differ from the true positions. In actual tracking applications, known perturbations to the inverse-square field would be compensated, and additional process noise would be needed to compensate for unknown disturbances associated with higher-order gravitational forces,[12] atmospheric drag, and other nonconservative forces.

Keplerian orbital elements may be determined from a current estimate of the state vector:

$$r_n = \|\boldsymbol{r}_n\|, \qquad v_n = \|\boldsymbol{v}_n\|$$

$$a_n = \frac{\mu r_n}{2\mu - r_n v_n^2}, \qquad c_n = \|\boldsymbol{r}_n \times \boldsymbol{v}_n\|, \qquad p_n = \frac{c_n^2}{\mu}$$

$$e_n = \sqrt{1 - \frac{p_n}{a_n}}, \qquad f_n = \cos^{-1}\left[\frac{p_n - r_n}{e_n r_n}\right]$$

where μ is the Earth's gravitational parameter. (In the interests of notational simplicity, state estimates are not explicitly identified as in the preceding section. Subscripts indicate that the Keplerian elements depend on the current state vector. These elements can change when the state estimate is updated by the EKF.) The true anomaly f_n is the central angle, measured from the perigee to the orbit position at t_n. The radial distance r_{n+1} may be determined from the equation of the orbit and the future true anomaly f_{n+1}:

$$r_{n+1} = \frac{p_n}{1 + e_n\cos(f_{n+1})}$$

$$f_{n+1} = 2\tan^{-1}\left[\sqrt{\frac{1+e_n}{1-e_n}}\tan\left(\frac{1}{2}E_{n+1}\right)\right]$$

where the eccentric anomaly E_{n+1} is specified implicitly by Kepler's equation:

$$E_{n+1} - e_n\sin(E_{n+1}) = M_n + \sqrt{\frac{\mu}{a_n^3}}(t_{n+1} - t_n)$$

$$M_n = E_n - e_n\sin(E_n)$$

$$E_n = 2\tan^{-1}\left[\sqrt{\frac{1-e_n}{1+e_n}}\tan\left(\frac{1}{2}f_n\right)\right]$$

For a Keplerian orbit, the transition matrix depends on the state vector and elapsed time:

$$\Phi(\boldsymbol{x}_n, t_{n+1} - t_n) = \begin{bmatrix} F_n I_3 & G_n I_3 \\ U_n I_3 & V_n I_3 \end{bmatrix}$$

The Lagrangian coefficients F_n, G_n, U_n, V_n depend on the change in true anomaly[13]:

$$F_n = 1 - \frac{r_{n+1}}{p_n}[1 - \cos(f_{n+1} - f_n)]$$

$$G_n = \frac{r_{n+1}r_n}{c_n}\sin(f_{n+1} - f_n)$$

$$U_n\left(= \frac{\mathrm{d}F_n}{\mathrm{d}t}\right) = \frac{1}{r_n}\left\{\frac{\boldsymbol{r}_n\cdot\boldsymbol{v}_n}{p_n}[1 - \cos(f_{n+1} - f_n)] - \sqrt{\frac{\mu}{p_n}}\sin(f_{n+1} - f_n)\right\}$$

$$V_n\left(= \frac{\mathrm{d}G_n}{\mathrm{d}t}\right) = 1 - \frac{r_n}{p_n}[1 - \cos(f_{n+1} - f_n)]$$

These scalar functions also depend on the Keplerian elements, which are nonlinear functions of the state vector. Kepler's equation must be solved (iteratively) to determine the change in the true anomaly from the elapsed time.

When the state estimate differs from the true state, the resulting errors in the transition matrix may be expressed as errors in the Lagrangian coefficients:

$$\delta\Phi_n = \begin{bmatrix} \delta F_n I_3 & \delta G_n I_3 \\ \delta U_n I_3 & \delta V_n I_3 \end{bmatrix}$$

