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Application of a Colored Noise Kalman Filter to a Radio-Guided Ascent Mission

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A set of general sequential filter equations is derived for nonlinear system dynamics and a nonlinear observation model, but is obtained with the assumption of a linear estimator. These equations, which are based upon previously developed formulas, include the colored measurement noise statistics and the statistics of nonestimated model parameter errors. Several simulations made with this filter are compared with the simulation results of an estimation procedure constructed with the utilization of the standard white noise assumptions. The difference between the white noise filter results and the colored noise filter results is found to be minimal. Instability occurred when the statistics of the effective exhaust velocity in the acceleration model were not properly accounted for.

Nomenclature

A = $\left.\frac{\partial h}{\partial T}\right|_{X = \hat{X}_{n/n-1},\ T = \hat{T}} - \rho \left.\frac{\partial h}{\partial T}\right|_{X = \hat{X}_{n/n-1},\ T = \hat{T}}$
B $\equiv E[\delta\hat{T}\delta\hat{T}^T]$
D = $\left.\frac{\partial \phi}{\partial p}\right|_{X = \hat{X}_{n/n-1},\ p = \hat{p}}$
E = the expectation operator
H_a = $H_n\Phi - \rho H_{n-1}$
H_n = $\left.\frac{\partial h}{\partial X}\right|_{X = \hat{X}_{n/n-1},\ T = \hat{T}}$
K_n = the filter gain matrix
L_n $\equiv E[\delta\hat{X}_{n/n}\delta\hat{T}^T]$
n = subscript associating variable with the nth time point
$n/n-1$ = subscript indicating that the variable is evaluated at the nth time point and is based upon measurements up to and including the $(n-1)$st
N_n = vector of measurement noise
p = dynamic model parameter vector
$\hat{p}$ = an estimate of p; $\delta\hat{p} = p - \hat{p}$
Q,R,S $\equiv E[\mu_n\mu_n^T]$, $E[W_nW_n^T]$, and $E[\delta\hat{p}\ \hat{p}^T]$, respectively
T = measurement model parameter vector
$\hat{T}$ = an estimate of T; $\delta\hat{T} = T - \hat{T}$
V_n = $E[\delta\hat{X}_{n/n}\delta\hat{p}^T]$
W_n = a white noise vector
X_n = the state vector
$\hat{X}_{n/i}$ = the estimate of the state vector at t_n given all measurements up to and including that at t_i
z_n = the measurement vector
ζ_n = effective measurement after differencing
$\hat{\zeta}_n$ = estimated effective measurement computed from $\hat{X}_{n/n-1}$
μ_n = white process noise vector
Γ,Φ = $\left.\frac{\partial \phi}{\partial \mu}\right|_{X = \hat{X}_{n/n-1},\ p = \hat{p}}$, and $\left.\frac{\partial \phi}{\partial X}\right|_{X = \hat{X}_{n/n-1},\ p = \hat{p}}$, respectively
ρ = correlation coefficient matrix representing the correlation between N_n and N_{n-1}

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Introduction

During radio-guided launch missions (Fig. 1), angle measurement noise is more bothersome at low radar line-of-sight elevation angles. Significantly, a greater portion of the launch guidance occurs at low-elevation angles when larger boost vehicles are used. Random fluctuations in the tropospheric index of refraction are slower and more severe in the low-altitude regions and thus affect not only the magnitude but also the autocorrelation of the angle errors

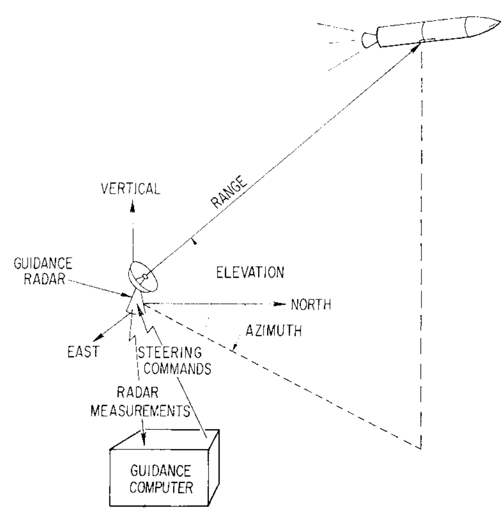

Fig. 1 Launch guidance radar system.

