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Asymptotically Optimal Attitude Filtering with Guaranteed Convergence

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This paper introduces some new concepts in attitude filtering to aid the formulation of simple filtering algorithms with guaranteed convergence from almost any initial condition, far outperforming the transient performance of standard extended Kalman filter formulations. The focus is on formulating a filter algorithm for estimating the four quaternion elements using unit vector observables to provide periodic updates of gyroscope-propagated attitude. It is assumed that the gyroscope rate measurement errors are solely due to zero-mean noise. This algorithm is suitable for coarse attitude determination using elementary sensors. Research into extending the approach to more complicated error models is ongoing.

Nomenclature

$\mathbf{A}(\boldsymbol{q})$	=	3×3 transformation matrix formed from the quaternion argument
$E\{\}$	=	expectation operator
$\boldsymbol{g}_k$	=	three-dimensional postupdate Gibbs vector attitude error representation
$\boldsymbol{g}_k^-$	=	three-dimensional preupdate Gibbs vector attitude error representation
$\hat{\boldsymbol{g}}_k$	=	three-dimensional estimate of the Gibbs vector attitude update
$\mathbf{I}_{n\times n}$	=	$n \times n$ identity matrix
$\mathbf{N}_k$	=	null projection matrix formed from the unit vector or quaternion observables
$\mathbf{P}_k$	=	3×3 matrix representation of the postupdate angular-error covariance tensor
$\mathbf{P}_k^+$	=	3×3 matrix representation of the postupdate pretangent-space mapped angular-error covariance tensor
$\mathbf{P}_k^-$	=	3×3 matrix representation of the preupdate angular-error covariance tensor
$\boldsymbol{q}$	=	four-element quaternion composed of vector part $\boldsymbol{Q}$ and scalar part q
$\boldsymbol{q}_k$	=	true inertial-to-body attitude quaternion
$\hat{\boldsymbol{q}}_k$	=	postupdate attitude quaternion estimate
$\bar{\boldsymbol{q}}_k$	=	direct measurement of the true quaternion
$\hat{\boldsymbol{q}}_k^-$	=	preupdate extrapolated attitude quaternion estimate
$\mathbb{R}^n$	=	n-dimensional Euclidean space
$\boldsymbol{r}_k$	=	reference measurement unit vector in inertial space
S^3	=	three-dimensional hypersphere embedded in $\mathbb{R}^4$
$\boldsymbol{s}_k$	=	measurement unit vector in the spacecraft body space
$T_{\boldsymbol{q}}S^3$	=	tangent space of S^3 at $\boldsymbol{q}$
$\bar{\mathbf{U}}_k$	=	$\mathbf{U}(\boldsymbol{q}_k)$
$\mathbf{U}_k$	=	$\mathbf{U}(\hat{\boldsymbol{q}}_k)$
$\mathbf{U}_k^-$	=	$\mathbf{U}(\hat{\boldsymbol{q}}_k^-)$
$\mathbf{U}(\boldsymbol{q})$	=	4×3 matrix, canonical map from $T_{\boldsymbol{q}}S^3$ to $\mathbb{R}^4$
δ_{nk}	=	Kronecker delta function (1 if $n = k$, else 0)
$\boldsymbol{\xi}_k$	=	small-angle approximation of the attitude error for the extended Kalman filter
σ_g	=	scalar propagation uncertainty parameter in units of angle
σ_m	=	scalar measurement uncertainty parameter in units of angle
$\boldsymbol{\Phi}(t,\tau)$	=	3×3 state error transition matrix from time τ to time t
$\boldsymbol{\Phi}_k^n$	=	$\boldsymbol{\Phi}(t_n, t_k)$
$\boldsymbol{\Psi}(t,\tau)$	=	4×4 quaternion transition matrix from time τ to time t
$\boldsymbol{\Psi}_k^n$	=	$\boldsymbol{\Psi}(t_n, t_k)$
$\hat{}$	=	estimated quantity

Subscripts

k	=	kth time step
0	=	initial value of the indicated quantity

Superscripts

T	=	transpose used to convert to row vectors when appropriate
$-$	=	quantity is extrapolated to the instant before an update
$+$	=	covariance estimate immediately following the update but before tangent-space mapping

I. Introduction

THE literature on various attitude filter formulations is rich and varied, as discussed in [1,2]. Most frequently, standard attitude estimation algorithms are based on linearization of the error kinematics in an extended Kalman filter (EKF). These algorithms often exhibit poor convergence properties when initial attitude estimation errors are large or when there are significant unmodeled error sources and many authors have proposed a variety of modified or significantly reformulated filter approaches to deal with this shortcoming. Of recent note, [3] uses a linear measurement model to formulate a filter that appears very robust to initial large attitude errors and mismodeling, but it is complex to implement and requires propagation of covariance elements for all four quaternion states, from which some singularity issues arise. Another recent algorithm that appears to be robust is detailed in [4], which approximates the actual multivariable probability distribution of the extrapolated state vector to capture the approximate true statistics of errors from the nonlinear propagation. However, this approach is also complex and requires propagating multiple quaternion estimates.

The aforementioned algorithms estimate an attitude quaternion as well as gyroscope bias states. The algorithm we will present here only estimates the attitude quaternion assuming zero gyroscope biases. Such modeling is fully adequate for many coarse attitude

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filtering applications wherein measurements from coarse attitude sensors, such as sun sensors, Earth horizon sensors, and magnetometers, are used to derive unit vector observables. The formulation of a simple algorithm with guaranteed convergence for the estimation of attitude and gyroscope biases is an area of active research.

The formalism that we introduce here is unique in that it introduces the concept of errors propagating between and being updated within the tangent spaces [5–8] of the unit quaternion manifold. This manifold is identified with, or *isomorphic* to, S^3, the three-dimensional hypersphere embedded in four-dimensional Euclidean space $\mathbb{R}^4$; that is, the set of points in $\mathbb{R}^4$ for which the square root of the sum of squares of the four coordinates is unity. The unit quaternion manifold attains a group structure via quaternion composition, which produces a new quaternion in S^3 from two other quaternions in S^3. To be considered a group composition, repeated compositions must be associative, and every quaternion must be invertible so that composition with its inverse yields the identity quaternion that, in this paper, is represented by a unit quaternion with the fourth element equal to unity and all others equal to zero.

It is helpful to visualize the tangent spaces of the unit quaternion manifold by visualizing the subgroup representing transformations about a particular axis in three-dimensional Euclidean space $\mathbb{R}^3$. Such a subgroup is isomorphic to S^1, the unit circle embedded in $\mathbb{R}^2$ (see Fig. 1). An element in the tangent space, which is isomorphic to $\mathbb{R}^1$, of length $\tan(\phi/2)$ may be associated with the transformation through an angle $\theta + \phi$ via radial projection of the tangent vector onto the unit circle. The tangent space of S^3 may be similarly associated with vectors in $\mathbb{R}^3$ for which the lengths are the tangents of the half-angles of the transformations they represent. Such vectors are called Gibbs vectors [9].

The formulation of this paper is possible because it has been found that when observables can be rendered as unit vector measurements in the spacecraft frame of reference, one may construct a linear measurement model for the error Gibbs vector in the local tangent space of the propagated quaternion estimate. It is natural, therefore, that the covariance estimate associated with the attitude estimation error should be represented as a second-rank tensor in the three-dimensional tangent space of the quaternion estimate. Each time the quaternion estimate is propagated or updated, the covariance estimate must be mapped from one tangent space to the next.

In this paper, we will apply these concepts to quaternion estimation assuming a simple gyroscope error model in which the propagation errors are represented as a series of zero-mean independent vectors defined in the local tangent space (i.e., what is commonly referred to as an angle random walk). Our goal is to create an algorithm with convergence virtually guaranteed, based on a simple observability criterion, from almost any initial state in the zero-noise case. The algorithm also produces a covariance estimate that describes, at least asymptotically and approximately, the second-order error statistics when the noise is "turned back on" and that allows, at least asymptotically and approximately, optimal weighting of the measurements to provide filter estimates. This is an important point that needs emphasizing. This is not intended to be a stochastically rigorous algorithm. The covariance estimate is an asymptotic approximation to the actual second-order statistics of the error under appropriate noise bounds. In the large-angular-error domain, it can be thought of as just a weighting matrix that may or may not reflect the true error statistics but is used to ensure a negative gradient direction in what is essentially a numerical search algorithm.

The resulting algorithm is relatively simple and requires little processing beyond the standard EKF. We will first present the algorithm in Sec. II and the derivation in Sec. III. A discussion of the strong convergence properties is in Secs. IV and V presents a performance comparison with an EKF formulation, which is shown to be a small-angular-error approximation of the new algorithm.

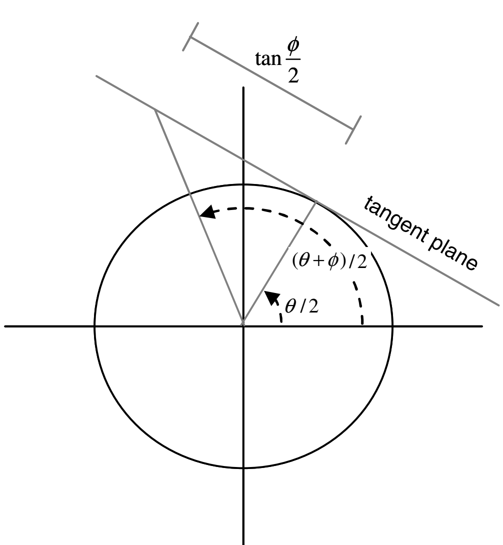

Fig. 1 Representation of additive transformations on S^1 via vectors in the tangent plane.

II. Kalman-Filter-Like Algorithm

As formulated in [10] and explained in Sec. III, given a measurement column unit vector $\boldsymbol{s}_k$ in the spacecraft body space and a calculated version of the same vector in inertial space $\boldsymbol{r}_k$, we can form the null projection matrix

$$\mathbf{N}_k = \frac{1}{2}\begin{bmatrix} (1+\boldsymbol{r}_k^T\boldsymbol{s}_k)\mathbf{I}_{3\times3} - (\boldsymbol{r}_k\boldsymbol{s}_k^T + \boldsymbol{s}_k\boldsymbol{r}_k^T) & \boldsymbol{r}_k\times\boldsymbol{s}_k \\ (\boldsymbol{r}_k\times\boldsymbol{s}_k)^T & 1-\boldsymbol{r}_k^T\boldsymbol{s}_k \end{bmatrix} \tag{1}$$

which has matrix rank 2 and in the case of perfect measurements, satisfies $\mathbf{N}_k\boldsymbol{q}_k = 0$, where $\boldsymbol{q}_k$ is the true attitude unit quaternion.