Because of the special structure of this matrix, the process-noise matrix takes the following form:

$$Q_n = \Lambda_n + \Phi_n\Omega_n^T + \Omega_n\Phi_n^T$$

$$\Lambda_n = \begin{bmatrix} E\left(\delta F_n^2\right)\boldsymbol{r}_n\boldsymbol{r}_n^T + E(\delta F_n\delta G_n)\left(\boldsymbol{r}_n\boldsymbol{v}_n^T + \boldsymbol{v}_n\boldsymbol{r}_n^T\right) + E\left(\delta G_n^2\right)\boldsymbol{v}_n\boldsymbol{v}_n^T & E(\delta F_n\delta U_n)\boldsymbol{r}_n\boldsymbol{r}_n^T + E(\delta F_n\delta V_n)\boldsymbol{r}_n\boldsymbol{v}_n^T + E(\delta G_n\delta U_n)\boldsymbol{v}_n\boldsymbol{r}_n^T + E(\delta G_n\delta V_n)\boldsymbol{v}_n\boldsymbol{v}_n^T \\ E(\delta F_n\delta U_n)\boldsymbol{r}_n\boldsymbol{r}_n^T + E(\delta G_n\delta U_n)\boldsymbol{r}_n\boldsymbol{v}_n^T + E(\delta F_n\delta V_n)\boldsymbol{v}_n\boldsymbol{r}_n^T + E(\delta G_n\delta V_n)\boldsymbol{v}_n\boldsymbol{v}_n^T & E\left(\delta U_n^2\right)\boldsymbol{r}_n\boldsymbol{r}_n^T + E(\delta U_n\delta V_n)\left(\boldsymbol{r}_n\boldsymbol{v}_n^T + \boldsymbol{v}_n\boldsymbol{r}_n^T\right) + E\left(\delta V_n^2\right)\boldsymbol{v}_n\boldsymbol{v}_n^T \end{bmatrix}$$

$$\Omega_n = \begin{bmatrix} \boldsymbol{r}_n E\left(\delta F_n\delta\boldsymbol{r}_n^T\right) + \boldsymbol{v}_n E\left(\delta G_n\delta\boldsymbol{r}_n^T\right) & \boldsymbol{r}_n E\left(\delta F_n\delta\boldsymbol{v}_n^T\right) + \boldsymbol{v}_n E\left(\delta G_n\delta\boldsymbol{v}_n^T\right) \\ \boldsymbol{r}_n E\left(\delta U_n\delta\boldsymbol{r}_n^T\right) + \boldsymbol{v}_n E\left(\delta V_n\delta\boldsymbol{r}_n^T\right) & \boldsymbol{r}_n E\left(\delta U_n\delta\boldsymbol{v}_n^T\right) + \boldsymbol{v}_n E\left(\delta V_n\delta\boldsymbol{v}_n^T\right) \end{bmatrix}$$

Expectations involving the Lagrangian coefficients may be expressed in terms of the elements of the covariance matrix.

An approximate analytic expression for the process-noise matrix may be derived after simplification of the Lagrangian coefficients. For short intervals of time, the Lagrangian coefficients may be approximated by truncated series expansions[14] in the elapsed time:

$$F_n \cong 1 - \tfrac{1}{2}\alpha_n T_n^2 + \tfrac{1}{2}\alpha_n\beta_n T_n^3 + \tfrac{1}{8}\alpha_n\left(\gamma_n - 5\beta_n^2 - \tfrac{2}{3}\alpha_n\right)T_n^4$$

$$G_n \cong T_n - \tfrac{1}{6}\alpha_n T_n^3 + \tfrac{1}{4}\alpha_n\beta_n T_n^4$$

$$U_n \cong -\alpha_n T_n + \tfrac{3}{2}\alpha_n\beta_n T_n^2 + \tfrac{1}{2}\alpha_n\left(\gamma_n - 5\beta_n^2 - \tfrac{2}{3}\alpha_n\right)T_n^3$$

$$V_n \cong 1 - \tfrac{1}{2}\alpha_n T_n^2 + \alpha_n\beta_n T_n^3$$

$$\alpha_n \equiv \frac{\mu}{r_n^3}, \qquad \beta_n \equiv \frac{1}{r_n^2}(\boldsymbol{r}_n \cdot \boldsymbol{v}_n)$$

$$\gamma_n \equiv \frac{1}{r_n^2}(\boldsymbol{v}_n \cdot \boldsymbol{v}_n), \qquad T_n \equiv t_{n+1} - t_n$$