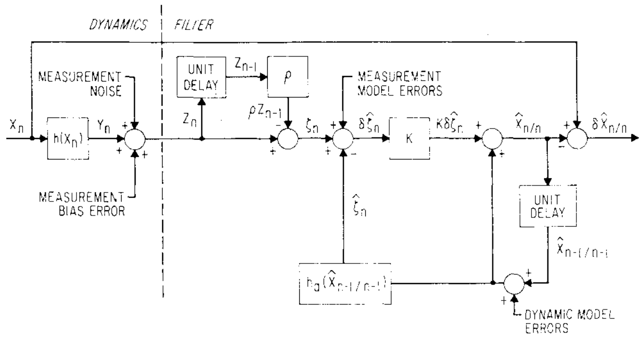

Fig. 2 Sequential filter block diagram.

when the radar energy traverses these regions. In fact, angle noise correlation times of several seconds are common at radar elevation angles of a few degrees. In addition, the present trend in guidance filtering has been toward more intensive use of Kalman filtering techniques. Since filters of this type require that the measurement noise characteristics be modeled, it is appropriate to investigate the behavior of such filters when various assumptions are made concerning the noise correlations.

A filter constructed to minimize the effects of measurement noise will itself introduce modeling errors that reflect the limits in human ability to describe the real world in exact mathematical terms. Errors of this sort can have an accumulative effect over a period of time and can cause the estimate to diverge. It is assumed for the purposes of this paper that the mathematical form of the system model is known or can be adequately represented by some series expansion; however, the values of certain parameters (e.g. measurement bias, or the numerical value of the gravitational constant) of the model are in error. This assumption differs significantly from the situation where the mathematical form itself is uncertain.

A general sequential filter is derived which utilizes the differencing scheme of Refs. 1 and 2 to account for sequentially correlated noise. Nonestimated model parameter uncertainties in both the state dynamic model and the measurement model are accounted for in the derivation. The various features of the general filter have been programed in a simulation of a launch guidance mission, and the behavior of the general filter is examined to determine whether its performance will warrant the consequent increase in the complexity of existing guidance programs.

Optimum Filter

An optimum linear sequential filter (see Fig. 2) operates on data samples that are available at discrete values of time. Its output at a particular time is an estimate of the system state based on all information available up to that time. It is optimum in a least-squares or minimum-variance sense, providing that the gaussian and linear propagation properties of the errors hold; and it is linear because the estimates are formed from a linear combination of the observation data.

At the beginning of each computation cycle there are available a) an estimate of the state vector and a covariance matrix of errors in the estimate and b) a set of measurements and a covariance matrix of measurement errors. These data are weighted according to the relative magnitudes of their respective covariances (data with larger covariances are weighted less) and are appropriately transformed to form a new state vector estimate. Next, a new state vector estimate error covariance matrix is computed by propagating the previous state vector estimate error covariance matrix and the measurement error covariance matrix through the filtering equations. Finally, the state vector estimate and the state vector estimate error covariance matrix are propagated to the beginning of the next computation cycle by the filter dynamic and error propagation models.

If the measurement noise is highly correlated (as it is in low-elevation radar tracking), the estimate errors will be correlated with the measurement noise through previous measurements. A filter based on white noise assumptions will ignore this fact and improperly propagate the covariance matrices, consequently causing improper weighting of the data. For example, suppose that the range from a radar to a fixed point is determined from range measurement samples and that these measurements possess noise with long-term drift characteristics. A white noise filter will treat the measurement noise as being statistically independent from sample to sample with a statistical mean that tends toward zero as the number of samples increases, when in actuality the measurement error will behave more as a bias if the total time span under consideration is short.

One means of accounting for the presence of colored noise is to augment the state vector to include the states of a hypothetical shaping filter whose input is white noise and whose output is noise of the required correlations. This approach is often prohibitive, primarily because of computer storage limitations, since for each increase in the state vector dimensions there is a corresponding increase in the dimensions of the state covariance matrices and in the complexity of the ensuing matrix arithmetic. Also, the state augmentation scheme can result in ill-conditioned matrices.[1] A second method[1,2]† utilizes a measurement differencing technique wherein the most recent past value of the measurement vector is weighted by the noise correlation coefficient matrix and subtracted from the present measurement vector. The correlated components of the measurement noise thus subtract out, leaving an effective measurement with only white noise errors. Furthermore, this method reduces the implementational difficulties associated with state augmentation. This method was used in the launch guidance simulation.

