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New State Update Equation for the Unscented Kalman Filter

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I. Introduction

A FUNDAMENTAL tenet of a Kalman–Bucy filter, usually referred to as the Kalman filter [1–3] (KF), is that the system to which it is applied can be described through a linear model. The simplicity of its formulation is so powerful that its concept has been extended to nonlinear systems in almost every imaginable application [3,4]. The unscented Kalman filter [5] (UKF) is among the latest and most compelling extensions of Kalman-type filters, particularly for systems that are markedly nonlinear. Despite its allure, however, the UKF has also been shown to diverge for highly nonlinear systems [6]. In the following sections, we first review both the KF and the UKF and then present a new formulation for the state update equation of the latter. In the theoretical derivation of the new coefficients for the new update equation, we consider several approximations. The error that arises from these approximations is compared with the error that arises from truncating the UKF update equation to the first order. Numerical simulations of a simple nonlinear model relevant to formation flying configurations show that the improved UKF can be several orders of magnitude more accurate than the standard UKF.

II. Kalman–Bucy Filter

To introduce the KF, let us consider a bidimensional linear system of Gaussian random variables x_k and z_k such that

$$x_k = a_{k-1}x_{k-1} + \Gamma_{k-1}v_{k-1} \tag{1}$$

$$z_k = A_k x_k + w_k \tag{2}$$

where subscript k denotes evaluation at time t_k; x_k is an n-dimensional state vector; $a_k \in \mathbb{R}^{n\times n}$ is a nonsingular state transition matrix; $\Gamma_k \in \mathbb{R}^{n\times r}$ is a deterministic matrix; $v_k \sim N_r(0, Q_k)$ is a state noise process vector, with v_j and v_k independent if $j \neq k$; z_k is an observation vector of dimension m; $A_k \in \mathbb{R}^{m\times n}$ is a deterministic matrix; and $w_k \sim N_m(0, R_k)$ is a measurement noise vector, with w_j and w_k independent if $j \neq k$. In addition, the state process x_0 is Gaussian and independent of z_0, and the noise processes v_k and w_k are mutually independent.

The KF is an optimal (in the sense of a minimum mean-square error) linear estimator of x_k conditioned to the σ-algebra of past observations $\mathcal{F}^z_j$, $j \in \{k-1, k\}$.

Theorem 1. [Kalman–Bucy] Let $t_{k-1} < t_k$, $k \geq 1$ be the epochs of two consecutive observations z_{k-1} and z_k. We define $\hat{x}^-_k := E(x_k|\mathcal{F}^z_{k-1})$ and $P^-_k := E((x_k - \hat{x}^-_k)^2|\mathcal{F}^z_{k-1})$, $\hat{x}^+_k := E(x_k|\mathcal{F}^z_k)$ and $P^+_k := E((x_k - \hat{x}^+_k)^2|\mathcal{F}^z_k)$. Then,

1) Between observations, the optimal estimate conditioned to the observed process is

$$\hat{x}^-_k = a_{k-1}\hat{x}^+_{k-1} \tag{3}$$

$$P^-_k = a_{k-1}P^+_{k-1}a^T_{k-1} + \Gamma_{k-1}Q_k\Gamma^T_{k-1} \tag{4}$$

2) At the observation epoch, the optimal state estimate is

$$\hat{x}^+_k = \hat{x}^-_k + K_k(z_k - A_k\hat{x}^-_k) \tag{5}$$

$$P^+_k = P^-_k - K_kA_kP^-_k \tag{6}$$

where $K_k = P^-_kA^T_k[A_kP^-_kA^T_k + R_k]^{-1}$ is the Kalman gain matrix. □

Equations (3) and (4) are known as propagation equations and (5) and (6) as update equations. These estimates are optimal only if both the process and observation models are linear. Although the estimates are not optimal for nonlinear systems, the results can be extended to nonlinear models by linearizing them using a Taylor expansion and making some general hypotheses about the random-noise probability distributions.