(The iteration associated with the solution of Kepler's equation may be avoided by expressing the Lagrangian coefficients by functions of elapsed time, rather than the change in true anomaly. However, this representation is accurate over short intervals of time, e.g., 1 s or less.) It follows that the errors in the coefficients may be expressed by

$$\begin{bmatrix} \delta F_n \\ \delta G_n \\ \delta U_n \\ \delta V_n \end{bmatrix} = B_n \begin{bmatrix} \delta\alpha_n \\ \delta\beta_n \\ \delta\gamma_n \end{bmatrix}, \qquad B_n = \begin{bmatrix} \tfrac{1}{2}T_n^2\left[-1 + \beta_n T_n + \tfrac{1}{4}\left(\gamma_n - 5\beta_n^2 - \tfrac{4}{3}\alpha_n\right)T_n^2\right] & \tfrac{1}{2}\alpha_n T_n^3\left(1 - \tfrac{5}{2}\beta_n T_n\right) & \tfrac{1}{8}\alpha_n T_n^4 \\ \tfrac{1}{6}T_n^3\left(-1 + \tfrac{3}{2}\beta_n T_n\right) & \tfrac{1}{4}\alpha_n T_n^4 & 0 \\ T_n\left[-1 + \tfrac{3}{2}\beta_n T_n + \tfrac{1}{2}\left(\gamma_n - 5\beta_n^2 - \tfrac{4}{3}\alpha_n\right)T_n^2\right] & \tfrac{3}{2}\alpha_n T_n^2\left(1 - \tfrac{10}{3}\beta_n T_n\right) & \tfrac{1}{2}\alpha_n T_n^3 \\ \tfrac{1}{2}T_n^2(-1 + 2\beta_n T_n) & \alpha_n T_n^3 & 0 \end{bmatrix}$$

Elements of the matrices Λ and Ω may be determined from the following identities:

$$\begin{bmatrix} E\left(\delta F_n^2\right) & E(\delta F_n \delta G_n) & E(\delta F_n \delta U_n) & E(\delta F_n \delta V_n) \\ & E\left(\delta G_n^2\right) & E(\delta G_n \delta U_n) & E(\delta G_n \delta V_n) \\ & & E\left(\delta U_n^2\right) & E(\delta U_n \delta V_n) \\ & & & E\left(\delta V_n^2\right) \end{bmatrix}$$

$$= B_n \begin{bmatrix} E\left(\delta\alpha_n^2\right) & E(\delta\alpha_n\delta\beta_n) & E(\delta\alpha_n\delta\gamma_n) \\ & E\left(\delta\beta_n^2\right) & E(\delta\beta_n\delta\gamma_n) \\ & & E\left(\delta\gamma_n^2\right) \end{bmatrix} B_n^T$$

$$E\left\{\begin{bmatrix} \delta F_n \\ \delta G_n \\ \delta U_n \\ \delta V_n \end{bmatrix} \delta\boldsymbol{r}_n^T\right\} = B_n E\left\{\begin{bmatrix} \delta\alpha_n \\ \delta\beta_n \\ \delta\gamma_n \end{bmatrix} \delta\boldsymbol{r}_n^T\right\}$$

$$E\left\{\begin{bmatrix} \delta F_n \\ \delta G_n \\ \delta U_n \\ \delta V_n \end{bmatrix} \delta\boldsymbol{v}_n^T\right\} = B_n E\left\{\begin{bmatrix} \delta\alpha_n \\ \delta\beta_n \\ \delta\gamma_n \end{bmatrix} \delta\boldsymbol{v}_n^T\right\}$$

To evaluate the several expectations in these identities, it is necessary to express the errors in α_n, β_n, γ_n as functions of errors in the state variables:

$$\delta\alpha_n = -\left(3\mu/r_n^4\right)\delta r_n$$

$$\delta\beta_n = \left(1/r_n^2\right)(\delta\boldsymbol{r}_n \cdot \boldsymbol{v}_n + \boldsymbol{r}_n \cdot \delta\boldsymbol{v}_n) - (2\beta_n/r_n)\delta r_n$$

$$\delta\gamma_n = \left(2/r_n^2\right)(\boldsymbol{v}_n \cdot \delta\boldsymbol{v}_n) - (2\gamma_n/r_n)\delta r_n$$