Assume that the measurement unit vector $\boldsymbol{s}_k$ has covariance

$$E\{(\boldsymbol{s}_k - E\{\boldsymbol{s}_k\})(\boldsymbol{s}_n - E\{\boldsymbol{s}_n\})^T\} = \sigma_m^2(\mathbf{I}_{3\times3} - \boldsymbol{s}_k\boldsymbol{s}_k^T)\delta_{nk} \tag{2}$$

where δ_{nk} is the Kronecker delta function and σ_m is a scalar uncertainty parameter. Define the map $\mathbf{U}(\boldsymbol{q})\colon T_{\boldsymbol{q}}S^3 \to \mathbb{R}^4$ (i.e., from the tangent space of S^3 at $\boldsymbol{q}$ to four-dimensional Euclidean space) as

$$\mathbf{U}(\boldsymbol{q}) = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{3}$$

For notational economy, let $\mathbf{U}_k^- = \mathbf{U}(\hat{\boldsymbol{q}}_k^-)$ and $\mathbf{U}_k = \mathbf{U}(\hat{\boldsymbol{q}}_k)$, where $\hat{\boldsymbol{q}}_k^-$ is the extrapolated unit quaternion estimate just before a measurement update and $\hat{\boldsymbol{q}}_k$ is the updated unit quaternion estimate.

The 4×4 quaternion transition matrix, denoted as $\boldsymbol{\Psi}_k^{k+1}$, and the 3×3 state error transition matrix $\boldsymbol{\Phi}_k^{k+1}$ are calculated in the usual way (see Sec. III). Let the delta-error uncertainty in the gyro-propagated attitude be zero mean and quantified by σ_g in units of angle. Let the 3×3 matrix $\mathbf{P}_k$ denote the covariance estimate, which is a second-rank tensor in the tangent space of the current estimated quaternion and in a sense discussed in later sections, quantifies the expected variation in our attitude estimate. The Kalman-filter-like algorithm proceeds as follows:

State and covariance propagation:

$$\hat{\boldsymbol{q}}_{k+1}^- = \boldsymbol{\Psi}_k^{k+1}\hat{\boldsymbol{q}}_k \tag{4a}$$

$$\mathbf{P}_{k+1}^- = \boldsymbol{\Phi}_k^{k+1}\mathbf{P}_k\boldsymbol{\Phi}_{k+1}^k + \sigma_g^2\mathbf{I}_{3\times3} \tag{4b}$$

State and covariance measurement update:

$$\hat{\boldsymbol{q}}_k \sim \left(\mathbf{I}_{4\times4} + \frac{1}{\sigma_m^2}\mathbf{U}_k^-\mathbf{P}_k^-\mathbf{U}_k^{-T}\mathbf{N}_k\right)^{-1}\hat{\boldsymbol{q}}_k^- \tag{4c}$$

$$\mathbf{P}_k = \mathbf{U}_k^T \mathbf{U}_k^- \left(\mathbf{P}_k^{--1} + \frac{1}{\sigma_m^2} \mathbf{U}_k^{-T} \mathbf{N}_k \mathbf{U}_k^- \right)^{-1} \mathbf{U}_k^{-T} \mathbf{U}_k \tag{4d}$$

where $\sim$ indicates similarity and should be taken to mean that the given vector quantity should be normalized to unit length.

If we were using a sensor that provided a measurement $\boldsymbol{q}_k$ of the true quaternion directly, we could, in a similar spirit, form a rank-3 null projection matrix $\mathbf{N}_k = \mathbf{I}_{4\times4} - \boldsymbol{q}_k \boldsymbol{q}_k^T$ or a weighted symmetric null matrix and the filter algorithm could proceed without any other modification. For multiple measurements with associated null matrices $\mathbf{N}_k^{(n)}$, $n = 1, 2, \ldots, N$ and measurement noise parameters $\sigma_{m(n)}^2$, we could replace

$$\frac{1}{\sigma_m^2} \mathbf{N}_k$$

with

$$\sum_{n=1}^{N} \frac{1}{\sigma_{m(n)}^2} \mathbf{N}_k^{(n)}$$

and proceed in the same way.

III. Derivation of the Algorithm

Let us assume the *negative cross-product* convention for unit quaternion group composition†

$$\boldsymbol{q}_{ac} = \boldsymbol{q}_{ab} \circ \boldsymbol{q}_{bc} = (q_{ab}\boldsymbol{Q}_{bc} + q_{bc}\boldsymbol{Q}_{ab} - \boldsymbol{Q}_{ab} \times \boldsymbol{Q}_{bc}, q_{ab}q_{bc} - \boldsymbol{Q}_{ab} \cdot \boldsymbol{Q}_{bc}) \tag{5}$$

where we used the shorthand $\boldsymbol{q} = (\boldsymbol{Q}, q)$ to split and group the 4-element quaternions into their 3-vector and scalar components [9]. The ordering of subscripts as shown previously is a helpful mnemonic to keep the order of quaternion composition straight: that is, $\boldsymbol{q}_{ab}$ represents the transformation from frame b to frame a, $\boldsymbol{q}_{bc}$ represents the transformation from frame c to frame b, and hence $\boldsymbol{q}_{ac}$ represents the transformation from frame c to frame a. Inversion of the quaternion reverses the order of the subscripts [e.g., $\boldsymbol{q}_{ba} = (-\boldsymbol{Q}_{ab}, q_{ab}) = \boldsymbol{q}_{ab}^{-1}$]. In the following, it will always be assumed that the quaternion in question represents the transformation from inertial-to-body space and a single subscript will be used to signify the time step at which the quaternion is valid. A handy formula involving unit quaternion composition and the Euclidean inner or dot product is

$$(\boldsymbol{p} \circ \boldsymbol{q}) \cdot \boldsymbol{r} = (\boldsymbol{r} \circ \boldsymbol{q}^{-1}) \cdot \boldsymbol{p} \tag{6}$$

for arbitrary unit quaternions $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$.

The columns of the 4×3 matrix

$$\mathbf{U}(\boldsymbol{q}) = [\boldsymbol{e}_1 \circ \boldsymbol{q} \quad \boldsymbol{e}_2 \circ \boldsymbol{q} \quad \boldsymbol{e}_3 \circ \boldsymbol{q}] \tag{7}$$

where $\boldsymbol{e}_1 = [1 \quad 0 \quad 0 \quad 0]^T$ and so forth, are orthogonal unit vectors in $\mathbb{R}^4$ that are tangent to the unit 3-sphere S^3 at $\boldsymbol{q}$. This matrix maps 3-vectors in $T_{\boldsymbol{q}}S^3$ to 4-vectors in $\mathbb{R}^4$ that are orthogonal to $\boldsymbol{q}$ in terms of the Euclidean inner or dot product; hence, we will refer to this matrix as the *tangent basis map*. Consider the two quaternions $\boldsymbol{q}$ and $\hat{\boldsymbol{q}}$ and the delta quaternion formed by $\delta\boldsymbol{q} = \boldsymbol{q} \circ \hat{\boldsymbol{q}}^{-1} = (\delta\boldsymbol{Q}, \delta q)$. Then the following identities hold for the tangent basis map:

$$\mathbf{U}^T(\boldsymbol{q})\mathbf{U}(\boldsymbol{q}) = \mathbf{I}_{3\times3} \tag{8a}$$

$$\mathbf{U}(\boldsymbol{q})\mathbf{U}^T(\boldsymbol{q}) = \mathbf{I}_{4\times4} - \boldsymbol{q}\boldsymbol{q}^T \tag{8b}$$

$$\mathbf{U}^T(\hat{\boldsymbol{q}})\boldsymbol{q} = \delta\boldsymbol{Q} \tag{8c}$$

$$\mathbf{U}^T(\hat{\boldsymbol{q}})\mathbf{U}(\boldsymbol{q}) = \delta q \mathbf{I}_{3\times3} + \operatorname{skew}(\delta\boldsymbol{Q}) \tag{8d}$$

where skew denotes the cross-product matrix of the 3-vector argument:

$$\operatorname{skew}(\boldsymbol{v}) = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{9}$$

It is useful to consider the set of Gibbs vectors [9] $\boldsymbol{g}$ in $T_{\boldsymbol{q}}S^3$ that represent rotations from $\boldsymbol{q}$ to any other quaternion $\boldsymbol{q}'$ via

$$\underline{g} = \frac{\mathbf{U}^T(\boldsymbol{q})\boldsymbol{q}'}{\boldsymbol{q} \cdot \boldsymbol{q}'} \tag{10}$$

The inverse relationship is

$$\boldsymbol{q}' = \pm \frac{\boldsymbol{q} + \mathbf{U}(\boldsymbol{q})\boldsymbol{g}}{\sqrt{1 + \boldsymbol{g} \cdot \boldsymbol{g}}} \tag{11}$$

The Gibbs vector representation has a singularity when the angle between $\boldsymbol{q}$ and $\boldsymbol{q}'$ is ± 180 deg, but because our final algorithm of Eqs. (4) does not explicitly rely upon the Gibbs vector representation of attitude, this does not concern us overmuch, so long as our initial attitude error has a magnitude of less than 180 deg. We will discuss this issue further in Sec. IV.