Basically, the model parameter errors can be treated in one of 3 ways in the estimation process:

1) They can be ignored, and the state vector estimate error covariance matrix, since it will not show the effect of propagating the parameter errors, will be smaller than it should be. Consequently, the state vector estimate that is present at the beginning of each computation cycle will be weighted more and the measurements weighted less than if the effects of the parameter errors were accounted for.

2) The parameters can be estimated along with the rest of the state vector elements in hopes of driving the bothersome errors to smaller values. In any event, the parameter error covariances will be included in the state vector estimate error covariance matrix and will hence influence the computation of the filter weights. That is, even if the parameter uncertainties are not reduced, they will be accounted for in the filter error analysis.

3) The error covariances of the parameter errors can be included in the filter covariance matrix propagations,[4] but unlike method 2, the parameters are not solved for, and thus the parameter errors never decrease. The procedure is equivalent to the situation where the parameter is solved for as in method 2 but where poor parameter estimates are obtained. Attempts have been made to approximate this third approach by the addition of an artificial "state noise" covariance matrix to the state error covariance matrix once during each computation cycle. Although such an approach is not a systematic one, it is obviously better than ignoring the parameter errors as the first method does, since it prevents the state error covariance matrix from becoming too small. Nevertheless, the correlation between the state estimate

† Recently, a third method was developed which does not depend upon state augmentation or measurement differencing. It was, however, developed too late to obtain results for the paper, and hence will be discussed in a subsequent report.

error and the parameter error is ignored, inasmuch as the state noise approach is based on the assumption of white noise parameter errors.

Derivation of the General Filter

Let the state dynamics be described by

$$X_n = \phi(X_{n-1},p,\mu_n) \tag{1}$$

and the relation of the measurement to the states by

$$z_n = h(X_n,T) + N_n \tag{2}$$

Let the measurement noise be generated by the Markov process

$$N_n = \rho N_{n-1} + qW_n \tag{3}$$

The filter dynamic model is assumed to be of the form

$$\hat{X}_{n/n-1} = \phi(\hat{X}_{n-1/n-1},\hat{p}) \tag{4}$$

and the filter measurement model of the form

$$\hat{z}_n = h(\hat{X}_{n/n-1},\hat{T}) \tag{5}$$

Let the measurements be differenced to give an effective measurement

$$\zeta_n = z_n - \rho z_{n-1} \tag{6}$$

The measurement estimate is

$$\hat{\zeta}_n = h(\hat{X}_{n/n-1},\hat{T}) - \rho h(\hat{X}_{n-1/n-1},\hat{T}) = h_a(\hat{X}_{n-1/n-1},\hat{T}) \tag{7}$$

Assume that the state vector estimates are obtained from a linear filter of the form

$$\hat{X}_{n/n} = \hat{X}_{n/n-1} + K_n[\zeta_n - \hat{\zeta}_n] \tag{8}$$

The error in the estimate is

$$\delta\hat{X}_{n/n} \equiv X_n - \hat{X}_{n/n} = \delta\hat{X}_{n/n-1} - K_n\delta\hat{\zeta}_n \tag{9}$$

where

$$\delta\hat{X}_{n/n-1} \equiv X_n - X_{n/n-1} \tag{10}$$

$$\delta\hat{\zeta}_n \equiv \zeta_n - \hat{\zeta}_n \tag{11}$$

The filter described by the foregoing equations is shown in Fig. 2.

By the Wiener-Hopf Equation (see Appendix) the filter gain is obtained as

$$K_n = E[\delta\hat{X}_{n/n-1}\delta\zeta_n^T]\{E[\delta\hat{\zeta}_n\delta\hat{\zeta}_n^T]\}^{-1} \tag{12}$$

To obtain an explicit expression for K_n it is first necessary to expand Eq. (10). Thus

$$\delta\hat{X}_{n/n-1} = \phi(X_{n-1},p,\mu_n) - \phi(\hat{X}_{n-1/n-1},\hat{p}) \tag{13}$$

which, if one assumes that the estimate is within a linear expansion of the state, becomes

$$\delta\hat{X}_{n/n-1} = \Phi\delta\hat{X}_{n-1/n-1} + D\delta\hat{p} + \Gamma\mu_n \tag{14}$$

Similarly, Eq. (11) can be written as

$$\begin{aligned}\delta\hat{\zeta}_n &= (H_n\Phi - \rho H_{n-1})\delta\hat{X}_{n-1/n-1} + H_nD\delta\hat{p} + A\delta\hat{T} + H_n\Gamma\mu_n + qW_n \\ &= H_a\delta\hat{X}_{n-1/n-1} + H_nD\delta\hat{p} + A\delta\hat{T} + H_n\Gamma\mu_n + qW_n \end{aligned} \tag{15}$$