Following substitution of these identities, it may be shown that

$$E\left(\delta\alpha_n^2\right) = \left(3\mu/r_n^5\right)^2 \boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n$$

$$E(\delta\alpha_n\delta\beta_n) = \left(3\mu/r_n^7\right)\left[2\beta_n\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n - \boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{v}_n - \boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n\right]$$

$$E(\delta\alpha_n\delta\gamma_n) = \left(6\mu/r_n^7\right)\left[\gamma_n\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n - \boldsymbol{v}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n\right]$$

$$E\left(\delta\beta_n^2\right) = -\left(1/r_n^4\right)\left[\boldsymbol{v}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{v}_n + 2\boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{v}_n + \boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{v}_n^T\right)\boldsymbol{r}_n + 4\beta_n\left[\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{v}_n + \boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n\right] - (2\beta_n)^2\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n\right]$$

$$E(\delta\beta_n\delta\gamma_n) = \left(2/r_n^4\right)\left[\boldsymbol{v}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{v}_n + \boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{v}_n^T\right)\boldsymbol{v}_n - 2\beta_n\boldsymbol{v}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n - \gamma_n\left[\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{v}_n + \boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n\right] + 2\beta_n\gamma_n\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n\right]$$

$$E\left(\delta\gamma_n^2\right) = \left(4/r_n^4\right)\left[\boldsymbol{v}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{v}_n^T\right)\boldsymbol{v}_n - 2\gamma_n\boldsymbol{v}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n + \gamma_n^2\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\boldsymbol{r}_n\right]$$

$$E\left(\delta\alpha_n\delta\boldsymbol{r}_n^T\right) = -\left(3\mu/r_n^5\right)\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)$$

$$E\left(\delta\beta_n\delta\boldsymbol{r}_n^T\right) = \left(1/r_n^2\right)\left[\boldsymbol{v}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right) + \boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right) - 2\beta_n\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\right]$$

$$E\left(\delta\gamma_n\delta\boldsymbol{r}_n^T\right) = \left(2/r_n^2\right)\left[\boldsymbol{v}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{r}_n^T\right) - \gamma_n\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{r}_n^T\right)\right]$$

$$E\left(\delta\alpha_n\delta\boldsymbol{v}_n^T\right) = -\left(3\mu/r_n^5\right)\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{v}_n^T\right)$$

$$E\left(\delta\beta_n\delta\boldsymbol{v}_n^T\right) = \left(1/r_n^2\right)\left[\boldsymbol{v}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{v}_n^T\right) + \boldsymbol{r}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{v}_n^T\right) - 2\beta_n\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{v}_n^T\right)\right]$$

$$E\left(\delta\gamma_n\delta\boldsymbol{v}_n^T\right) = \left(2/r_n^2\right)\left[\boldsymbol{v}_n^T E\left(\delta\boldsymbol{v}_n\delta\boldsymbol{v}_n^T\right) - \gamma_n\boldsymbol{r}_n^T E\left(\delta\boldsymbol{r}_n\delta\boldsymbol{v}_n^T\right)\right]$$

The process-noise matrix Q_n compensates for the gravity-gradient effect that arises when the state estimate differs from the true state (which is unknown). Consequently, Q_n depends on the error covariance matrix P_n, which is a measure of such differences. It follows that Q_n should approach zero as the state estimate converges to the true state, or when the covariance matrix is identically zero. Referring to the preceding expectations, it is clear that all elements of Q_n depend on the elements of P_n, and it follows that

$$Q_n \Rightarrow 0 \qquad \text{as} \qquad P_n \Rightarrow 0$$

IV. Convergence of Velocity Errors

Performance of a linear tracking filter determines the best performance that can be expected from the nonlinear filter. It is shown that proper initialization of the covariance matrix affects the time required to achieve a specified accuracy in the velocity estimates. In particular, the correlations of position and velocity errors at filter initialization have a very important role.

For simplicity, a two-state, linear kinematic model is analyzed. The state variables are position x and velocity u, gravity is neglected, and the radar measures position with accuracy r. With these simplifications, the covariance matrix before a radar update, the covariance matrix after an update, and the measurement noise matrix are given by

$$M = \begin{bmatrix} M_{xx} & M_{xu} \\ M_{xu} & M_{uu} \end{bmatrix}, \qquad P = \begin{bmatrix} P_{xx} & P_{xu} \\ P_{xu} & P_{uu} \end{bmatrix}, \qquad R = r$$

After the radar update, it may be shown that

$$P_{xx} = \frac{r M_{xx}}{M_{xx} + r}, \qquad P_{xu} = \frac{r M_{xu}}{M_{xx} + r}, \qquad P_{uu} = M_{uu} - \frac{(M_{xu})^2}{M_{xx} + r}$$

