Pseudolinear and State-Dependent Riccati Equation Filters for Angular Rate Estimation

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I. Introduction

THERE has been a recent trend toward the use of small inexpensive satellites for Earth and space exploration. In view of this, it is desired to eliminate the use of expensive gyroscopes onboard. There is, however, still a need to know the satellites' angular rate vector, especially for attitude control. Whereas the attitude determination task requires high-precision angular rate measurements, low-precision angular rate measurements are adequate for control loop damping. Similar necessity arises even in gyroscope equipped satellites when performing high rate maneuvers whose angular rate is outside of the range of the onboard gyroscopes.

Satellites usually utilize vector measurements for attitude determination. By the use of these vectors, it is possible to determine the attitude of the satellite where the attitude can be expressed by various attitude parameterizations. To find the satellite angular rate, one can differentiate the attitude in whatever parameters it is given and use the kinematics equation that relates the derivative of the attitude with the satellite angular rate and compute the latter.[1] In fact, this approach is being used in the solar, anomalous, and magnetospheric particle explorer (SAMPEX) satellite.[2] Another approach may also be adopted to the problem of angular rate computation where the vector measurements themselves are differentiated.[3–6] Because the measured vectors and, thus, the attitude parameters are noisy, the differentiation of the attitude introduces a considerable noise component in the computed angular rate vector. To overcome this noise, the computed rate components can be filtered by a low-pass filter of the form $G(s) = (1/\tau)/(s + 1/\tau)$. This, however, introduces a delay in the computed rate. An example of it is shown in Fig. 1, which presents the angular rate components of the Rossi x-ray timing explorer satellite (RXTE) that were computed and filtered when the algorithm just mentioned[1] was applied. (The RXTE satellite was launched on 30 December 1995. The data used were real magnetometer measurements, whose error standard deviation was 1 mG, and sun sensor measurements, whose error standard deviation was 0.1 deg. We chose a segment of data starting 4 January 1996 at 21 h, 51 min, and 43.148 s. At this time the spacecraft performed a slew maneuver about its z axis.) As will be discussed in the ensuing, the delay can be eliminated when using an active filter such as a Kalman filter. Note that the reference rates are gyroscope measurements, and because these measurements are very accurate, we call their output *true* angular velocity components.

As will be shown, the dynamics model of a spacecraft (SC) is a nonlinear model; therefore, a linear Kalman filter (KF) is not suitable, and some kind of a nonlinear estimator is needed for estimating ω. The extended KF (EKF) is, then, the natural choice.[7] The EKF is based on linearization about estimated values; therefore, one may expect convergence problems when the estimation error happens to be large. Consequently, other filtering approaches were sought. One such approach is that of the extended interlaced KF (EIKF),[6] mentioned before, where three linear KFs are run in parallel.

In this work we further investigate two filtering methods for estimating ω, where the nonlinear problem is treated as a linear filtering problem. Although these methods use derivatives of the vector measurements (similar to Refs. 3–6), the filters are unique. They are based on the ability to decompose the nonlinear angular rate dependent part of the satellite dynamics equation into a product of an angular rate dependent matrix and the angular rate vector itself. They are the pseudolinear Kalman (PSELIKA) and the continuous discrete (CD) state-dependent algebraic Riccati equation (SDARE) filters.

II. Angular Rate Measurement

Let $\boldsymbol{b}_n$ denote a column vector, whose elements are the components of some nth abstract vector $\boldsymbol{a}_n$ when resolved in the body b coordinate system, and let $\boldsymbol{r}_n$ be a column vector, whose elements are the components of the same abstract vector when resolved in the reference i system. From vector kinematics it is known that

$$\dot{\boldsymbol{b}}_n - D_b^i \dot{\boldsymbol{r}}_n = [\boldsymbol{b}_n \times]\boldsymbol{\omega} \tag{1}$$

where the dot denotes a simple time derivative, $[\boldsymbol{b}_n \times]$ is the cross product skew-symmetric matrix of $\boldsymbol{b}_n$, and D_b^i is the matrix that transforms vectors from the reference to the body coordinates. Note that D_b^i has to be known and that all elements of Eq. (1) other than the satellite angular rate vector $\boldsymbol{\omega}$ are known. Define

$$z_n = \dot{\boldsymbol{b}}_n - D_b^i \dot{\boldsymbol{r}}_n \tag{2}$$

and

$$C_n = \begin{bmatrix} 0 & -b_{nz} & b_{ny} \\ b_{nz} & 0 & -b_{nx} \\ -b_{ny} & b_{nx} & 0 \end{bmatrix} \tag{3}$$