After one uses Eqs. (14) and (15) in Eq. (12), and performs the indicated expected values, K_n can be determined from

$$\begin{aligned}E[\delta\hat{X}_{n/n-1}\delta\zeta_n^T] = {}& \Phi P_{n-1}H_a^T + \Phi V_{n-1}D^TH_n^T + \\ & \Phi L_{n-1}A^T + DV_{n-1}^TH_a^T + DSD^TH_n^T + \Gamma Q\Gamma^TH_n^T \end{aligned} \tag{16}$$

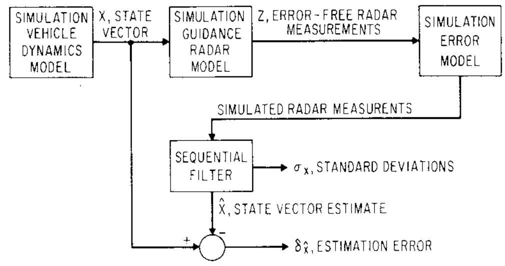

Fig. 3 Simulation block diagram.

and

$$\begin{aligned}E[\delta\hat{\zeta}_n\delta\hat{\zeta}_n^T] = {}& H_aP_{n-1}H_a^T + qRq^T + H_n\Gamma Q\Gamma^TH_n^T + \\ & H_nDV_{n-1}^TH_a^T + H_aV_{n-1}D^TH_n^T + H_nDSD^TH_n^T + \\ & AL_{n-1}^TH_a^T + H_aL_{n-1}A^T + ABA^T \end{aligned} \tag{17}$$

where

$$\begin{aligned}P_n = {}& (\Phi - K_nH_a)P_{n-1}(\Phi - K_nH_a)^T + K_nqRq^TK_n^T + \\ & (I - K_nH_n)\Gamma Q\Gamma^T(I - K_nH_n)^T + (I - K_nH_n)DSD^T \times \\ & (I - K_nH_n)^T + (\Phi - K_nH_a)V_{n-1}D^T(I - K_nH_n)^T + \\ & (I - K_nH_n)DV_{n-1}^T(\Phi - K_nH_a)^T + K_nABA^TK_n^T - \\ & (\Phi - K_nH_a)L_{n-1}A^TK_n^T - K_nAL_{n-1}^T(\Phi - K_nH_a)^T \end{aligned} \tag{18}$$

$$L_n = (\Phi - K_nH_a)L_{n-1} - K_n AB \tag{19}$$

$$V_n = (\Phi - K_nH_a)V_{n-1} + (I - K_nH_n)DS \tag{20}$$

Equations (16–20) along with Eq. (12) constitute the general sequential filter.

Simulation Results and Conclusions

In radio-guided ascent missions a tracking radar measures the position and/or velocity of the launch vehicle (Fig. 1). The guidance computer determines, according to a programed guidance law, what vehicle steering is necessary to satisfy the mission requirements; and steering commands are transmitted to the vehicle via the guidance radar link. In addition, the radar transmits discrete commands such as engine cutoff as soon as the computer determines the necessity of such events.

A simulation (Fig. 3) for guidance of a vehicle during its last two stages has been constructed to assess the performance of the general sequential filter in a simulated radar noise environment, and in the presence of uncertainties in both the filter state model parameters and filter measurement parameters. The vehicle is assumed to be guided perfectly, hence the state vector in the simulation is independent of the filter estimates. The performance factors are the computed residuals of the state estimation error and the computed state estimation error covariance matrix. A position radar, measuring range, azimuth, and elevation, are simulated. The simulation error model adds random noise to the simulated measurements which possesses the following 0.5-sec correlations: range noise correlation = 0.1; azimuth noise correlation = 0.95; and elevation noise correlation = 0.97. The estimated state is a 12-element vector comprising: x,y,z = a 3-element ECI Cartesian position vector; $\dot{x},\dot{y},\dot{z}$ = a 3-element ECI Cartesian velocity vector; c^* = effective engine exhaust velocity; t_m = engine mass depletion time; i_x,i_z = components of a roll-axis-oriented unit vector; and Ω_x,Ω_y = pitch and yaw axis drift rates.