The next step in the filtering problem is to create a linear measurement model. Let an observation unit vector in spacecraft body space at step k be given by $\boldsymbol{s}_k$ and in inertial space (via mathematical model) by $\boldsymbol{r}_k$. Then, with perfect measurements, the true quaternion lies in the two-dimensional subspace of $\mathbb{R}^4$ spanned by the basis vectors [10]:

$$\boldsymbol{b}_k^{(1)} = \frac{\begin{bmatrix} \boldsymbol{s}_k \times \boldsymbol{r}_k \\ 1 + \boldsymbol{r}_k \cdot \boldsymbol{s}_k \end{bmatrix}}{\sqrt{2(1 + \boldsymbol{r}_k \cdot \boldsymbol{s}_k)}}, \qquad \boldsymbol{b}_k^{(2)} = \frac{\begin{bmatrix} \boldsymbol{r}_k + \boldsymbol{s}_k \\ 0 \end{bmatrix}}{\sqrt{2(1 + \boldsymbol{r}_k \cdot \boldsymbol{s}_k)}} \tag{12}$$

The first basis vector represents a direct rotation about the cross axis that will transform $\boldsymbol{r}_k$ to $\boldsymbol{s}_k$. The second represents a 180-deg rotation about the bisecting vector that forms the central axis of a cone equidistant from $\boldsymbol{r}_k$ and $\boldsymbol{s}_k$. A linear combination and normalization of these two basis vectors can be represented as the direct rotation composed with a rotation about $\boldsymbol{s}_k$. The set of all such combinations is thus the set of all quaternions that transform $\boldsymbol{r}_k$ to $\boldsymbol{s}_k$. The construction breaks down when $\boldsymbol{r}_k \cdot \boldsymbol{s}_k = -1$, which signifies a 180-deg transformation. In this case, the true quaternion must lie in the plane defined by the set of quaternions with the vector part perpendicular to $\boldsymbol{r}_k$ and $\boldsymbol{s}_k$ and the scalar part equal to zero.

It follows that there are two orthogonal null unit 4-vectors that can be constructed that are ideally orthogonal to the true quaternion:

$$\boldsymbol{n}_k^{(1)} = \frac{\begin{bmatrix} \boldsymbol{r}_k \times \boldsymbol{s}_k \\ 1 - \boldsymbol{r}_k \cdot \boldsymbol{s}_k \end{bmatrix}}{\sqrt{2(1 - \boldsymbol{r}_k \cdot \boldsymbol{s}_k)}}, \qquad \boldsymbol{n}_k^{(2)} = \frac{\begin{bmatrix} \boldsymbol{r}_k - \boldsymbol{s} \\ 0 \end{bmatrix}}{\sqrt{2(1 - \boldsymbol{r}_k \cdot \boldsymbol{s}_k)}} \tag{13}$$

These vectors have a singularity when $\boldsymbol{r}_k = \boldsymbol{s}_k$, but the rank-2 orthogonal projection matrix, $\mathbf{N}_k = \boldsymbol{n}_k^{(1)}\boldsymbol{n}_k^{(1)T} + \boldsymbol{n}_k^{(2)}\boldsymbol{n}_k^{(2)T}$ of Eq. (1), onto the span of $\boldsymbol{n}_k^{(1)}$ and $\boldsymbol{n}_k^{(2)}$ does not. Thus, we can always find a pair of well-defined orthogonal null unit vectors simply by evaluating appropriate eigenvectors of $\mathbf{N}_k$ associated with an eigenvalue of unity. We shall call the null vectors so obtained $\tilde{\boldsymbol{n}}_k^{(1)}$ and $\tilde{\boldsymbol{n}}_k^{(2)}$ and note that we still have $\mathbf{N}_k = \tilde{\boldsymbol{n}}_k^{(1)}\tilde{\boldsymbol{n}}_k^{(1)T} + \tilde{\boldsymbol{n}}_k^{(2)}\tilde{\boldsymbol{n}}_k^{(2)T}$. Our linear measurement is created by taking advantage of the fact that the true quaternion is orthogonal to these null vectors; that is, defining $\mathbf{M}_k = [\tilde{\boldsymbol{n}}_k^{(1)} \quad \tilde{\boldsymbol{n}}_k^{(2)}]^T$, with perfect measurements, we have

†References [9,10] use the *positive cross-product* convention, in which the order of composition and of the subscripts is reversed.

$$\mathbf{M}_k \boldsymbol{q}_k = 0 \tag{14}$$

Via Eq. (11), this can be rewritten as

$$0 = \mathbf{M}_k(\hat{\boldsymbol{q}}_k^- + \mathbf{U}_k^- \boldsymbol{g}_k^-) \tag{15}$$

or, with $\boldsymbol{y}_k = -\mathbf{M}_k \hat{\boldsymbol{q}}_k^-$ and $\mathbf{H}_k = \mathbf{M}_k \mathbf{U}_k^-$, we have a linear measurement model

$$\boldsymbol{y}_k = \mathbf{H}_k \boldsymbol{g}_k^- + \tfrac{1}{2}\boldsymbol{v}_k \tag{16}$$

in which we have inserted a noise term $\boldsymbol{v}_k$ that is assumed for small $\boldsymbol{g}_k^-$ to have covariance $E\{\boldsymbol{v}_k \boldsymbol{v}_k^T\} \approx \sigma_m^2 \mathbf{I}_{2\times 2}$, representing the uncertainty in the measurement unit vector $\boldsymbol{s}_k$, as defined in Eq. (2). Note that the factor of $\frac{1}{2}$in Eq. (16) is related to the fact that quaternions and Gibbs vectors are defined in terms of half-angles and we have a preference for defining our noise term using full angles.

The quaternion transition matrix $\boldsymbol{\Psi}_k^{k+1} = \boldsymbol{\Psi}(t_{k+1}, t_k)$ can be computed via a suitable numerical integration of the matrix differential equation

$$\dot{\boldsymbol{\Psi}}(t,\tau) = \begin{bmatrix} -\mathrm{skew}(\boldsymbol{\omega}) & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \boldsymbol{\Psi}(t,\tau), \qquad \boldsymbol{\Psi}(\tau,\tau) = \mathbf{I}_{4\times 4} \tag{17}$$

and the state error transition matrix $\boldsymbol{\Phi}_k^{k+1} = \boldsymbol{\Phi}(t_{k+1}, t_k)$ can be computed via a suitable numerical integration of the matrix differential equation

$$\dot{\boldsymbol{\Phi}}(t,\tau) = -\mathrm{skew}(\boldsymbol{\omega})\boldsymbol{\Phi}(t,\tau), \qquad \boldsymbol{\Phi}(\tau,\tau) = \mathbf{I}_{3\times 3} \tag{18}$$

where $\boldsymbol{\omega}$ is the gyro angular rate. Note that

$$\mathbf{U}\left(\boldsymbol{\Psi}_k^{k+1}\mathbf{u}\right)\boldsymbol{\Phi}_k^{k+1} = \boldsymbol{\Psi}_k^{k+1}\mathbf{U}(\mathbf{u}) \tag{19}$$

for any unit 4-vector $\boldsymbol{u}$. This identity is easy to prove if one notes that for an arbitrary 3-vector $\boldsymbol{v}$, $\mathbf{U}(\boldsymbol{u})\boldsymbol{v} = \boldsymbol{v} \circ \boldsymbol{u}$, where the quaternion composition with a 3-vector takes place according to Eq. (5) by extending $\boldsymbol{v}$ to a 4-vector with a fourth element of zero. Then defining $\boldsymbol{\Delta q}_k$ such that $\boldsymbol{\Psi}_k^{k+1}\boldsymbol{u} = \boldsymbol{\Delta q}_k \circ \boldsymbol{u}$ and $\boldsymbol{\Phi}_k^{k+1}\boldsymbol{v} = \boldsymbol{\Delta q}_k \circ \boldsymbol{v} \circ \boldsymbol{\Delta q}_k^{-1}$, both sides of Eq. (19) applied to $\boldsymbol{v}$ are equal to

$$\mathbf{U}\left(\boldsymbol{\Psi}_k^{k+1}\boldsymbol{u}\right)\boldsymbol{\Phi}_k^{k+1}\boldsymbol{v} = \boldsymbol{\Psi}_k^{k+1}\mathbf{U}(\boldsymbol{u})\boldsymbol{v} = \boldsymbol{\Delta q}_k \circ \boldsymbol{v} \circ \boldsymbol{u} \tag{20}$$

Our goal now is to formulate a Kalman-filter-like algorithm with assured convergence, not necessarily a stochastically rigorous one, but one for which the covariance estimate is asymptotically accurate to first order. It may appear that this is no better than what can be expected of an extended Kalman filter, but it differs in two crucial respects:

1) The update itself is based on a linear measurement and is performed without approximation in the tangent space, then mapped down to S^3.

2) The covariance estimate is mapped from one tangent space to the next, so that it consistently remains a second-rank tensor in the local tangent space of the attitude error.

We loosely define the preupdate small-angle error covariance $\mathbf{P}_k^-$ as the expectation

$$\mathbf{P}_k^- = 4E\left\{\frac{\boldsymbol{g}_k^- \boldsymbol{g}_k^{-T}}{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}\right\} \tag{21}$$

and assume that it increases between updates according to Eq. (4b). Although this propagation is not rigorous, an argument could be made that Eq. (4b) propagates an upper bound for the actual covariance that is asymptotically accurate to first order if the filter converges to a reasonably bounded state. We will not delve too deeply into these matters because they detract from the task at hand, which is to formulate a filter with superior convergence properties. However, we will use Eq. (21) as a guide to how our filter should be implemented.