Many authors [1,2] have used the coincidence of the term $A_k\hat{x}^-_k$ in Eq. (5) with the expectation of the observation process $\hat{z}_k = E(z_k|z_{k-1})$ to design extensions of this filter that take into account higher-order moments of the probability distribution functions, or higher-order terms of the Taylor series expansion of the models, or both. Alternatively, the term $A_k\hat{x}^-_k$ can be also interpreted as the noiseless observation corresponding to the state $\hat{x}^-_k$, that is, after Eq. (2). These two interpretations are fully equivalent for linear models, but they are not when the models are not linear. This different interpretation of Eq. (5) will be used to define a different state update equation.

III. Unscented Kalman Filter

Building on the formulation of the KF, Julier and Uhlmann [5] developed the UKF for the following (nonlinear) system:

$$x_k = f(x_{k-1}, v_{k-1}) \tag{7}$$

$$z_k = h(x_k) + w_k \tag{8}$$

where f is the dynamic model and h is the measurement model, and both are almost linear functions (i.e., $\|\frac{\partial f}{\partial x}\|$, $\|\frac{\partial h}{\partial x}\| \ll \epsilon$). The UKF incorporates quite ingeniously multiple moments of the probability

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density function and higher-order terms of the Taylor series expansion of the dynamic and measurement models using an unscented transformation [7], thus increasing its accuracy over the KF.

A. Unscented Transformation

The purpose of the unscented transformation is to propagate the mean and covariance of a process through nonlinear functions using a set of "well chosen" points known as "sigma points." As stated in [5], the unscented transformation builds on the principle that "it is easier to approximate a probability distribution than it is to approximate an arbitrary nonlinear function."

The unscented transformation operates on a random variable x, with mean $\hat{x}$ and covariance P_x, and a second random variable y, related to x through the nonlinear transformation $y = g(x)$, as follows. First, it selects a set of p (sigma) points around $\hat{x}$ and a set of $p + 1$ weights (i.e., including one for $\hat{x}$). Second, it computes the image of the p points through g. And third, it calculates the statistics $\hat{y}$ and P_y using this set of point images and their weights. Several strategies exist for the selection of the sigma points and weights such as the minimal skew simplex points [8] and the scaled sigma set [9]. Depending on the number of points and their spatial distribution around $\hat{x}$, this set may retain the information of several moments of the random variable x. The transformation resembles Monte Carlo simulation methods. However, whereas in Monte Carlo simulations the statistics of y are calculated using a large set of samples, in an unscented transformation the y statistics are calculated as the weighted image of a reduced set of "well chosen" sigma points p. (For an n-dimensional variable, a typical choice for p is $2n$.)

B. Linear Approximation in the Unscented Kalman Filter Update Equations

The basis of the UKF is the unscented transformation, which is applied through the dynamic model f, in the propagation step, and through the measurement model h, in the update step. We summarize the implementation of the UKF update equations as an introductory step to the improved UKF that will be presented.

Borrowing from the form of the KF update equations (5) and (6), the UKF approximates the conditional expectation $\hat{x}_k^+ = E(x_k|\mathcal{F}_k^z)$ as a power series in measurement residuals (or innovations) of the form $z_k - \hat{z}_k$ truncated at the first order for computational tractability. The updated estimate covariance is assumed to be independent of the residuals. Thus, the UKF update equations are as follows:

$$\hat{x}_k^+ = B_0 + B_1(z_k - \hat{z}_k) \tag{9}$$

$$P_k^+ = B_2 \tag{10}$$

where B_0, B_1, and B_2 are deterministic vectors and matrices (at t_k). Their values are determined by imposing that

$$E\{x_k|z_{k-1}\} = E\{\hat{x}_k^+|z_{k-1}\} \tag{11}$$

$$E\{x_k(z_k - \hat{z}_k)^T|z_{k-1}\} = E\{\hat{x}_k^+(z_k - \hat{z}_k)^T|z_{k-1}\} \tag{12}$$

$$E\{(x_k - \hat{x}_k^+)(x_k - \hat{x}_k^+)^T|z_{k-1}\} = E\{P_k^+|z_{k-1}\} \tag{13}$$

which can be demonstrated to result in $B_0 = \hat{x}_k^-$, $B_1 = P_{xz,k}P_{zz,k}^{-1}$, and $B_2 = P_{k-1}^- - P_{xz,k}P_{zz,k}^{-1}P_{xz,k}^T$, where