The following 7 simulations are performed:

a) A white noise filter is assumed, and the simulation error model generates measurement noise with the foregoing correlations.

b) A colored noise filter (correlated measurement noise with the foregoing correlations) is assumed, and the simula-

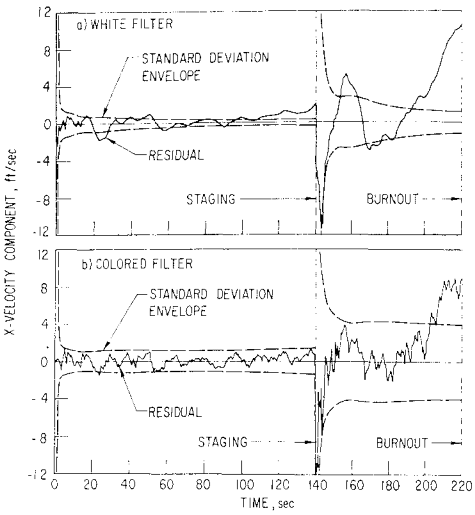

Fig. 4 Velocity residual, white and colored noise filter.

tion error model generates measurement noise with these same correlations.

c) A white noise filter is used, and the simulation error model generates bias errors only.

d) The filter error model is based on an assumption of measurement bias errors, and the simulation error model generated bias errors only.

e) A white noise filter is used, the simulated radar measurements are error-free, and c^* is omitted from the estimated state vector. No compensation for c^* uncertainties is attempted.

f) A white noise filter is used, the simulated radar measurements are error-free, and c^* is included in the estimation vector.

g) This simulation is identical with (e), except that the c^* error statistics are included in the filter equations.

Results of simulations a and b are given in Figs. 4a and 4b. The behavior of the x-components of the velocity estimate residual is typical of the behavior of the other eleven state vector elements. The x-velocity standard deviation envelopes obtained from the state estimate error covariance matrices of the filters are also shown. The average deviations of the residual about the zero mean value are about the same in the two simulations, and there is little difference in the velocity residual at burnout ($\sim$15% smaller with the colored noise filter). The standard deviation envelope, however, is larger in the colored noise case and is more representative of the

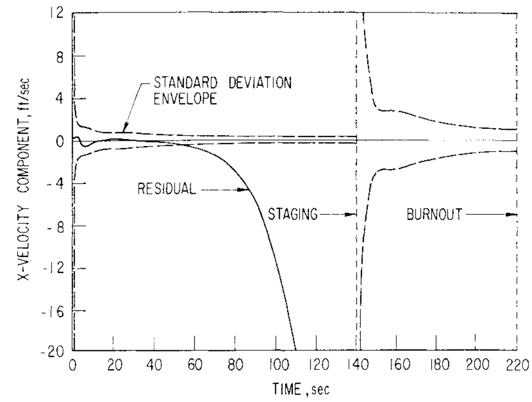

Fig. 6 Velocity residual, white noise filter, C* not estimated.

estimate errors. The apparent drift of the velocity residual is a consequence of the particular sample of the noise generator used in the simulation and is not a divergence of the estimate. The results of simulations c and d showing the effects of measurement bias errors on the output of the white noise filter and the bias filter are given in Fig. 5; the differences here are indeed small, but the computed standard deviation envelope is larger when the bias filter is used.

An intuitive argument is offered to explain why so little improvement in the state estimates is obtained after implementing the colored noise and bias filters. It is seen in Fig. 2 that the effective measurement ζ_n is differenced with its predicted value $\hat{\zeta}_n$; the difference $\delta\hat{\zeta}_n$ is then operated on by the gain matrix K. Suppose ρ is the null matrix (white noise filter) and the system geometry is very slowly changing. If the measurement error is highly correlated random noise or a bias, the ζ_n errors will be correlated with the $\hat{\zeta}_n$ errors. As a result of differencing ζ_n and $\hat{\zeta}_n$, the correlated errors will subtract out in much the same way as they did in the colored noise filter.

Figure 6 shows the results of simulation e in which c^* was omitted from the state estimation vector and the c^* uncertainties were ignored; the filter velocity estimates become unstable after $\sim$60 sec.

Figure 7 shows the results of simulation f in which c^* was estimated and the results of simulation g in which c^* was not estimated but the c^* errors were accounted for. The position, velocity, and attitude estimate residuals are nearly the same in both simulations.