Without prior information at target acquisition, the accuracy of the position measurement is usually better than the uncertainty in the prior position estimate:

$$r \ll M_{xx}: \qquad P_{xx} \cong r, \qquad P_{xu} \cong \frac{r M_{xu}}{M_{xx}}, \qquad P_{uu} \cong M_{uu} - \frac{(M_{xu})^2}{M_{xx}}$$

The first update reduces the target position uncertainty to the radar measurement accuracy. The second and third equations may be expressed in terms of the prior correlation coefficient ρ:

$$\rho \equiv \frac{M_{xu}}{\sqrt{M_{xx} M_{uu}}}: \qquad P_{xu} \cong r\rho\sqrt{\frac{M_{uu}}{M_{xx}}}, \qquad P_{uu} \cong M_{uu}(1 - \rho^2)$$

When the filter is initialized with a diagonal covariance matrix ($\rho = 0$), the first update does not correct velocity errors. Although residual velocity errors cause position errors that are corrected on subsequent updates, the convergence process is slower compared to the situation when initial position and velocity errors are correlated (Fig. 2).

Velocity covariance decreases at a rate that depends on the initial correlation of errors and the radar measurement accuracy (Fig. 2). Covariance propagation (between updates) increases correlation of position and velocity errors, whereas filter updates reduce correlations. The interaction of these effects determines the duration of the convergence process. Faster convergence occurs for correlations greater than 0.99 (in absolute value) and as the radar measurement accuracy improves. These results represent the best performance that can be achieved because nonlinear effects have been omitted in the two-state linear filter.

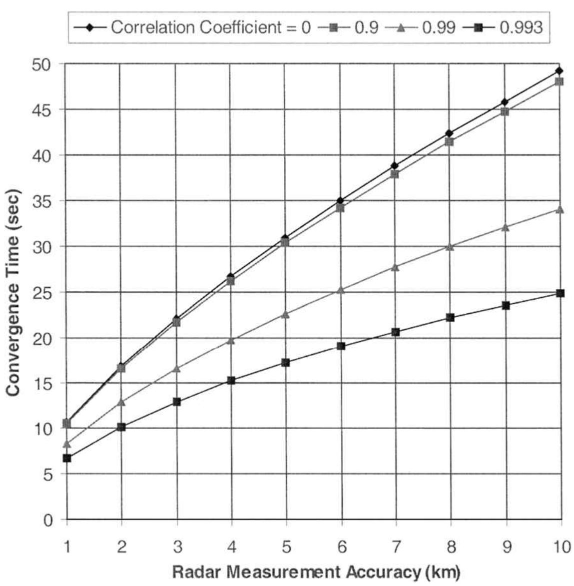

Fig. 2 Effects of tracking accuracy and filter initialization on filter convergence time. The time interval required to reduce initial velocity errors by an order of magnitude (from 1000 to 100 m/s) is determined for a linear, two-state filter. Convergence times decrease as radar tracking accuracy improves, and with increasing correlation of position and velocity errors at filter initialization. The most significant improvements occur when correlation coefficients exceed 0.99 (in absolute value).

V. Fidelity of the EKF Covariance Matrix

The EKF covariance matrix is the lowest-order approximation of the statistical properties of the actual errors in the estimates. In the formulation of the EKF, nonlinear expectations are approximated by the lowest-order terms of a series expansion, evaluated with the current estimate.[10] As nonlinear effects can generate non-Gaussian statistics for the state variables, the EKF covariance matrix may not adequately characterize EKF performance.

The most reliable evaluation of EKF performance is determined by Monte Carlo simulations of the EKF and its nonlinear truth model. For this assessment a phased-array radar tracks a spherical target in a circular, inclined, low-Earth Keplerian orbit. The Keplerian model assures that the actual dynamics are consistent with the filter formulation. It was beyond the scope of this study to include higher-order gravitational perturbations and atmospheric drag in the evaluation.

The measurement accuracy of the radar is characterized by a theoretical model that specifies rms errors in range and angular variables as functions of the range resolution of the radar, radar bandwidth and beamwidth, and the signal-to-noise ratio of the radar return.[15] For example, range and angular variances are inversely proportional to the signal-to-noise ratio. Signal-to-noise ratio is modulated by the range from the tracking station to the target, and the target cross section. For simplicity, a spherical target with constant cross section (1 m^2) is assumed.