Applying the Kalman filter formalism, it is advantageous to compute the Kalman gain from the postupdate covariance estimate in the form

$$\mathbf{P}_k^+ = \left(\mathbf{P}_k^{--1} + \frac{1}{\sigma_m^2}\mathbf{H}_k^T \mathbf{H}_k\right)^{-1} = \left(\mathbf{P}_k^{--1} + \frac{1}{\sigma_m^2}\mathbf{U}_k^{-T}\mathbf{N}_k \mathbf{U}_k^-\right)^{-1} \tag{22}$$

$$\mathbf{K}_k = \frac{1}{\sigma_m^2}\mathbf{P}_k^+ \mathbf{H}_k^T \tag{23}$$

After each measurement update, we incorporate the estimated Gibbs vector into the updated quaternion. Hence, our preupdate Gibbs vector estimate is $\hat{\boldsymbol{g}}_k^- = 0$ and the update estimate is

$$\hat{\boldsymbol{g}}_k = \mathbf{K}_k \boldsymbol{y}_k = -\frac{1}{\sigma_m^2}\mathbf{P}_k^+ \mathbf{U}_k^{-T}\mathbf{N}_k \hat{\boldsymbol{q}}_k^- \tag{24}$$

We use this Gibbs vector estimate to update the quaternion according to Eq. (11):

$$\begin{aligned} \hat{\boldsymbol{q}}_k &= \frac{\hat{\boldsymbol{q}}_k^- + \mathbf{U}_k \hat{\boldsymbol{g}}_k}{\sqrt{1 + \hat{\boldsymbol{g}}_k \cdot \hat{\boldsymbol{g}}_k}} = \frac{\left[\mathbf{I}_{4\times 4} - (1/\sigma_m^2)\mathbf{U}_k^- \mathbf{P}_k^+ \mathbf{U}_k^{-T}\mathbf{N}_k\right]}{\sqrt{1 + \hat{\boldsymbol{g}}_k \cdot \hat{\boldsymbol{g}}_k}}\hat{\boldsymbol{q}}_k^- \\ &= \frac{\left[\mathbf{I}_{4\times 4} + (1/\sigma_m^2)\mathbf{U}_k^- \mathbf{P}_k^- \mathbf{U}_k^{-T}\mathbf{N}_k\right]^{-1}}{\sqrt{1 + \hat{\boldsymbol{g}}_k \cdot \hat{\boldsymbol{g}}_k}}\hat{\boldsymbol{q}}_k^- \end{aligned} \tag{25}$$

Note from Eq. (11) that

$$\boldsymbol{q}_k = \frac{\hat{\boldsymbol{q}}_k + \mathbf{U}_k \boldsymbol{g}_k}{\sqrt{1 + \boldsymbol{g}_k \cdot \boldsymbol{g}_k}} = \frac{\hat{\boldsymbol{q}}_k^- + \mathbf{U}_k^- \boldsymbol{g}_k^-}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}} \tag{26}$$

where $\boldsymbol{g}_k^-$ is the preupdate Gibbs representation of the estimation error and $\boldsymbol{g}_k$ is the postupdate representation. Multiplying both sides by $\mathbf{U}_k^T$ and using Eq. (8a), we obtain

$$\begin{aligned} \frac{\boldsymbol{g}_k}{\sqrt{1 + \boldsymbol{g}_k \cdot \boldsymbol{g}_k}} &= \frac{\mathbf{U}_k^T \mathbf{U}_k^- (\boldsymbol{g}_k^- - \hat{\boldsymbol{g}}_k)}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}} \\ &= \mathbf{U}_k^T \mathbf{U}_k^- (\mathbf{I}_{3\times 3} - \mathbf{K}_k \mathbf{H}_k)\frac{\boldsymbol{g}_k^-}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}} \end{aligned} \tag{27}$$

Thus, although $\mathbf{P}_k^+$ is the covariance estimate of

$$\frac{2(\boldsymbol{g}_k^- - \hat{\boldsymbol{g}}_k)}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}}$$

the covariance estimate $\mathbf{P}_k$ of

$$\frac{2\boldsymbol{g}_k}{\sqrt{1 + \boldsymbol{g}_k \cdot \boldsymbol{g}_k}}$$

requires pre- and postmultiplying $\mathbf{P}_k^+$ by $\mathbf{U}_k^T \mathbf{U}_k^-$ and its transpose, respectively:

$$\mathbf{P}_k = \mathbf{U}_k^T \mathbf{U}_k^- \mathbf{P}_k^+ \mathbf{U}_k^{-T} \mathbf{U}_k \tag{28}$$

This result has a certain differential geometric appeal. If we consider $\boldsymbol{g}_k^-$ and $\hat{\boldsymbol{g}}_k$ to be vectors in the tangent space of the preupdate quaternion $T_{\hat{\boldsymbol{q}}_k^-}S^3$ and thus $\mathbf{P}_k^-$ and $\mathbf{P}_k^+$ to be second-rank tensors in the tangent space of $T_{\hat{\boldsymbol{q}}_k^-}S^3$, it is clear that we need to transform $\mathbf{P}_k^+$ to the tangent space of the updated quaternion $T_{\hat{\boldsymbol{q}}_k}S^3$ to maintain a consistent local representation of the covariance estimate. The map $\mathbf{U}_k^T \mathbf{U}_k^-$ transforms the Gibbs vector from $T_{\hat{\boldsymbol{q}}_k^-}S^3$ to $\mathbb{R}^4$, then from $\mathbb{R}^4$ to $T_{\hat{\boldsymbol{q}}_k}S^3$. This mapping of the covariance estimate from propagated to update quaternion tangent spaces is crucial to ensuring large-angle convergence of the algorithm. In accordance with Eq. (8d), $\mathbf{U}_k^T \mathbf{U}_k^-$ can be determined directly to be

$$\mathbf{U}_k^T \mathbf{U}_k^- = \frac{\mathbf{I}_{3\times 3} - \mathrm{skew}(\hat{\boldsymbol{g}}_k)}{\sqrt{1 + \hat{\boldsymbol{g}}_k \cdot \hat{\boldsymbol{g}}_k}} \tag{29}$$

from which we may easily confirm from Eq. (27) that $\boldsymbol{g}_k = \boldsymbol{g}_k^- \circ \hat{\boldsymbol{g}}_k^{-1}$, where Gibbs vector composition proceeds analogously to the quaternion composition of Eq. (5),

$$\boldsymbol{g}_{ac} = \boldsymbol{g}_{ab} \circ \boldsymbol{g}_{bc} = \frac{\boldsymbol{g}_{ab} + \boldsymbol{g}_{bc} - \boldsymbol{g}_{ab} \times \boldsymbol{g}_{bc}}{1 - \boldsymbol{g}_{ab} \cdot \boldsymbol{g}_{bc}} \tag{30}$$

and $\hat{\boldsymbol{g}}_k^{-1} = -\hat{\boldsymbol{g}}_k$.

IV. Convergence Proof

In this section, we will demonstrate near-global asymptotic convergence of the filter algorithm in the case of zero propagation and measurement errors, with the parameters σ_g and σ_m chosen arbitrarily to have nonzero finite values. By near-global, we mean that the algorithm converges for every initial quaternion estimate that has at least a nonzero projection along the true quaternion: in other words, for all initial conditions except those contained in a set of measure zero in $\mathbb{R}^4$. (In the real world, noise and biases always assure that this condition will be met at some time in the evolution of the filter with probability 1.)

Our goal is to provide sufficient conditions under which the filter is guaranteed to converge to the true attitude quaternion. Most references taht discuss conditions for global asymptotic convergence of nonlinear algorithms are framed in the context of continuous systems [11,12]. However, most of the concepts carry over to the discrete time universe and we will add results from the theory of numerical sequences [13] to shore up the foundations when it is needed. We will first define the following observability grammian [14]:

$$\mathcal{O}_n^k = \frac{1}{\sigma_m^2} \sum_{i=n}^{k} \boldsymbol{\Phi}_i^k (\mathbf{I}_{3\times3} - \boldsymbol{s}_i \boldsymbol{s}_i^T) \boldsymbol{\Phi}_k^i \tag{31}$$

which is not quite the observability grammian one would expect in traditional filtering theory if one were attempting to bound the covariance estimates, but which will suit our purposes well because it is blessedly independent of the Gibbs vector estimate $\hat{\boldsymbol{g}}_k$.

In the forthcoming analysis, we will prove that the quaternion estimate of the filter is asymptotically convergent to the true quaternion in the noise-free case if the following three sufficient conditions are met:

1) The initial quaternion estimate $\hat{\boldsymbol{q}}_0$ has some component along the true quaternion $\boldsymbol{q}_0$ (i.e., $\hat{\boldsymbol{q}}_0 \cdot \boldsymbol{q}_0 \neq 0$). This ensures that the initial Gibbs vector $\boldsymbol{g}_0$ is bounded in magnitude, in accordance with Eq. (10).

2) The initial covariance is chosen proportional to the identity matrix, $\mathbf{P}_0 = \sigma_0^2 \mathbf{I}_{3\times3}$, for some scalar initial uncertainty parameter σ_0. This may be overly restrictive, but it is the most likely mode of use and it readily ensures that the Gibbs vector will remain bounded for all k.

3) There exists a finite constant integer N and a scalar constant $\alpha > 0$ such that

$$\alpha \mathbf{I} \leq \mathcal{O}_{k-N}^k \tag{32}$$

for all $k \geq N$; that is, the matrix has full rank.

We shall require some results on definite matrices and singular values [15,16].

Definition 1: A square matrix $\mathbf{G}$ is said to be positive definite or $\mathbf{G} > 0$ if for any vector $\boldsymbol{x} \neq \mathbf{0}$, $\boldsymbol{x}^T\mathbf{G}\boldsymbol{x} > 0$. If $\boldsymbol{x}^T\mathbf{G}\boldsymbol{x} \geq 0$, it is said to be positive semidefinite or nonnegative definite. We say $\mathbf{G}_1 > \mathbf{G}_2$ if $\mathbf{G}_1 - \mathbf{G}_2 > 0$ and $\mathbf{G}_1 \geq \mathbf{G}_2$ if $\mathbf{G}_1 - \mathbf{G}_2 \geq 0$. A matrix $\mathbf{G}$ is negative definite (semidefinite) if for any vector $\boldsymbol{x} \neq \mathbf{0}$, $\boldsymbol{x}^T\mathbf{G}\boldsymbol{x} < 0$ ($\boldsymbol{x}^T\mathbf{G}\boldsymbol{x} \leq 0$). We say $\mathbf{G}_1 < \mathbf{G}_2$ if $\mathbf{G}_1 - \mathbf{G}_2 < 0$ and $\mathbf{G}_1 \leq \mathbf{G}_2$ if $\mathbf{G}_1 - \mathbf{G}_2 \leq 0$.

Definition 2: The spectral norm of a matrix $\mathbf{G}$ is the maximum singular value denoted as $\bar{\sigma}(\mathbf{G})$, which is the square root of the maximum eigenvalue of $\mathbf{G}^T\mathbf{G}$. As a matrix norm, $\bar{\sigma}(\mathbf{G})$ satisfies the triangle inequality

$$\bar{\sigma}(\mathbf{G}_1 + \mathbf{G}_2) \leq \bar{\sigma}(\mathbf{G}_1) + \bar{\sigma}(\mathbf{G}_2) \tag{33a}$$

as well as

$$\bar{\sigma}(\mathbf{G}_1\mathbf{G}_2) \leq \bar{\sigma}(\mathbf{G}_1)\bar{\sigma}(\mathbf{G}_2) \tag{33b}$$

Definition 3:The minimum singular value of a matrix $\mathbf{G}$ is the square root of the minimum eigenvalue of $\mathbf{G}^T\mathbf{G}$, or of $\mathbf{G}\mathbf{G}^T$, whichever has lower dimension.