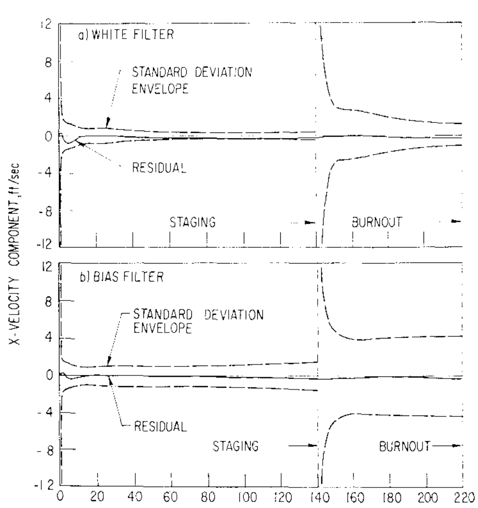

Fig. 5 Velocity residual, white noise and bias filters.

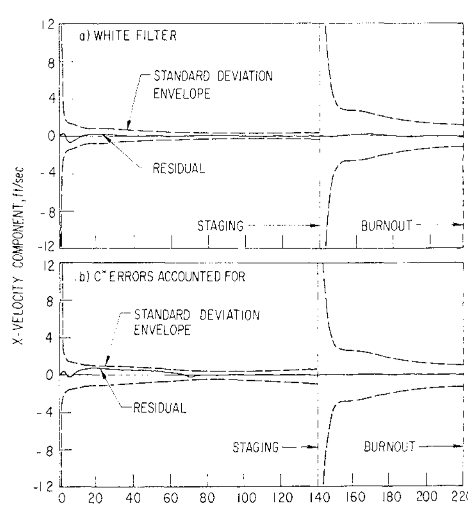

Fig. 7 Velocity residual, white noise filter and c* errors accounted for.

In conclusion, the degrees of improvement obtained with either the colored-noise or the bias filter would not warrant the increased complexity of the guidance equations. State vector estimates which do not contain c^* as one of their elements can be satisfactorily obtained if the c^* error statistics are properly accounted for in the filter equations. Some form of compensation for the omission of this parameter from the estimate is mandatory if the filter is to remain stable.

In general, the results demonstrate that a) caution must accompany the desire for optimality in a practical application of theoretical results, and b) the optimal approach, when used as an analysis tool, provides useful goals or limitations upon which a practical design can be based.

Appendix: Wiener-Hopf Equation

Define the inner product of two real-valued vectors a and b to be

$$\langle a, b\rangle \equiv X_1{}^T E(ab^T) X_2 \tag{A1}$$

where a is an n-vector and b is an m-vector. Let X_1 be any real n-vector and X_2 any real m-vector such that

$$X_1 \subset X_2 \text{ for } n < m$$
$$X_1 \supset X_2 \text{ for } m < n$$
$$X_1 = X_2 \text{ for } m = n$$

An inner product of two real vectors must satisfy

$$\langle \alpha X + \beta y, z\rangle = \alpha \langle X, z\rangle + \beta \langle y, z\rangle \tag{A2}$$

where Φ and π are real scalars

$$\langle X, X\rangle \geq 0; \ \langle X, X\rangle = 0 \Longrightarrow X = 0 \tag{A3}$$

Satisfaction of condition (A2) by Eq. (A1) is easily demonstrated. Condition (A3) merely states, for the inner product defined, the positive-definite property of a covariance matrix.

The vector $\delta\hat{X}_{n/n}$ of Eq. (9) in the text is the estimation error and represents the additional information required to specify the state exactly. The vector $\delta\hat{\zeta}_n$ represents the information, in addition to that obtained from past data, provided by the most recent observation. If $\delta\hat{X}_{n/n}$ is an optimal estimate in a least squares sense, $\delta\hat{X}_{n/n}$ is orthogonal to $\delta\hat{\zeta}_n$; i.e.,

$$\langle \delta\hat{X}_{n/n}, \delta\hat{\zeta}_n\rangle = 0 \tag{A4}$$

But if $X_1{}^T E(ab^T) X_2 = 0$ for all X_1 and X_2, then $E(ab^T) = 0$, so that

$$E[\delta\hat{X}_{n/n}\delta\hat{\zeta}_n{}^T] = 0 = E[(\delta\hat{X}_{n/n-1} - K_n\delta\hat{\zeta}_n)\delta\hat{\zeta}_n{}^T] = 0 \tag{A5}$$

If one solves for K_n, the Wiener-Hopf Equation, Eq. (12), is obtained.

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