During the observation interval, rms measurement accuracies of the radar can exhibit significant variations in amplitude, particularly in the two directions perpendicular to the radar line of sight. These variations must be carefully modeled to weight the radar measurements in the batch and recursive tracking filters properly. Consequently, the instantaneous signal-to-noise ratio must be monitored by the radar itself, and this information is used to modulate the measurement noise matrix using the forementioned theoretical model.

The EKF is initialized with KBF estimates of the target state vector and covariance matrix. (Statistical consistency of the a priori covariance matrix with the measurements is ensured by processing data with the KBF. Alternatively, an initial covariance matrix could be specified with unity correlation coefficients, but this approach might result in an ill-conditioned matrix.) An ECI state estimate is generated after three KBF iterations using the same set of measurements collected over 15 s. The corresponding ECI covariance matrix has large principal correlations of position and velocity errors (shown in bold type):

$$\begin{array}{cccccc}
\sigma_{xx} & -0.844 & 0.854 & \mathbf{-0.998} & 0.981 & -0.980 \\
 & \sigma_{yy} & -0.999 & 0.809 & \mathbf{-0.929} & 0.931 \\
 & & \sigma_{zz} & -0.820 & 0.936 & \mathbf{-0.937} \\
 & & & \sigma_{uu} & -0.968 & 0.967 \\
 & & & & \sigma_{vv} & -0.999 \\
 & & & & & \sigma_{ww}
\end{array}$$

Earlier results (refer to Fig. 2) have demonstrated that velocity errors converge more quickly when the initial covariance matrix contains large correlations of position and velocity errors.

The fidelity of the EKF covariance matrix was determined for an ensemble of 100 Monte Carlo simulations. A probability distribution function (or PDF) was constructed using actual errors in the velocity estimates at a specified time after acquisition, without process noise (see Fig. 3) and with gravity-gradient noise (see Fig. 4). In each case, the actual PDF may be compared with the Gaussian PDF, which is determined from the eigenvalues of the EKF velocity covariance. Without process noise, the EKF covariance matrix underestimates actual errors at large PDF values (e.g., above 50%). With gravity-

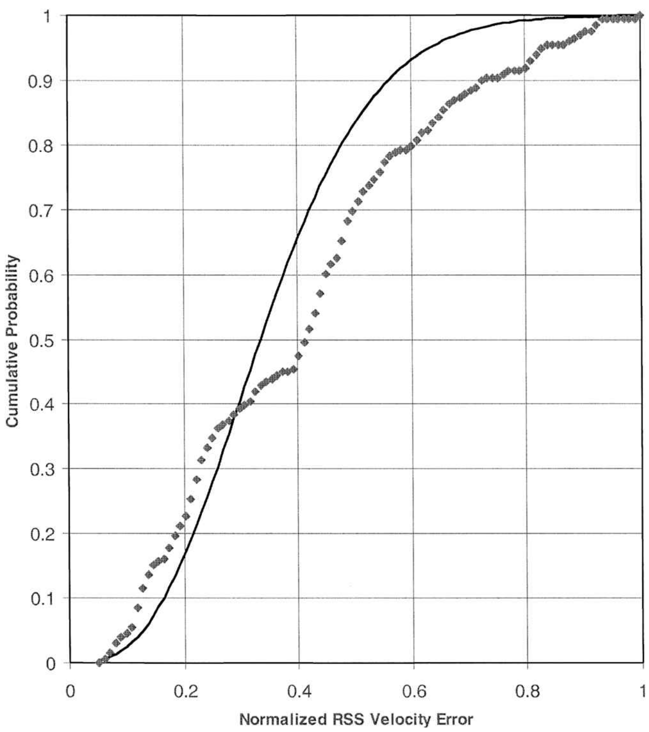

Fig. 3 Probability distribution functions for velocity errors, without process noise. Probability distribution functions are determined for the actual distribution of velocity errors and theoretical Gaussian distribution of errors based on the filter covariance matrix. The solid line reflects the rss velocity error determined from the eigenvalues of the filter covariance matrix, at a specified time (600 s) after target acquisition. The dots represent the rss error determined by comparing the true velocity to the filter estimate. Accuracy results are normalized by the maximum velocity error observed.