$$P_{xz,k} := E\{x_k(z_k - \hat{z}_k)^T|z_{k-1}\} \tag{14}$$

$$P_{zz,k} := E\{(z_k - \hat{z}_k)(z_k - \hat{z}_k)^T|z_{k-1}\} \tag{15}$$

(Further details on the derivation of the B coefficients can be found in [2] and the references therein.) The resulting update equations are as follows:

$$\hat{x}_k^+ = \hat{x}_k^- + P_{xz,k}P_{zz,k}^{-1}(z_k - \hat{z}_k) \tag{16}$$

$$P_k^+ = P_k^- - P_{xz,k}P_{zz,k}^{-1}P_{xz,k}^T \tag{17}$$

Thus, Eqs. (16) and (17) coincide with the KF update equations (5) and (6) in the case of linear systems. The novelty introduced by the UKF lies in the way the terms involved in these equations are computed and in the application of the unscented transformation through the measurement model h to extend the KF to nonlinear systems.

IV. New Formulation of the Measurement Residuals for an Improved Unscented Kalman Filter

We present a different formulation for calculating the measurement residuals while keeping the linear form of the UKF update equations. As discussed, the unscented transformation incorporates high-order terms of the nonlinear measurement model h in the computation of $\hat{z}_k$. However, the state estimate update of the UKF, Eq. (16), is a linear approximation and, thus, does not incorporate the high-order terms of h or of the measurement distribution z. Alternatively, if the measurement residuals are computed as $z_k - h(\hat{x}_k^-) \approx z_k - \hat{z}_k + \frac{1}{2}\frac{\partial^2 h(\hat{x}_k^-)}{\partial x^2}P_k^-$, the second-order terms of the measurement model are incorporated into the state estimate update as part of the residuals.

Paralleling the derivation leading to Eqs. (16) and (17), we now derive a new filter by assuming that the state update equation can be approximated by a power series expansion on $z_k - h(\hat{x}_k^-)$ instead of on $z_k - \hat{z}_k$. Similar to (9) and (10), the linear approximation of the new update equations are as follows:

$$\hat{x}_k^+ = B_0^* + B_1^*(z_k - h(\hat{x}_k^-)) \tag{18}$$

$$P_k^+ = B_2^* \tag{19}$$

and, for simplicity, we will assume that the coefficients B_0^*, B_1^*, and B_2^* are B_0, B_1, and B_2, respectively. We evaluate in turn both sides of Eqs. (11–13) to assess the error introduced by the new state update equations:

1) Error in the verification of Eq. (11):

$$E\{x_k|z_{k-1}\} = \hat{x}_k^- \tag{20}$$

$$\begin{aligned} E\{\hat{x}_k^+|z_{k-1}\} &= E\{\hat{x}_k^- + P_{xz,k}P_{zz,k}^{-1}[z_k - h(\hat{x}_k^-)]|z_{k-1}\} \\ &= \hat{x}_k^- + P_{xz,k}P_{zz,k}^{-1}E\{z_k - h(\hat{x}_k^-)|z_{k-1}\} \\ &= \hat{x}_k^- + P_{xz,k}P_{zz,k}^{-1}E\{z_k - \hat{z}_k + \hat{z}_k - h(\hat{x}_k^-)|z_{k-1}\} \\ &= \hat{x}_k^- + P_{xz,k}P_{zz,k}^{-1}(\hat{z}_k - h(\hat{x}_k^-)) \\ &= \hat{x}_k^- + \mathbf{P_{xz,k}P_{zz,k}^{-1}}\left(\frac{\mathbf{1}}{\mathbf{2}}\frac{\partial^2\mathbf{h}(\hat{\mathbf{x}}_\mathbf{k}^-)}{\partial\mathbf{x}^2}\mathbf{P_k^-} + \mathcal{O}(\mathbf{3})\right) \end{aligned} \tag{21}$$

Thus, the expectation of both expressions is the same to the first order, and their difference is at the second-order level (i.e., the second derivative of h and the second-order moments of the distribution of x), which is comparable with the error introduced by the linear truncation of the UKF state update equation.