Then Eq. (1) can be written as

$$z_n = C_n \boldsymbol{\omega} \tag{4}$$

We realize that $\boldsymbol{\omega}$ cannot be determined from Eq. (4) because $C_n = [\boldsymbol{b}_n \times]$ is not invertible. We may, however, use z_n as an input to an estimator. The subscript n in Eq. (4) denotes the order of the measured vector. We use the subscript k to denote a discrete time

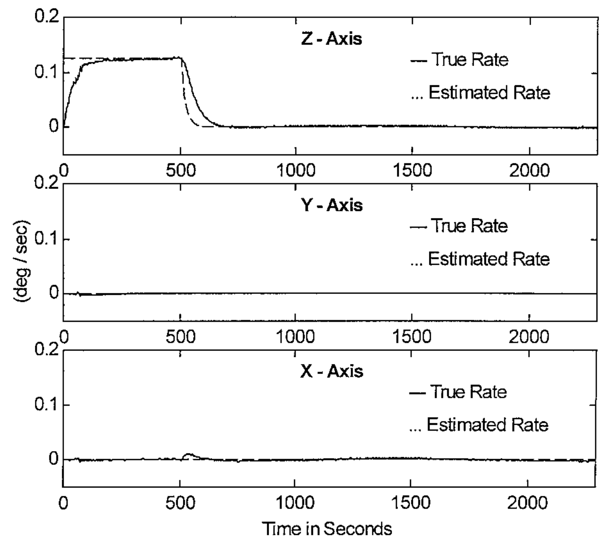

Fig. 1 True and estimated filtered RXTE angular rates components obtained from Direction Cosine Matrix (DCM) differentiation and filtration with τ = 40 s.

Received 19 May 1998; revision received 29 March 1999; accepted for publication 7 April 1999.

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t_k, where the estimator performs a measurement update; thus, if at time t_k we use two vector measurements to update the filter, then we actually have $z_{k,1}$ and $z_{k,2}$. For simplicity of notation we drop the second subscript, noting that at each time point t_k we may have more than one update or no update at all. Also, to use z as a measurement in an estimator, we need to add some white noise to the measurements to account for measurement noise. Therefore, instead of Eq. (4) we use

$$z_k = C_k \omega_k + v_k \tag{5}$$

III. Pseudolinear SC Dynamics Model

To implement the ordinary PSELIKA and SDARE filter, we need to convert the SC nonlinear dynamics model that describes the propagation of ω into a model in which the nonlinear ω-dependent term is decomposed into a product of an ω-dependent matrix and ω itself. To do this, consider the angular dynamics of a constant mass SC, which is given in the following equation[8]:

$$I\dot{\omega} + \dot{h} + \omega \times (I\omega + h) = T \tag{6}$$

where I is the moment of inertia matrix, ω is the angular velocity of the SC with respect to inertial space, h is the angular momentum of the reaction wheels, and T is the external torque applied to the SC. Because I is nonsingular, we may write Eq. (6) as

$$\dot{\omega} = I^{-1}[(I\omega + h)\times]\omega + I^{-1}(T - \dot{h}) \tag{7}$$

Defining

$$F(\omega) = I^{-1}[(I\omega + h)\times] \tag{8a}$$

and

$$u(t) = I^{-1}(T - \dot{h}) \tag{8b}$$

Eq. (7) can now be written as

$$\dot{\omega} = F(\omega)\omega + u(t) \tag{9}$$

Note that the transformation of Eq. (7) to Eq. (9) was done to express the nonlinear terms in the components of ω as a product of a matrix and ω, which led to the form shown in Eq. (9). It can be seen that the path from Eq. (7) to Eq. (9) is not unique. It stems from the decomposition of $I\omega \times \omega$ into $[I\omega\times]\omega$ not being unique. It can be shown that there are eight possible ways to decompose the vector; consequently, there are eight *primary* dynamics matrices that express the angular dynamics of an SC; that is, we have $F_i(\omega)\ i = 1, 2, \ldots, 8$ for which $F_1(\omega)\omega = F_2(\omega)\omega = \cdots = F_8(\omega)\omega$, although

$$F_1(\omega) \neq F_2(\omega) \neq \cdots \neq F_8(\omega) \tag{10}$$

Although there are only eight primary representations, one can generate infinite secondary dynamics matrices as linear combinations of the primary matrices by forming

$$E_i(\omega) = \sum_{j=1}^{8} \alpha_{i,j} F_j(\omega) \qquad \text{where} \qquad \sum_{j=1}^{8} \alpha_{i,j} = 1$$
$$i = 1, 2, \ldots, \to \infty \tag{11}$$