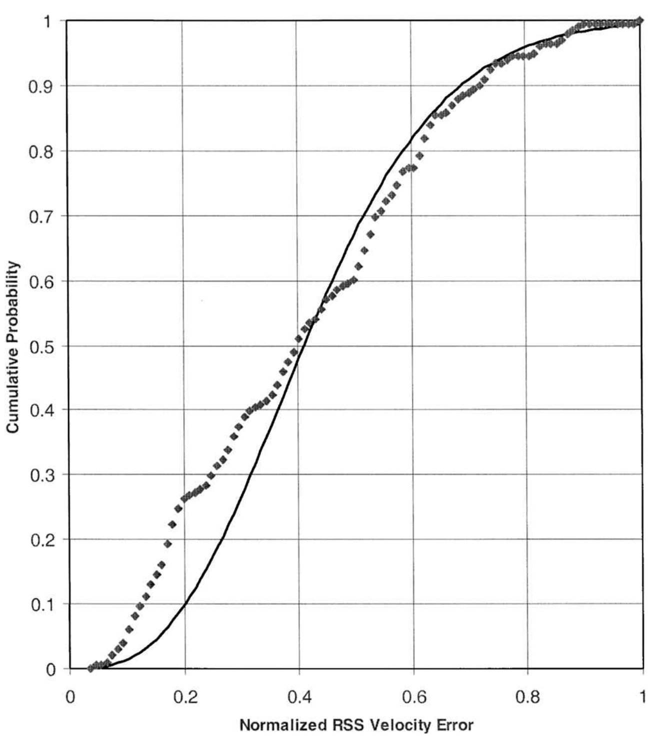

Fig. 4 Probability distribution functions for velocity errors, with process noise. Probability distribution functions are determined for the actual distribution of velocity errors and theoretical Gaussian distribution of errors based on the filter covariance matrix. The solid line reflects the rss velocity error determined from the eigenvalues of the filter covariance matrix, at a specified time (600 s) after target acquisition. The dots represent the rss error determined by comparing the true velocity to the filter estimate. Accuracy results are normalized by the maximum velocity error observed.

gradient noise, this problem is corrected, and the Gaussian PDF is conservative (because it overestimates the true errors) at smaller PDF values. (Based on other results not reported here, covariance fidelity is sensitive to the radar measurement accuracy, which can fluctuate. As radar measurement accuracy improves, the EKF covariance matrix is conservative because it tends to overestimate the statistics of actual errors in the estimate.)

VI. Conclusions

Performance of nonlinear recursive tracking filters can be improved using batch initialization and process-noise adaptation. These techniques were applied to improve radar tracking of a nonmaneuvering target during orbital flight. Simulation results demonstrated an improvement in the convergence of velocity errors during the filter transient response and improved agreement of the filter covariance with actual errors in the estimates.

After position measurements are collected during a short interval following target acquisition, a Keplerian batch filter generates a state estimate and a covariance matrix with significant, nonzero correlations of position and velocity errors. Highly correlated errors allow the recursive filter to correct velocity errors using the first position measurement. Consequently, this initialization technique is strongly recommended to accelerate the convergence of velocity errors compared to a more conventional initialization method that uses a diagonal covariance matrix.

In many nonlinear applications, the covariance matrix of a recursive filter is not consistent with the statistics of the actual errors in the state estimates obtained by Monte Carlo simulation. It was shown that covariance fidelity may be improved using a model-referenced technique for adaptation of the process-noise matrix. This matrix compensates for errors in the nonlinear transition matrix that arise when the state estimate differs from the true state. The noise matrix is "tuned" with an adaptive procedure that uses the most recent state estimate, covariance matrix, and gravity-gradient model for an inverse-square gravitational field. As the state estimate becomes more accurate, all elements of the process-noise matrix eventually approach zero with the covariance matrix.

Appendix: Radar Measurement Model

A phased-array radar measures the spherical coordinates of the target position vector relative to the tracking station.[1] These coordinates are the slant range distance S from tracking station to the target, and the direction-sine (or u, v) components of the unit line-of-sight vector on the radar face. Two coordinate axes are locally horizontal and downward, respectively, and the third axis (or w coordinate) is along the array boresight (or perpendicular to the face, pointing outward).