If $\mathbf{G}$ is square and has full rank, then $\underline{\sigma}(\mathbf{G})$ is related to $\bar{\sigma}(\mathbf{G}^{-1})$ by

$$\underline{\sigma}(\mathbf{G}) = \frac{1}{\bar{\sigma}(\mathbf{G}^{-1})} \tag{34a}$$

From Eqs. (33b) and (34a), one may thus deduce that for square matrices $\mathbf{G}_1$ and $\mathbf{G}_2$,

$$\underline{\sigma}(\mathbf{G}_1\mathbf{G}_2) \geq \underline{\sigma}(\mathbf{G}_1)\underline{\sigma}(\mathbf{G}_2) \tag{34b}$$

Equation (34b) works even if one or both of $\mathbf{G}_1$ or $\mathbf{G}_2$ are singular, because then both sides of the inequality are zero. In the case in which $\mathbf{G}$ is a nonnegative definite symmetric matrix, $\bar{\sigma}(\mathbf{G})$ is the same as the maximum eigenvalue and $\underline{\sigma}(\mathbf{G})$ is the minimum eigenvalue; hence,

$$\bar{\sigma}(\mathbf{G}) = \max_{\|\boldsymbol{u}\|=1} \boldsymbol{u}^T\mathbf{G}\boldsymbol{u} \tag{35a}$$

$$\underline{\sigma}(\mathbf{G}) = \min_{\|\boldsymbol{u}\|=1} \boldsymbol{u}^T\mathbf{G}\boldsymbol{u} \tag{35b}$$

Thus,

$$\underline{\sigma}(\mathbf{G})\mathbf{I} \leq \mathbf{G} \leq \bar{\sigma}(\mathbf{G})\mathbf{I} \tag{36}$$

and for any scalar $\eta \geq 0$,

$$\bar{\sigma}(\mathbf{G} + \eta\mathbf{I}) = \bar{\sigma}(\mathbf{G}) + \eta \tag{37a}$$

$$\underline{\sigma}(\mathbf{G} + \eta\mathbf{I}) = \underline{\sigma}(\mathbf{G}) + \eta \tag{37b}$$

It additionally follows that if $\mathbf{G}_1$ and $\mathbf{G}_2$ are nonnegative definite and symmetric, the counterpart to Eq. (33a) is

$$\underline{\sigma}(\mathbf{G}_1 + \mathbf{G}_2) \geq \underline{\sigma}(\mathbf{G}_1) + \underline{\sigma}(\mathbf{G}_2) \tag{38}$$

For any square $\mathbf{G}$, when a compatible matrix $\mathbf{F}$ is unitary ($\mathbf{F}\mathbf{F}^T = \mathbf{F}^T\mathbf{F} = \mathbf{I}$),

$$\bar{\sigma}(\mathbf{F}\mathbf{G}\mathbf{F}^T) = \bar{\sigma}(\mathbf{G}) \tag{39a}$$

$$\underline{\sigma}(\mathbf{F}\mathbf{G}\mathbf{F}^T) = \underline{\sigma}(\mathbf{G}) \tag{39b}$$

Proposition 1: The 2-norm or root-sum-squared magnitude of the error Gibbs vector update of Eq. (24) is bounded if the covariance $\mathbf{P}_k^-$ is bounded from above.

Proof: The proof is an elementary application of Eq. (35a) and other results on singular values listed previously.

$$\|\hat{\boldsymbol{g}}_k\| = \frac{1}{\sigma_m^2}\sqrt{\hat{\boldsymbol{q}}_k^{-T}\mathbf{N}_k\mathbf{U}_k^{-}\mathbf{P}_k^{+2}\mathbf{U}_k^{-T}\mathbf{N}_k\hat{\boldsymbol{q}}_k^{-}} \leq \frac{\bar{\sigma}(\mathbf{P}_k^{+}\mathbf{U}_k^{-T}\mathbf{N}_k)}{\sigma_m^2} \leq \frac{\bar{\sigma}(\mathbf{P}_k^{-})}{\sigma_m^2} \tag{40}$$

Proposition 2: If $\mathbf{P}_k^-$ is bounded both above and below, then so is $\mathbf{P}_k$.

Proof: The proof is again an elementary application of the results on singular values listed previously. We first note that in evaluating the maximum and minimum singular values of $\mathbf{U}_k^T\mathbf{U}_k^-$ as expressed in Eq. (29) and taking into account the result of Proposition 1, we find

$$\bar{\sigma}(\mathbf{U}_k^T\mathbf{U}_k^-) = 1 \tag{41}$$

$$\underline{\sigma}(\mathbf{U}_k^T\mathbf{U}_k^-) = \frac{1}{\sqrt{1+\hat{\boldsymbol{g}}_k\cdot\hat{\boldsymbol{g}}_k}} \geq \frac{\sigma_m^2}{\sqrt{\sigma_m^4+\bar{\sigma}^2(\mathbf{P}_k^-)}} \tag{42}$$

For the upper bound of $\mathbf{P}_k$,

$$\bar{\sigma}(\mathbf{P}_k) \leq \frac{\bar{\sigma}^2(\mathbf{U}_k^T\mathbf{U}_k^-)}{\underline{\sigma}[\mathbf{P}_k^{-\,-1}+(1/\sigma_m^2)\mathbf{U}_k^{-T}\mathbf{N}_k\mathbf{U}_k^-]} \leq \bar{\sigma}(\mathbf{P}_k^-) \tag{43}$$

For the lower bound of $\mathbf{P}_k$, and noting from Eq. (4b) that $\underline{\sigma}(\mathbf{P}_k^-) \geq \sigma_g^2$, we find

$$\underline{\sigma}(\mathbf{P}_k) \geq \frac{\underline{\sigma}^2(\mathbf{U}_k^T\mathbf{U}_k^-)}{\bar{\sigma}[\mathbf{P}_k^{-\,-1}+(1/\sigma_m^2)\mathbf{U}_k^{-T}\mathbf{N}_k\mathbf{U}_k^-]} \geq \frac{\sigma_m^2\sigma_g^2}{\sigma_m^2+\sigma_g^2}\frac{\sigma_m^4}{\sigma_m^4+\bar{\sigma}^2(\mathbf{P}_k^-)} \tag{44}$$

We note that $\bar{\sigma}(\mathbf{P}_k^-) \leq \bar{\sigma}(\mathbf{P}_{k-1}) + \sigma_g^2 \leq \bar{\sigma}(\mathbf{P}_{k-1}^-) + \sigma_g^2$, hence $\bar{\sigma}(\mathbf{P}_k^-) \leq \bar{\sigma}(\mathbf{P}_0^-) + k\sigma_g^2$, and so the upper bound on $\mathbf{P}_k^-$ does not increase very rapidly and is finite over any finite interval. Because the actual quaternion evolves according to

$$\boldsymbol{q}_{\boldsymbol{k+1}} = \boldsymbol{\Psi}_k^{k+1}\boldsymbol{q}_k \tag{45}$$

we can use Eqs. (10) and (19) to find

$$\boldsymbol{g}_{k+1}^- = \frac{\mathbf{U}^T(\hat{\boldsymbol{q}}_{\boldsymbol{k+1}}^-)\boldsymbol{q}_{k+1}}{\hat{\boldsymbol{q}}_{\boldsymbol{k+1}}^-\cdot\boldsymbol{q}_{k+1}} = \frac{\mathbf{U}^T(\boldsymbol{\Psi}_k^{k+1}\hat{\boldsymbol{q}}_k)\boldsymbol{\Psi}_k^{k+1}\boldsymbol{q}_k}{(\boldsymbol{\Psi}_k^{k+1}\hat{\boldsymbol{q}}_k)\cdot(\boldsymbol{\Psi}_k^{k+1}\boldsymbol{q}_k)} = \boldsymbol{\Phi}_k^{k+1}\boldsymbol{g}_k \tag{46}$$

Then from Eqs. (22), (23), and (27–29),

$$\frac{\boldsymbol{g}_k}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}} = \mathbf{P}_k(\mathbf{U}_k^T\mathbf{U}_k^-)^{-T}\mathbf{P}_k^{-\,-1}\frac{\boldsymbol{g}_{\boldsymbol{k}}^-}{\sqrt{1+\boldsymbol{g}_{\boldsymbol{k}}^-\cdot\boldsymbol{g}_{\boldsymbol{k}}^-}} \tag{47}$$

We shall have need for the tangent basis map from the tangent space of the true quaternion $T_{\boldsymbol{q}_k}S^3$ to $\mathbb{R}^4$, which we denote as $\bar{\mathbf{U}}_k = \mathbf{U}(\boldsymbol{q}_k)$. We note, similarly to Eq. (29), that

$$\bar{\mathbf{U}}_k^T\mathbf{U}_k^- = \frac{\mathbf{I}_{3\times3}-\text{skew}(\boldsymbol{g}_k^-)}{\sqrt{1+\boldsymbol{g}_k^-\cdot\boldsymbol{g}_k^-}} \tag{48}$$

Two useful relations that can be obtained from $\boldsymbol{g}_k = \boldsymbol{g}_k^-\circ\hat{\boldsymbol{g}}_k^{-1}$ according to the Gibbs vector composition rule of Eq. (30) are

$$\frac{1+\hat{\boldsymbol{g}}_k\cdot\hat{\boldsymbol{g}}_k}{(1+\boldsymbol{g}_k^-\cdot\hat{\boldsymbol{g}}_k)^2} = \frac{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}{1+\boldsymbol{g}_k^-\cdot\boldsymbol{g}_k^-} = \frac{(1-\boldsymbol{g}_{\boldsymbol{k}}\cdot\hat{\boldsymbol{g}}_k)^2}{1+\hat{\boldsymbol{g}}_k\cdot\hat{\boldsymbol{g}}_k} \tag{49}$$