This can be easily proven when noting that

$$E_i(\omega)\omega = \sum_{j=1}^{8} \alpha_{i,j} F_j(\omega)\omega = \sum_{j=1}^{8} \alpha_{i,j} f(\omega)$$
$$= f(\omega) \sum_{j=1}^{8} \alpha_{i,j} = f(\omega) \tag{12}$$

The ability to form infinite secondary dynamics matrices from two basic matrices of nonlinear systems was already noted by Cloutier et al.[9,10] We conclude then that the representation of the SC dynamics equation in the form

$$\dot{x} = F(x)x + u(t) \tag{13}$$

is not unique and that there are exactly eight different ways to express the nonlinear SC dynamics equation by basic state-dependent linear equations and infinite secondary equations. To account for modeling error we add a white noise vector $w(t)$ to the dynamics model given by Eq. (9) and obtain

$$\dot{\omega} = F(\omega)\omega + u(t) + w(t) \tag{14}$$

IV. Angular Rate Computation

When the dynamics equation [Eq. (14)] and the corresponding measurement equation [Eq. (5)] have been obtained, we convert the dynamics equation to a discrete form and obtain the couple

$$\omega_{k+1} = F(\omega_k)\omega_k + u_k + w_k \tag{15a}$$

$$z_k = C_k \omega_k + v_k \tag{15b}$$

We can now select a suitable estimator to estimate ω. Two estimators are used: the PSELIKA and the SDARE filters.

A. Pseudolinear Filtering

The PSELIKA filter is an ordinary discrete KF in which the coefficients are functions of the estimated state. In the computation of the estimate, we use the best available estimate to evaluate those coefficient. Therefore, whenever $F(\omega_k)$ of Eq. (15a) is used, we substitute ω_k by $\hat{\omega}$, the present available estimate of ω at the time it is needed. The estimated vs the true angular rate components, which were measured by the satellite gyroscopes, are presented in Fig. 2. By comparing the results presented in Fig. 2 to those presented in Fig. 1, we realize that the PSELIKA algorithm performs quite well and the estimates are obtained without a delay.

B. CD SDARE Filtering

Recently the use of the SDARE for nonlinear regulator control of special systems was presented by Cloutier et al.[9,10] Following the duality between linear optimal controller design and the linear optimal estimator design, Pappano and Friedland[11] and Mracek et al.[12] suggested an SDARE filtering algorithm for special continuous-time nonlinear systems. The special systems for which the algorithm was suggested are of the form

$$\dot{x} = F(x)x + u(t) + w(t) \tag{16a}$$

$$z = C(x)x + v(t) \tag{16b}$$

where, as before, $w(t)$ and $v(t)$ are zero-mean white noise vectors, whose covariance matrices are $Q(t)\delta(t-\tau)$ and $R(t)\delta(t-\tau)$, respectively. The SDARE filter for these systems, as suggested in Refs. 11 and 12, is

$$\dot{\hat{x}} = F(\hat{x})\hat{x} + u(t) + K(\hat{x})[z(x) - C(\hat{x})\hat{x}] \tag{17a}$$

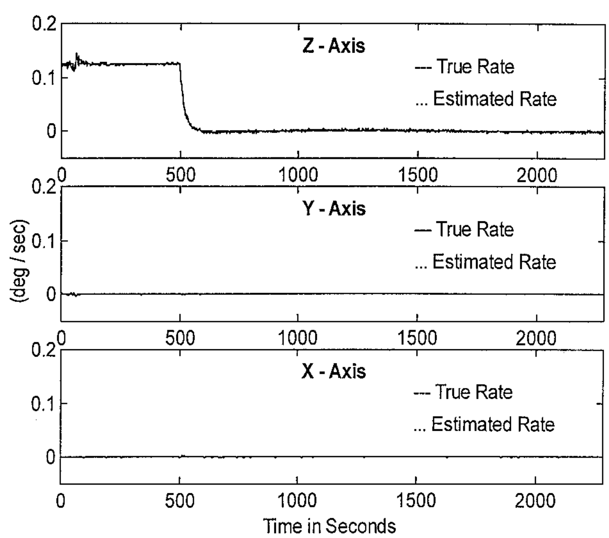

Fig. 2 True and estimated body rates using the CD SDARE filter.