The target's Suv coordinates are transformed to measurements of its geocentric position vector in ECI coordinates:

$$\boldsymbol{r} = \boldsymbol{r}_s + \boldsymbol{s}, \qquad \boldsymbol{r}_s = \begin{bmatrix} R_s \cos L_s \cos \alpha_s \\ R_s \cos L_s \sin \alpha_s \\ R_s \sin L_s \end{bmatrix}$$

$$\boldsymbol{s} = \Re^T(L_s, \alpha_s, \gamma_f, \psi_f) S \boldsymbol{\lambda}, \qquad \boldsymbol{\lambda} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

where the location of the tracking station is specified by its geocentric radius R_s, geocentric latitude L_s, and inertial right ascension α_s. The rotation matrix from the ECI frame to the Suv frame may be constructed by the following sequence of rotations:

$$\Re(L_s, \alpha_s, \gamma_f, \psi_f) = \Re_1(\gamma_f)\Re_2(\psi_f)\Re_3(\pi/2)\Re_2(-L_s)\Re_3(\alpha_s)$$

where γ_f, ψ_f denote the elevation and azimuth of the array boresight. In this expression, subscripts on each rotation matrix identify the axis of rotation (1, 2, or 3), and the signed arguments identify the rotation angle and sense of the rotation (positive or negative).

Measurement errors may be modeled by additive, zero-mean, Gaussian random variables in Suv coordinates. (In the interests of simplicity, the parameters specifying the location of the tracking

station and the orientation of the array boresight are assumed to be error free. However, the analysis may be generalized easily to account for errors in these quantities.) As the ECI position measurement is a nonlinear function of these variables, the corresponding error in the ECI pseudomeasurement vector may be approximated by:

$$\delta \boldsymbol{s} \cong \Re^T (S\delta\boldsymbol{\lambda} + \delta S\boldsymbol{\lambda}) + \text{ hot}$$

The measurement-noise matrix in ECI coordinates may be determined by forming the expectation:

$$R = E\{\delta\boldsymbol{s}\delta\boldsymbol{s}^T\} = \Re^T[S^2\Theta + S(\Xi\boldsymbol{\lambda}^T + \boldsymbol{\lambda}\Xi^T) + E(\delta S^2)\boldsymbol{\lambda}\boldsymbol{\lambda}^T]\Re$$

$$\Theta = E\{\delta\boldsymbol{\lambda}\delta\boldsymbol{\lambda}^T\} = \begin{bmatrix} E(\delta u^2) & E(\delta u\delta v) & E(\delta u\delta w) \\ E(\delta u\delta v) & E(\delta v^2) & E(\delta v\delta w) \\ E(\delta u\delta w) & E(\delta v\delta w) & E(\delta w^2) \end{bmatrix}$$

$$\Xi = E\{\delta S\delta\boldsymbol{\lambda}^T\} = [E(\delta S\delta u) \quad E(\delta S\delta w) \quad E(\delta S\delta w)]$$

In calculating these expectations, it is important that errors in the third radar coordinate w are not independent of the u, v errors because of the following definition of the line-of-sight unit vector:

$$u^2 + v^2 + w^2 = 1$$

Consequently, errors in w are correlated to errors in u, v:

$$\delta w = -(1/w)(u\delta u + v\delta v)$$

Singularities are precluded by the physical constraint that $w > 0$. It follows that

$$E(\delta S\delta w) = -(u/w)E(\delta S\delta u) - (v/w)E(\delta S\delta v)$$

$$E(\delta u\delta w) = -(u/w)E(\delta u^2) - (v/w)E(\delta u\delta v)$$

$$E(\delta v\delta w) = -(u/w)E(\delta u\delta v) - (v/w)E(\delta v^2)$$

$$E(\delta w^2) = (u/w)^2E(\delta u^2) + (2uv/w^2)E(\delta u\delta v) + (v/w)^2E(\delta v^2)$$

The foregoing expectations may be evaluated using a specified set of statistics for the measurement errors and with estimates of the S, u, v coordinates of the target. The latter quantities may be determined from the most recent measurements, rather than from the state vector, to avoid mathematical problems associated with nonlinear stochastic systems.[16] When measurement errors are uncorrelated in Suv coordinates, the ECI measurement noise matrix may be shown to have the form:

$$R = \Re^T\left[(S\sigma_{uv})^2\Psi + \sigma_s^2\boldsymbol{\lambda}\boldsymbol{\lambda}^T\right]\Re, \qquad \boldsymbol{\lambda} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1 & 0 & -u/w \\ 0 & 1 & -v/w \\ -u/w & -v/w & -1 \end{bmatrix}$$

Generally, the standard deviation σ_s of range measurement errors is much smaller (by at least three orders of magnitude) than the product $S\sigma_{uv}$ associated with angular measurement errors. Moreover, measurement variances usually depend on bandwidth and beamwidth of the radar and the signal-to-noise ratio of the radar return.[15]

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