Using Eqs. (29), (48), and (49), we find the updated error vector $\boldsymbol{g}_k$ is an eigenvector of $\bar{\mathbf{U}}_k^T\mathbf{U}_k^-(\mathbf{U}_k^T\mathbf{U}_k^-)^{-1}$ such that

$$\bar{\mathbf{U}}_k^T\mathbf{U}_k^-(\mathbf{U}_k^T\mathbf{U}_k^-)^{-1}\boldsymbol{g}_k = \sqrt{\frac{1+\hat{\boldsymbol{g}}_k\cdot\hat{\boldsymbol{g}}_k}{1+\boldsymbol{g}_{\boldsymbol{k}}^-\cdot\boldsymbol{g}_{\boldsymbol{k}}^-}}\boldsymbol{g}_k \tag{50}$$

With $\mathbf{N}_k$ as in Eq. (1), and expressing the true quaternion as a linear combination of basis vectors from Eq. (12), it is straightforward to derive

$$\bar{\mathbf{U}}_k^T\mathbf{N}_k\bar{\mathbf{U}}_k = \mathbf{I}_{3\times3} - \boldsymbol{s}_k\boldsymbol{s}_{\boldsymbol{k}}^T \tag{51}$$

thus,

$$\begin{aligned}\mathbf{N}_k &= (\mathbf{I}_{4\times4}-\boldsymbol{q}_k\boldsymbol{q}_k^T)\mathbf{N}_k(\mathbf{I}_{4\times4}-\boldsymbol{q}_k\boldsymbol{q}_k^T)\\ &= \bar{\mathbf{U}}_k\bar{\mathbf{U}}_k^T\mathbf{N}_k\bar{\mathbf{U}}_k\bar{\mathbf{U}}_k^T = \bar{\mathbf{U}}_k(\mathbf{I}_{3\times3}-\boldsymbol{s}_k\boldsymbol{s}_{\boldsymbol{k}}^T)\bar{\mathbf{U}}_k^T\end{aligned} \tag{52}$$

Propositions 1 and 2 allow us to invert $\mathbf{P}_k$ to form a Lyapunov function that, with the foregoing equations, allows us to prove asymptotic convergence from any allowable initial condition in the following Theorem:

Theorem: If

1) The initial Gibbs vector $\boldsymbol{g}_0$ is bounded in magnitude.

2) The initial covariance is chosen proportional to the identity matrix, $\mathbf{P}_0 = \sigma_0^2\mathbf{I}_{3\times3}$.

3) There exists a finite constant integer N and a scalar constant $\alpha > 0$ such that $\alpha\mathbf{I} \leq \mathcal{O}_{k-N}^k$ for all $k \geq N$.

Then the quaternion output of the filter of Eqs. (4) asymptotically converges to the true quaternion in the noise-free case and the covariance output is asymptotically uniformly bounded both above and below.

Proof: The proof will proceed by the following steps:

1) Provide a Lyapunov function in $\boldsymbol{g}_k$ that can be shown to be monotonically decreasing; hence, it provides an upper bound for $\boldsymbol{g}_k$ and approaches a lower bound at which point $\hat{\boldsymbol{g}}_k \to 0$.

2) Show that the observability condition then requires $\boldsymbol{g}_k \to 0$; hence $\boldsymbol{q}_{\boldsymbol{k}} \to \hat{\boldsymbol{q}}_k$.

3) Demonstrate that the asymptotic form of the covariance equations for $\boldsymbol{g}_k = 0$, along with the asymptotic controllability [14] and observability grammians, ensures that the asymptotic covariance is bounded both above and below.

The symbol $\to$ should be taken to signify that *for every $\varepsilon > 0$, there exists a step k at which the norm of the difference of what is on the left and right sides of this symbol is less than ε*. We form the Lyapunov function

$$V_k = \bar{\sigma}(\mathbf{P}_k)\frac{\boldsymbol{g}_k^T\mathbf{P}_k^{-1}\boldsymbol{g}_k}{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k} \geq \bar{\sigma}(\mathbf{P}_k)\underline{\sigma}(\mathbf{P}_k^{-1})\frac{\boldsymbol{g}_k^T\boldsymbol{g}_k}{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k} = \frac{\boldsymbol{g}_k^T\boldsymbol{g}_k}{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k} \tag{53}$$

Under the assumption $\mathbf{P}_0 = \sigma_0^2\mathbf{I}_{3\times3}$ and $\boldsymbol{g}_0$ bounded:

$$V_0 = \frac{\boldsymbol{g}_0^T\boldsymbol{g}_0}{1+\boldsymbol{g}_0\cdot\boldsymbol{g}_0} < 1 \tag{54}$$

Hence, if V_k is strictly nonincreasing, then $\|\boldsymbol{g}_k\| \leq \sqrt{V_0/(1-V_0)}$ for all k.

The Lyapunov function of Eq. (53) is radially unbounded [11,12] as a function of $\boldsymbol{g}_k/\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}$. Hence, if over some interval N,

$$V_k - V_{k-N} < -\gamma\left\|\frac{\boldsymbol{g}_k}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}}\right\|^2$$

for some constant $\gamma > 0$, $\boldsymbol{g}_k/\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}$ is globally asymptotically stable to the origin. It is easiest to evaluate the Lyapunov increments in stages. We begin with

$$V_{k+1}^- = \bar{\sigma}(\mathbf{P}_{k+1}^-)\frac{\boldsymbol{g}_{k+1}^{-T}\mathbf{P}_{k+1}^{-\,-1}\boldsymbol{g}_{k+1}^-}{1+\boldsymbol{g}_{k+1}^-\cdot\boldsymbol{g}_{k+1}^-} \tag{55}$$

Using Eqs. (4b) and (19) and the orthonormality of $\boldsymbol{\Phi}_k^{k+1}$, we find

$$\begin{aligned}V_{k+1}^- &= \bar{\sigma}(\mathbf{P}_k+\sigma_g^2\mathbf{I}_{3\times3})\\ &\times\frac{\boldsymbol{q}_k^T\boldsymbol{\Psi}_{k+1}^k\mathbf{U}(\boldsymbol{\Psi}_k^{k+1}\hat{\boldsymbol{q}}_k)\boldsymbol{\Phi}_k^{k+1}(\mathbf{P}_k+\sigma_g^2\mathbf{I}_{3\times3})^{-1}\boldsymbol{\Phi}_{k+1}^k\mathbf{U}^T(\boldsymbol{\Psi}_k^{k+1}\hat{\boldsymbol{q}}_k)\boldsymbol{\Psi}_k^{k+1}\boldsymbol{q}_k}{(\boldsymbol{q}_k^T\boldsymbol{\Psi}_{k+1}^k\boldsymbol{\Psi}_k^{k+1}\hat{\boldsymbol{q}}_k)^2(1+\boldsymbol{g}_{k+1}^-\cdot\boldsymbol{g}_{k+1}^-)}\\ &= \frac{\boldsymbol{g}_{\boldsymbol{k}}^T}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}}\left(\frac{\mathbf{P}_k+\sigma_g^2\mathbf{I}_{3\times3}}{\bar{\sigma}(\mathbf{P}_k)+\sigma_g^2}\right)^{-1}\frac{\boldsymbol{g}_k}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}}\end{aligned} \tag{56}$$

Hence,

$$\begin{aligned}V_{k+1}^- - V_k &= \frac{\boldsymbol{g}_{\boldsymbol{k}}^T}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}}\left[\left(\frac{\mathbf{P}_k+\sigma_g^2\mathbf{I}_{3\times3}}{\bar{\sigma}(\mathbf{P}_k)+\sigma_g^2}\right)^{-1}\right.\\ &\quad\left.-\left(\frac{\mathbf{P}_k}{\bar{\sigma}(\mathbf{P}_k)}\right)^{-1}\right]\frac{\boldsymbol{g}_k}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}}\\ &= -\sigma_g^2\frac{\boldsymbol{g}_{\boldsymbol{k}}^T}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}}(\mathbf{P}_k^2+\sigma_g^2\mathbf{P}_k)^{-1/2}[\bar{\sigma}(\mathbf{P}_k)\mathbf{I}_{3\times3}-\mathbf{P}_k]\\ &\quad\times(\mathbf{P}_k^2+\sigma_g^2\mathbf{P}_k)^{-1/2}\frac{\boldsymbol{g}_k}{\sqrt{1+\boldsymbol{g}_k\cdot\boldsymbol{g}_k}}\end{aligned} \tag{57}$$

where $(\mathbf{P}_k^2 + \sigma_g^2 \mathbf{P}_k)^{1/2}$ is the symmetric square root of the positive definite symmetric matrix. In this form, $V_{k+1}^- - V_k \leq 0$ is immediately obvious. Because $\bar{\sigma}(\mathbf{P}_k)\mathbf{I}_{3\times3} - \mathbf{P}_k$ is positive semidefinite, for Eq. (57) to approach zero, $(\mathbf{P}_k^2 + \sigma_g^2 \mathbf{P}_k)^{-1/2}\boldsymbol{g}_k$ must approach zero or become a null vector of $\bar{\sigma}(\mathbf{P}_k)\mathbf{I}_{3\times3} - \mathbf{P}_k$. But $(\mathbf{P}_k^2 + \sigma_g^2 \mathbf{P}_k)^{-1/2}$ commutes with $\bar{\sigma}(\mathbf{P}_k)\mathbf{I}_{3\times3} - \mathbf{P}_k$ and hence $\boldsymbol{g}_k$ must approach zero or an eigenvector of $\mathbf{P}_k$ associated with its maximum eigenvalue $\bar{\sigma}(\mathbf{P}_k)$.