where

$$K(\hat{x}) = PC^T(\hat{x})R^{-1}(t) \tag{17b}$$

and where P is the solution of the following algebraic Riccati equation:

$$F(\hat{x})P + PF^T(\hat{x}) - PC^T(\hat{x})R^{-1}(t)C(\hat{x})P + Q(t) = 0 \tag{17c}$$

Our rate estimation problem is somewhat different. The main difference between our problem and that handled in Refs. 11 and 12 is that their measurement equation was continuous and ours is discrete. This implies that we have a CD-time filtering problem. (Another minor difference is that our measurement equation is linear.) The question is then: *Is the continuous* SDARE *filter also valid for this case where the measurements are discrete*? We maintain that the SDARE filter is valid for this case. Next we present the CD SDARE filter that suits our problem, and then we explain the logic behind it. The CD SDARE is as follows.

Choose an approximate value for the initial estimate of the rate vector. For example, in the absence of such initial estimate simply choose $\hat{\omega}_0 = 0$. Between measurements propagate only the state estimate by solving the differential equation $\dot{\hat{\omega}} = F(\hat{\omega})\hat{\omega} + \boldsymbol{u}(t)$ between t_k and t_{k+1}, where $\hat{\omega}(t_k) = \hat{\omega}_{k/k}$ and $\hat{\omega}_{k+1/k} = \hat{\omega}(t_{k+1})$. When a measurement takes place at time t_{k+1}, solve the following algebraic Riccati equation for P_{k+1}:

$$F(\hat{\omega}_{k+1/k})P_{k+1} + P_{k+1}F^T(\hat{\omega}_{k+1/k}) - P_{k+1}C_{k+1}^T R_{k+1}^{-1} C_{k+1} P_{k+1} + Q_{k+1} = 0 \tag{18}$$

By the use of P_{k+1}, compute the gain matrix

$$K_{k+1} = P_{k+1}C_{k+1}^T R_{k+1}^{-1} \tag{19}$$

and finally compute the updated state estimate

$$\hat{\omega}_{k+1/k+1} = \hat{\omega}_{k+1/k} + K_{k+1}[z_{k+1} - C_{k+1}\hat{\omega}_{k+1/k}] \tag{20}$$

To explain the suitability of this algorithm to the CD case, we note the following:

1) In the case where both the dynamics and the measurements are continuous, the solution of the following matrix Riccati equation at any time point t_{k+1},

$$F(\hat{\omega})P + PF^T(\hat{\omega}) - PC^T R^{-1}(t)CP + Q(t) = \dot{P} \tag{21}$$

yields the value of P at that time point.

2) If at a certain time point, t_{k+1}, P reaches steady state, the value of P at that time point, denoted by P_{k+1}, solves the equation

$$F(\hat{\omega}_{k+1/k})P_{k+1} + P_{k+1}F^T(\hat{\omega}_{k+1/k}) - P_{k+1}C^T R_{k+1}^{-1} C P_{k+1} + Q_{k+1} = 0 \tag{22}$$

The logic behind the use of the last equation in the CD SDARE filter is that we assume that, although the measurements obtained and processed before and at time t_{k+1} were discrete, we can regard their influence as if they were continuous. In other words, we assume that there is an entirely continuous system that is equivalent to our CD system; therefore, we may apply Eq. (22) to obtain P_{k+1}. Of course, the values of $R(t)$ and $Q(t)$ are not equal to the corresponding R_{k+1} and Q_{k+1} matrices, and as is the normal procedure of the design of a KF, they have to be determined by tuning. The estimates of the angular rate components that were generated when the CD SDARE filter was applied to the data used before are very similar but slightly better than those generated by the PSELIKA filter, which are presented in Fig. 2.