2) Error in the verification of Eq. (12):

$$E(x_k(z_k - h(\hat{x}_k^-))^T|z_{k-1}) = P_{xz} + \hat{x}_k^-(\hat{z}_k - h(\hat{x}_k^-))^T \tag{22}$$

$$
\begin{aligned}
&E\{\hat{x}_k^+(z_k - h(\hat{x}_k^-))^T|z_{k-1}\} = \hat{x}_k^-(\hat{z}_k - h(\hat{x}_k^-))^T \\
&+ P_{xz}P_{zz}^{-1}E((z_k - h(\hat{x}_k^-))(z_k - h(\hat{x}_k^+))^T|z_{k-1}\} \\
&= \hat{x}_k^-(\hat{z}_k - h(\hat{x}_k^-))^T + P_{xz}P_{zz}^{-1}(P_{zz} + (\hat{z}_k - h(\hat{x}_k^-))(\hat{z}_k - h(\hat{x}_k^-))^T) \\
&= P_{xz} + \hat{x}_k^-(\hat{z}_k - h(\hat{x}_k^-))^T + \frac{1}{4}\mathbf{P_{xz}P_{zz}^{-1}} \\
&\cdot \left(\frac{\partial^2 \mathbf{h}(\hat{\mathbf{x}}_\mathbf{k}^-)}{\partial \mathbf{x}^2}(\mathbf{P_k^-})^2 \frac{\partial^2 \mathbf{h}(\hat{\mathbf{x}}_\mathbf{k}^-)^\mathbf{T}}{\partial \mathbf{x}^2} + \mathcal{O}(\mathbf{3})\right)
\end{aligned} \tag{23}
$$

As before, the expectation of both expressions is the same to the first order.

3) Error in the verification of Eq. (13):

$$
\begin{aligned}
&E\{(x_k - \hat{x}_k^+)(x_k - \hat{x}_k^+)^T|z_{k-1}\} \\
&= E\{[x_k - \hat{x}_k^- - P_{xz}P_{zz}^{-1}(z_k - h(\hat{x}_k^-))] \\
&\cdot [x_k - \hat{x}_k^- - P_{xz}P_{zz}^{-1}(z_k - h(\hat{x}_k^-))]^T|z_{k-1}\} = P_k^- - P_{xz,k}P_{zz,k}^{-1}P_{xz,k}^T \\
&+ \{P_{xz}P_{zz}^{-1}(\hat{z}_k - h(\hat{x}_k^-))(\hat{z}_k - h(\hat{x}_k^-))^T P_{zz}^{-1}P_{xz}^T\} \\
&= P_k^- - P_{xz,k}P_{zz,k}^{-1}P_{xz,k}^T + \frac{1}{4}\mathbf{P_{xz}P_{zz}^{-1}} \\
&\cdot \left(\frac{\partial^2 \mathbf{h}(\hat{\mathbf{x}}_\mathbf{k}^-)}{\partial \mathbf{x}^2}(\mathbf{P_k^-})^2 \frac{\partial^2 \mathbf{h}(\hat{\mathbf{x}}_\mathbf{k}^-)^\mathbf{T}}{\partial \mathbf{x}^2}\right)\mathbf{P_{zz}^{-1}P_{xz}^T}
\end{aligned} \tag{24}
$$

$$
E\{P_k^+|z_{k-1}\} = P_k^- - P_{xz,k}P_{zz,k}^{-1}P_{xz,k}^T \tag{25}
$$

As before, the expectation of both expressions is identical up to the second-order terms.

In summary, the error introduced by assuming B_0^*, B_1^*, and B_2^* as B_0, B_1, and B_2, respectively, in the new state update equation is second order, and so is the truncation error of the update. Using this approximation, the filter update equations can be written as follows:

$$
\hat{x}_k^+ = \hat{x}_k^- + P_{xz,k}P_{zz,k}^{-1}(z_k - h(\hat{x}_k^-)) \tag{26}
$$

$$
P_k^+ = P_k^- - P_{xz,k}P_{zz,k}^{-1}P_{xz,k}^T \tag{27}
$$

where the computation of each of these terms can be done via the unscented transformation. For identification purposes, we label the UKF that uses the new update equations the improved UKF (IUKF). Both these filters become identical when the observation models are linear. The difference between the UKF and IUKF lies in the measurement residuals, that is, Eq. (26). Although differences between the UKF and IUKF are minor, the latter builds on different approximations of the update equation, that is, on the power series expansion with respect to different terms. We will show that the performance of this improved filter for a simple nonlinear model is superior to that of the standard UKF, thus demonstrating that the update process of the IUKF may effectively incorporate higher-order terms.