The increment $V_k - V_k^-$ is found by invoking Eq. (27). Along with Eq. (28), we find V_k is

$$V_k = \bar{\sigma}(\mathbf{P}_k)\frac{\boldsymbol{g}_k^T \mathbf{P}_k^{-1} \boldsymbol{g}_k}{1 + \boldsymbol{g}_k \cdot \boldsymbol{g}_k} = \bar{\sigma}(\mathbf{P}_k)\frac{(\boldsymbol{g}_k^- - \hat{\boldsymbol{g}}_k)^T \mathbf{P}_k^{+-1}(\boldsymbol{g}_k^- - \hat{\boldsymbol{g}}_k)}{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-} \tag{58}$$

From Eqs. (22) and (24),

$$\begin{aligned}\boldsymbol{g}_k^- - \hat{\boldsymbol{g}}_k &= \left(\mathbf{I}_{3\times3} - \frac{1}{\sigma_m^2}\mathbf{P}_k^+ \mathbf{U}_k^{-T}\mathbf{N}_k \mathbf{U}_k^-\right)\boldsymbol{g}_k^- \\ &= \left(\mathbf{I}_{3\times3} + \frac{1}{\sigma_m^2}\mathbf{P}_k^- \mathbf{U}_k^{-T}\mathbf{N}_k \mathbf{U}_k^-\right)^{-1}\boldsymbol{g}_k^-\end{aligned} \tag{59}$$

and so we find

$$V_k = \bar{\sigma}(\mathbf{P}_k)\frac{\boldsymbol{g}_k^{-T}[\mathbf{P}_k^- + (1/\sigma_m^2)\mathbf{P}_k^- \mathbf{U}_k^{-T}\mathbf{N}_k \mathbf{U}_k^- \mathbf{P}_k^-]^{-1}\boldsymbol{g}_k^-}{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-} \tag{60}$$

The increment is thus equal to

$$\begin{aligned}V_k - V_k^- = \frac{\boldsymbol{g}_k^{-T}}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}}&\left[\bar{\sigma}(\mathbf{P}_k)\left(\mathbf{P}_k^- + \frac{1}{\sigma_m^2}\mathbf{P}_k^- \mathbf{U}_k^{-T}\mathbf{N}_k \mathbf{U}_k^- \mathbf{P}_k^-\right)^{-1}\right. \\ &\left. - \bar{\sigma}(\mathbf{P}_k^-)\mathbf{P}_k^{-^{-1}}\right]\frac{\boldsymbol{g}_k^-}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}}\end{aligned} \tag{61}$$

Because $\bar{\sigma}(\mathbf{P}_k) \leq \bar{\sigma}(\mathbf{P}_k^-)$, and taking into account Eq. (27) once again, along with Eqs. (47–52), we obtain

$$\begin{aligned}V_k - V_k^- \leq \frac{\bar{\sigma}(\mathbf{P}_k^-)}{\sigma_m^2}\frac{\boldsymbol{g}_k^{-T}}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}}&\left[\left(\mathbf{P}_k^- + \frac{1}{\sigma_m^2}\mathbf{P}_k^- \mathbf{U}_k^{-T}\mathbf{N}_k \mathbf{U}_k^- \mathbf{P}_k^-\right)^{-1}\right. \\ \left. - \mathbf{P}_k^{-^{-1}}\right]\frac{\boldsymbol{g}_k^-}{\sqrt{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}} &\leq -\frac{\bar{\sigma}(\mathbf{P}_k^-)}{\sigma_m^2}\left(\frac{1 + \hat{\boldsymbol{g}}_k \cdot \hat{\boldsymbol{g}}_k}{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}\right) \\ \times \frac{\boldsymbol{g}_k^T}{\sqrt{1 + \boldsymbol{g}_k \cdot \boldsymbol{g}_k}}(\mathbf{I}_{3\times3} - \boldsymbol{s}_k \boldsymbol{s}_k^T)\frac{\boldsymbol{g}_k}{\sqrt{1 + \boldsymbol{g}_k \cdot \boldsymbol{g}_k}}&\end{aligned} \tag{62}$$

By adding Eq. (57) to a one-step-advanced version of Eq. (62), it is seen that V_k is a monotonically decreasing sequence and, as such, must approach a limit [13] in which $V_{k+1}^- - V_k$ and $V_k - V_k^-$ both go to zero. We conclude that $\boldsymbol{g}_k$ must approach $\lambda_k \boldsymbol{s}_k$ for some scalar parameter λ_k, and $\boldsymbol{g}_k$ must approach a maximal eigenvector of $\mathbf{P}_k$. However, if $\boldsymbol{g}_k \to \lambda_k \boldsymbol{s}_k$ is a maximal eigenvector of $\mathbf{P}_k$, then

$$V_k \to \frac{\lambda_k^2}{1 + \lambda_k^2} \tag{63}$$

but $V_k \to$ a constant, which implies that $\lambda_k \to \lambda$, a constant.

If $\boldsymbol{g}_k \to \lambda \boldsymbol{s}_k$, then from Eq. (46), we must have $\boldsymbol{g}_k^- \to \lambda \boldsymbol{\Phi}_{k-1}^k \boldsymbol{s}_{k-1}$. Furthermore, if $[\bar{\sigma}(\mathbf{P}_k)\mathbf{I}_{3\times3} - \mathbf{P}_k]\boldsymbol{g}_k \to 0$, it is not difficult, using Eqs. (4b) and (46), to deduce that $[\bar{\sigma}(\mathbf{P}_k^-)\mathbf{I}_{3\times3} - \mathbf{P}_k^-]\boldsymbol{g}_k^- \to 0$ (i.e., that $\boldsymbol{g}_k^-$ approaches a maximal eigenvector of $\mathbf{P}_k^-$). From Eqs. (29), (47), and (49), then

$$\begin{aligned}\boldsymbol{g}_k &\to \frac{\bar{\sigma}(\mathbf{P}_k)}{\bar{\sigma}(\mathbf{P}_k^-)}(\mathbf{U}_k^T \mathbf{U}_k^-)^{-T}\boldsymbol{g}_k^- \\ &= \frac{\bar{\sigma}(\mathbf{P}_k)}{\bar{\sigma}(\mathbf{P}_k^-)}\sqrt{\frac{1 + \boldsymbol{g}_k^- \cdot \boldsymbol{g}_k^-}{1 + \boldsymbol{g}_k \cdot \boldsymbol{g}_k}}\frac{\boldsymbol{g}_k^- + (\boldsymbol{g}_k^- \cdot \hat{\boldsymbol{g}}_k)\hat{\boldsymbol{g}}_k + \boldsymbol{g}_k^- \times \hat{\boldsymbol{g}}_k}{1 + \boldsymbol{g}_k^- \cdot \hat{\boldsymbol{g}}_k} \\ &\to \frac{\bar{\sigma}(\mathbf{P}_k)}{\bar{\sigma}(\mathbf{P}_k^-)}\frac{\boldsymbol{g}_k^- + (\boldsymbol{g}_k^- \cdot \hat{\boldsymbol{g}}_k)\hat{\boldsymbol{g}}_k + \boldsymbol{g}_k^- \times \hat{\boldsymbol{g}}_k}{1 + \boldsymbol{g}_k^- \cdot \hat{\boldsymbol{g}}_k}\end{aligned} \tag{64}$$

For

$$\boldsymbol{g}_k = \boldsymbol{g}_k^- \circ \hat{\boldsymbol{g}}_k^{-1} = \frac{\boldsymbol{g}_k^- - \hat{\boldsymbol{g}}_k + \boldsymbol{g}_k^- \times \hat{\boldsymbol{g}}_k}{1 + \boldsymbol{g}_k^- \cdot \hat{\boldsymbol{g}}_k} \tag{65}$$

to approach Eq. (64), it is necessary (carry out the dot product with $\boldsymbol{g}_k^- \times \hat{\boldsymbol{g}}_k$ on both equations) that

$$\frac{\bar{\sigma}(\mathbf{P}_k)}{\bar{\sigma}(\mathbf{P}_k^-)} \to 1 \tag{66}$$

and $\boldsymbol{g}_k^- \cdot \hat{\boldsymbol{g}}_k \to -1$, or $\hat{\boldsymbol{g}}_k \to 0$. But from Eqs. (22) and (24),

$$\begin{aligned}\boldsymbol{g}_k^- \cdot \hat{\boldsymbol{g}}_k &= \frac{1}{\sigma_m^2}\boldsymbol{g}_k^{-T}\mathbf{P}_k^+ \mathbf{U}_k^{-T}\mathbf{N}_k \mathbf{U}_k^- \boldsymbol{g}_k^- = \boldsymbol{g}_k^{-T}(\mathbf{I}_{3\times3} - \mathbf{P}_k^+ \mathbf{P}_k^{--1})\boldsymbol{g}_k^- \\ &\to \boldsymbol{g}_k^{-T}\left(\mathbf{I}_{3\times3} - \frac{1}{\bar{\sigma}(\mathbf{P}_k^-)}\mathbf{P}_k^+\right)\boldsymbol{g}_k^- > 0\end{aligned} \tag{67}$$

and so we must have $\hat{\boldsymbol{g}}_k \to 0$. Thus, we approach a condition in which $\boldsymbol{g}_k \to \boldsymbol{\Phi}_{k-1}^k \boldsymbol{g}_{k-1}$. We claim that this implies $\boldsymbol{g}_k \to 0$, due to the observability requirement $\alpha\mathbf{I} \leq \mathcal{O}_{k-N}^k$.

Suppose not. Then $\hat{\boldsymbol{g}}_k \to 0$ implies

$$V_k - V_k^- \to -\gamma_k \tag{68}$$

where, from Eq. (62), because $\bar{\sigma}(\mathbf{P}_m^-) \geq \sigma_g^2$

$$\gamma_k \geq \frac{\sigma_g^2}{\sigma_m^2(1 + \lambda^2)^2}\boldsymbol{g}_k^T(\mathbf{I}_{3\times3} - \boldsymbol{s}_k \boldsymbol{s}_k^T)\boldsymbol{g}_k \tag{69}$$

Thus,

$$\sum_{i=k-N}^{k}(V_i - V_i^-) \to -\sum_{i=k-N}^{k}\gamma_i \tag{70}$$

and using the observability requirement $\alpha\mathbf{I} \leq \mathcal{O}_{k-N}^k$,

$$\begin{aligned}\sum_{i=k-N}^{k}\gamma_i &\geq \frac{\sigma_g^2}{\sigma_m^2(1 + \lambda^2)^2}\sum_{i=k-N}^{k}\boldsymbol{g}_i^T(\mathbf{I}_{3\times3} - \boldsymbol{s}_i \boldsymbol{s}_i^T)\boldsymbol{g}_i \\ &\to \frac{\sigma_g^2}{\sigma_m^2(1 + \lambda^2)^2}\boldsymbol{g}_k^T\left[\sum_{i=k-N}^{k}\boldsymbol{\Phi}_i^k(\mathbf{I}_{3\times3} - \boldsymbol{s}_i \boldsymbol{s}_i^T)\boldsymbol{\Phi}_k^i\right]\boldsymbol{g}_k \\ &= \frac{\sigma_g^2}{(1 + \lambda^2)^2}\boldsymbol{g}_k^T \mathcal{O}_{k-N}^k \boldsymbol{g}_k \geq \frac{\alpha\sigma_g^2\lambda^2}{(1 + \lambda^2)^2}\end{aligned} \tag{71}$$

But $\gamma_k \to 0$, which implies that $\lambda = 0$; therefore, if $\boldsymbol{g}_k$ does not approach zero, there is a contradiction. Hence, $\boldsymbol{g}_k \to 0$ and $\boldsymbol{q}_k \to \hat{\boldsymbol{q}}_k$.