V. Conclusions

We examined two new algorithms for estimating the angular rate vector of an SC. The two algorithms were based on the ability to decompose the nonlinear SC dynamics into a product of an angular rate dependent matrix and the angular rate vector itself. Using this nonunique decomposition, the ordinary KF was applied successfully. We named this algorithm PSELIKA. In addition, based on this decomposition, a new variant of the SDARE filter was introduced, which we called CD SDARE.

The two algorithms were applied to real data obtained from the RXTE satellite. The measurements on which the estimates were based were derived from the differentiation of a three-axis magnetometer and sun sensor readings. When comparing the estimates of the angular rate vector generated by these two filters to the estimate obtained when passing the derivatives through a simple low-pass filter, the advantage of using these filters was obvious. The estimates yielded by the two algorithms introduced no delays, whereas the latter computational technique introduced either a considerable delay or excessive noise. A comparison between the PSELIKA filter and CD SDARE filter reveals that the estimates obtained by the latter are less noisy than those obtained by the former. The advantage of using these filters vs the use of the EKF is in their robustness to large error in the determination of the initial estimate and to model mismatch; however, more work needs to be done on the comparison between the PSELIKA and CD SDARE filters themselves, but this is beyond the scope of this Note.

Acknowledgment

The authors wish to thank Ruth Azor of MBT-Israel Aircraft Industries for helpful discussions, mainly on the derivation of the primary forms of the decomposed dynamics equations.

References

[1] Harman, R. R., and Bar-Itzhack, I. Y., "Angular Rate Computation Using Attitude Differentiation," Technion—Israel Inst. of Technology, TR TAE-881, Haifa, Israel, Dec. 1997.

[2] Flately, T. W., Forden, J. K., Henretty, D. A., Lightsey, E. G., and Markley, F. L., "On-Board Attitude Determination and Control Algorithm for SAMPEX," *Proceedings of the Flight Mechanics/Estimation Theory Symposium*, NASA Goddard Space Flight Center, Greenbelt, MD, 1990.

[3] Natanson, G., "A Deterministic Method for Estimating Attitude from Magnetometer Data Only," *Proceedings of the 43rd International Astronautical Federation Congress*, Paper IFA-92-0036, Washington, DC, Aug.–Sept. 1992.

[4] Challa, M., Natanson, G., Deutschmann, J., and Galal, K., "A PC-Based Magnetometer-Only Attitude and Rate Determination System for Gyroless Spacecraft," *Proceedings of the Flight Mechanics/Estimation Theory Symposium*, NASA Goddard Space Flight Center, Greenbelt, MD, 1995.

[5] Challa, M., Kotaru, S., and Natanson, G., "Magnetometer-Only Attitude and Rate Estimates During the Earth Radiation Budget Satellite 1987 Control Anomaly," AIAA Paper 97-3615, Aug. 1997.

[6] Azor, R., Bar-Itzhack, I. Y., and Harman, R. R., "Satellite Angular Rate Estimation from Vector Measurements," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 3, 1998, pp. 450–457.

[7] Gelb, A. (ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974, p. 19.

[8] Wertz, J. R. (ed.), *Spacecraft Attitude Dynamics and Control*, Reidel, Dordrecht, The Netherlands, 1984, p. 523.

[9] Cloutier, J. R., D'Souza, C. N., and Mracek, C. P., "Nonlinear Regulation and Nonlinear H_∞ Control Via the State-Dependent Riccati Equation Technique: Part 1, Theory," *Proceedings of the First International Conference on Nonlinear Problems in Aviation and Aerospace*, Embry-Riddle Aeronautical Univ. Press, Daytona Beach, FL, 1996, pp. 117–142.

[10] Cloutier, J. R., D'Souza, C. N., and Mracek, C. P., "Nonlinear Regulation and Nonlinear H_∞ Control Via the State-Dependent Riccati Equation Technique: Part 2, Examples," *Proceedings of the First International Conference on Nonlinear Problems in Aviation and Aerospace*, Embry-Riddle Aeronautical Univ. Press, Daytona Beach, FL, 1996.

[11] Pappano, V., and Friedland, B., "SDRE Observer for an Induction Machine," *Proceedings of the 1997 American Control Conference*, June 1997.

[12] Mracek, C. P., Cloutier, J. R., and D'Souza, C. N., "A New Technique for Nonlinear Estimation," *Proceedings of the IEEE Conference on Control Applications*, IEEE Publications, Piscataway, NJ, 1996.