V. Origins of the Improved Unscented Kalman Filter

The modification to the UKF described herein was inspired by the excellent results of the UKF modification presented in [6] and labeled UFKz. In that paper, the performance of several filters were compared using numerical simulations, which showed that the stability and the positioning accuracy of the UKFz were significantly better than those of the UKF. The UKFz strategy approximated $\hat{z}_k$ by $h(\hat{x}_k^-)$, which not only modified the computation of the measurement residuals, but also the computation of the matrix

$$
P_{zz,k} = \sum_{0}^{p+1} W_i(\mathcal{Z}_i - \hat{z}_k)(\mathcal{Z}_i - \hat{z}_k)^T
$$

where W_i are the weights associated to the sigma points $\mathcal{Z}_i$. As a result, the innovation covariances of the UKFz are larger than those of the UKF, because the two covariances can be shown to be approximately related by

$$
P_{zz,k}^{\mathrm{UKFz}} \approx P_{zz,k}^{\mathrm{UKF}} + \frac{1}{4}\left\{\frac{\partial^2 h}{\partial x^2}(P_k^-)^2\frac{\partial^2 h^T}{\partial x^2}\right\}
$$

Unlike the UKFz case, the covariances $P_{zz,k}^{\mathrm{IUKF}}$ and $P_{zz,k}^{\mathrm{UKF}}$ are identical.

The additional term in the covariance of the UKFz with respect to that of the UKF acts as a small "bump up" of the observation noise level. Simulations suggest that this additional term is not necessary in practice (as will be shown). Although bump-up strategies can be very effective for avoiding divergence in certain types of filters (e.g., the extended Kalman filter, or EKF), they reduce the convergence speed of the uncertainty matrix P_k^+. Thus, their implementation should be considered only when there exists a significant probability of divergence or when the convergence speed of the state estimate and the convergence speed of the uncertainty matrix is unbalanced [6].

VI. Numerical Experiment

To assess the accuracy of the new filter, we compared the performance of the UKF and IUKF for a highly nonlinear measurement model system using numerical simulations. The experiment involves the estimation of a fixed position in Cartesian coordinates $x = (x_1, x_2)$ in $\mathbb{R}^2$ (i.e., a system with no dynamics, which can be defined as $\dot{x} = f(t, x) = 0$ and $Q \equiv 0$). This scenario effectively isolates the prediction phase from the update phase, which facilitates the comparison between these filters because the proposed modification only affects the state update equation. The nonlinear measurement model considered contains two observables: distance $r = \sqrt{x_1^2 + x_2^2}$ and azimuth $\theta = \arctan(x_2/x_1)$. This experiment has often been used to evaluate the performance of the EKF [6,10–12]. It is particularly well suited for filter evaluation because of its simplicity and nonlinearity.