Because $\hat{\boldsymbol{g}}_k \to 0$, the asymptotic form of the covariance update of Eq. (28) is

$$\mathbf{P}_k \to \left[\mathbf{P}_k^{-^{-1}} + \frac{1}{\sigma_m^2}(\mathbf{I}_{3\times3} - \boldsymbol{s}_k \boldsymbol{s}_k^T)\right]^{-1} \tag{72}$$

This covariance update corresponds to the system for which the observability grammian is Eq. (31), which is clearly bounded above for a finite sum, and so

$$\alpha\mathbf{I} \leq \mathcal{O}_{k-N}^k \leq \frac{N}{\sigma_m^2}\mathbf{I} \tag{73}$$

The covariance extrapolation of Eq. (4b) coupled with Eq. (72) corresponds to a system for which the controllability grammian $\mathcal{C}_n^k$ [14] is trivially bounded above and below over a finite sum, because the dimension of the noise source is the same as the dimension of the state:

$$\sigma_g^2 N\mathbf{I} \leq \mathcal{C}_{k-N}^k \leq \sigma_g^2 N\mathbf{I} \tag{74}$$

Hence, according to [14], the covariance $\mathbf{P}_k$ is asymptotically bounded above and below by

$$\left(\frac{\sigma_m^2\sigma_g^2}{\sigma_m^2/N+\sigma_g^2 N}\right)\mathbf{I} \le \left(\mathcal{C}_{k-N}^{k^{-1}} + \mathcal{O}_{k-N}^{k}\right)^{-1} \le \mathbf{P}_k \le \mathcal{O}_{k-N}^{k^{-1}} + \mathcal{C}_{k-N}^{k} \le \left(\frac{1}{\alpha}+\sigma_g^2 N\right)\mathbf{I} \tag{75}$$

V. Comparison with Extended Kalman Filter

An extended Kalman filter mechanization for the preceding filtering problem starts by transforming the inertial unit vector $\boldsymbol{r}_k$ into body space via the transformation matrix from the propagated quaternion. The cross product of this with $\boldsymbol{s}_k$ then forms the approximate measurement error equation:

$$\boldsymbol{s}_k \times \mathbf{A}(\hat{\boldsymbol{q}}_k^-)\boldsymbol{r}_k \approx (\mathbf{I}_{3\times3} - \boldsymbol{s}_k\boldsymbol{s}_k^T)\boldsymbol{\xi}_k \tag{76}$$

where, for an arbitrary unit quaternion $\boldsymbol{q} = (\boldsymbol{Q}, q)$,

$$\mathbf{A}(\boldsymbol{q}) = \mathbf{I}_{3\times3} + 2\,\mathrm{skew}(\boldsymbol{Q})[\mathrm{skew}(\boldsymbol{Q}) - q\mathbf{I}_{3\times3}] \tag{77}$$

and $\boldsymbol{\xi}_k$ is a small-angle error vector between $\hat{\boldsymbol{q}}_k^-$ and $\boldsymbol{q}_k$; that is, approximately

$$\boldsymbol{q}_k \sim (\tfrac{1}{2}\boldsymbol{\xi}_k, 1) \circ \hat{\boldsymbol{q}}_k^- \tag{78}$$

using the quaternion composition rule of Eq. (5). An extended Kalman filter, or EKF, could then be implemented as follows:

State and covariance propagation:

$$\hat{\boldsymbol{q}}_{k+1}^- = \boldsymbol{\Psi}_k^{k+1}\hat{\boldsymbol{q}}_k \tag{79a}$$

$$\mathbf{P}_{k+1}^- = \boldsymbol{\Phi}_k^{k+1}\mathbf{P}_k\boldsymbol{\Phi}_{k+1}^k + \sigma_g^2\mathbf{I}_{3\times3} \tag{79b}$$

State and covariance measurement update:

$$\mathbf{P}_k = \left[\mathbf{I}_{3\times3} + \frac{1}{\sigma_m^2}\mathbf{P}_k^-(\mathbf{I}_{3\times3} - \boldsymbol{s}_k\boldsymbol{s}_k^T)\right]^{-1}\mathbf{P}_k^- \tag{79c}$$

$$\hat{\boldsymbol{\xi}}_k = \frac{1}{\sigma_m^2}\mathbf{P}_k[\boldsymbol{s}_k \times \mathbf{A}(\hat{\boldsymbol{q}}_k^-)\boldsymbol{r}_k] \tag{79d}$$

$$\hat{\boldsymbol{q}}_k \sim (\tfrac{1}{2}\hat{\boldsymbol{\xi}}_k, 1) \circ \hat{\boldsymbol{q}}_k^- \tag{79e}$$

From the discussion of Sec. IV, it is apparent that the EKF of Eqs. (79) are an approximation to the filter of Eqs. (4) when $\mathbf{U}_k \approx \mathbf{U}_k^- \approx \bar{\mathbf{U}}_k$.

For the simulation results shown next, the spacecraft rate was selected to be -0.06 deg/s about the $[0 \quad 1 \quad 0]$ or pitch axis, with noisy gyroscope measurements contributing a propagation error of $\sigma_g = 0.001$ deg independently on each axis per 1-s update interval. Measurements were taken at the update interval of the $[0 \quad 0 \quad 1]$ or yaw axis in body space corrupted by noise on the order of $\sigma_m = 1$ deg about each cross axis. Thus, the simulation would be representative of an Earth-pointing spacecraft in low-Earth orbit using a horizon sensor for attitude. The initial covariances were set to the 3×3 identity matrix times 4 times the sine squared of the initial half-angular error, in accordance with Eq. (21) with the appropriate substitutions: that is,

$$\mathbf{P}_0 = 4E\left\{\frac{\boldsymbol{g}_0\boldsymbol{g}_0^T}{1+\boldsymbol{g}_0\cdot\boldsymbol{g}_0}\right\} \tag{80}$$

The slowest convergence rate is naturally on the yaw axis. In Fig. 2, we see the absolute value of the yaw estimation error for each filter when the initial attitude errors were identically 5 deg on each axis, and the associated one-sigma uncertainties in the estimates are the square roots of the (3,3) elements of the covariance estimates. The result for the filter of Eqs. (4) is denoted by S^3, referring to the unit quaternion manifold S^3. In general, the filter estimation errors and their estimated uncertainties are virtually identical, as might be expected because the two algorithms are approximately equivalent for small angles. In Fig. 3, however, when the initial error is 90 deg about each axis, the performance difference between the two filters is stark, with the EKF failing adequately to converge within the timeline of the simulation. It is partly because the EKF fails initially to converge and the norm of the covariance estimate quickly settles out to a small value that the EKF is unable to converge rapidly after the startup transient.

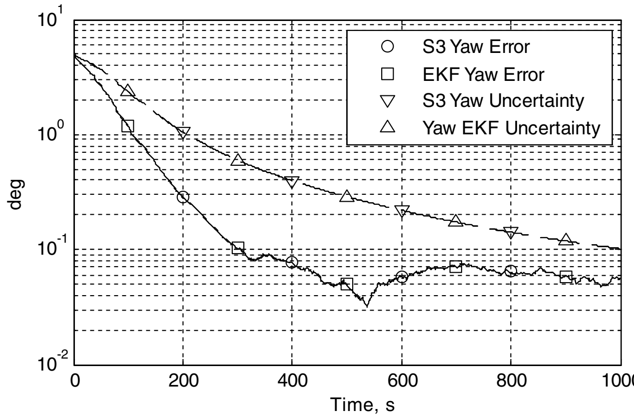

Fig. 2 Comparison of estimation errors for small initial errors.

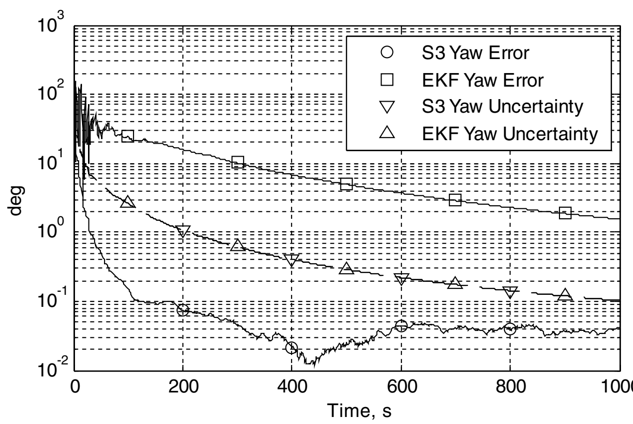

Fig. 3 Comparison of estimation errors for large initial errors.

VI. Conclusions

By use of a linear measurement model in the tangent space of the current attitude estimate and a mapping of covariance tensors from one tangent space to the next, we formulated an attitude filter algorithm with robust asymptotic convergence from almost any initial condition. We demonstrated the superiority of the approach to a standard extended Kalman filter, which does not employ a linear measurement model or consistently map the covariance between the tangent spaces and which performs poorly with large initial attitude errors.

The convergence of the filter algorithm in the noise-free case is absolutely guaranteed by Lyapunov analysis, whereas such convergence properties are not generally guaranteed for most nonlinear filters applied to this particular problem. Future investigations should focus on extending the formulation for more complicated gyroscope error models including biases, scale factors, and misalignments.

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