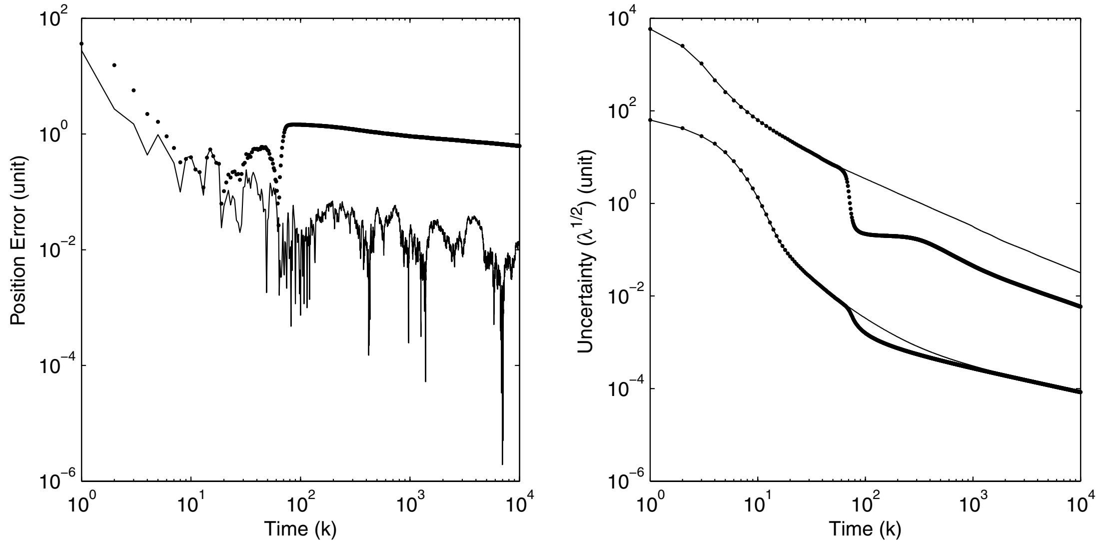
In these simulations, we used the following set of values for the various parameters necessary to initialize and run the filters: the true position $\tilde{x}_{\mathrm{true}} = (100, 100)$ units, the covariance matrix of the measurement noise R is diagonal with a radial noise variance σ_r^2 of $2.5\mathrm{E}-5$ square units and an angular noise variance σ_θ^2 of $6\mathrm{E}-3$ square radians, the a priori state estimate $\hat{x}_0 = (20, 80)$, and the a priori state uncertainty $P_0 = \sigma^2 I$ where $\sigma = 100$. We used a random number generating function to simulate the observations vector z_k, that is, $z_k = h(\tilde{x}_{\mathrm{true}}) + \mathrm{rand}(0, R_k)$ for every k. (The choice of parameter values has no bearing on these results; other sets of values tested produced results that are consistent with this example.)

Figure 1a shows that the accuracy of the position estimates of both filters improves gradually, approximately tracking each other, up to epoch $k \approx 80$. However, although the accuracy of the IUKF continues to improve after that epoch, the UKF diverges, with a resulting error that is about 2 orders of magnitude larger than that of the IUKF. Figure 1b shows a smooth convergence of the IUKF uncertainty matrix P_k, as one may expect from the gradual convergence in position error. This is in marked contrast to the UKF, whose uncertainty suddenly decreases at about $k \approx 80$ (precisely when the filter diverges).

Consistent with the results presented in [6], these simulations demonstrate a significant performance improvement of the IUKF, relative to the UKF, for nonlinear observation models.

VII. Conclusions

We have derived a new equation for the state update of the UKF, one of a large collection of filters that extend the formulation of the Kalman filter to nonlinear systems. Despite its robustness, the UKF has also been shown to diverge for systems that are highly nonlinear. The proposed new state update equation incorporates high-order terms of the observation model into the residuals. The modification builds on the assumption that the update state estimate can be approximated by a power series expansion of measurement residuals, as is the case of the UKF and other Kalman-type filters. However, whereas the measurement residuals of the UKF are calculated using the expected value of the observation process, the



a) b)

Fig. 1 Simulation results: a) temporal variation of the position error for the UKF (dots) and the IUKF (continuous line), and b) temporal variation of the state uncertainty estimate for the UKF (dots) and the IUKF (continuous lines). In b), for each filter, higher/lower value curves are the square root of the maximum/minimum eigenvalues $\lambda_{\{1,2\}}$ of the state covariance matrix P_k^+. Note the logarithmic scale in both axes and the scale change in the vertical axis.

measurement residuals in the new filter are calculated using the actual observation model and the propagated state. We have shown that the measurement residuals thus calculated incorporate the second-order terms of the measurement model into the state estimate update. We have also performed numerical simulations using a simple nonlinear model that is relevant to formation flying applications and showed that the new filter can be 2 orders of magnitude more accurate than the standard UKF. We therefore conclude that our theoretical and numerical results indicate an improvement of the estimation performance of the new filter with respect to the UKF for observation models that are not linear.

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