Orbit Computation with the Vinti Potential and Universal Variables

BASSFORD C. GETCHELL*
National Security Agency, Fort Meade, Md.

When the Vinti potential and spheroidal coordinates are used in the solution of the equations of motion of an Earth satellite, the Hamilton-Jacobi differential equation is separable, permitting computation of the free flight path in closed form. If the parameters are chosen to include the effect of the J_2 and J_3 terms of the Earth's gravitational potential, the Vinti potential also allows for a major part of the effect of the term in J_4. The model can serve as an improved reference orbit in perturbation theory, or it can be used to compute moderately accurate trajectories (extremely accurate over short time intervals) with a considerable saving of machine time over numerical integration. The development in universal variables is valid for orbits of eccentricity greater than unity as well as for periodic orbits. The main purpose of this paper is to generalize the results of Vinti and others to include the nonperiodic cases. Included is a description of a tested method for practical calculation of trajectories and a numerical example for illustration.

Introduction

THE most accurate models of the Earth's gravitational potential are obtained by analysis of the observed motions of artificial satellites. Theories of motion that conform closely to such models generally require the use of complicated equations and lengthy calculations. Important simplifications can be obtained with a system that permits the separation of the Hamilton–Jacobi equation. The simplest of these is the well-known solution of the two-body or Keplerian problem in equatorial polar coordinates, which is often used as a reference orbit in perturbation theories. The Vinti[1] potential with spheroidal coordinates is the most accurate of the small number of separable solutions proposed for this problem and can be used without modification in many applications. As a reference orbit it might have the advantage of permitting the use of a first-order correction that would be unsatisfactory with a less accurate model.

The integrals of motion of the two-body problem lead to solutions in terms of trigonometric functions when the eccentricity e is less than unity, and to the elementary hyperbolic functions for $e > 1$. Neither solution is satisfactory when the orbit is nearly parabolic. These inconveniences can be avoided by the introduction of special "universal" variables that preserve the same form of the solution in all cases. The series expansions of the corresponding elliptic integrals of the Vinti theory can likewise be generalized with the help of universal variables. The resulting expressions have been developed for their usefulness in numerical calculation, with particular attention to the elimination of inaccuracies due to indeterminate forms which occur with special inclinations and eccentricities. Excluded from consideration are the nearly linear trajectories with small latus rectum. Vinti's solution has no singularity at the so-called "critical inclination," $\cos I = (\frac{1}{5})^{1/2}$.

Equations of Motion

Let x, y, z be the vehicle coordinates in a right-handed rectangular equatorial system with x axis directed toward the equinox of epoch. If α is the right ascension of the vehicle the spheroidal coordinates ρ, σ, and α are related to x, y, z by

$$D^2 = x^2 + y^2 = r^2 - z^2 = (\rho^2 + c^2)(1 - \sigma^2) \quad (1)$$

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* Mathematician. Member AIAA.

$x = D\cos\alpha$, $y = D\sin\alpha$, and $z = \rho\sigma - \delta$, where $D \geq 0$. Here c and σ are parameters of the Vinti potential

$$V = -\mu(\rho + \delta\sigma)/(\rho^2 + c^2\sigma^2) \quad (2)$$

and μ is the gravitation constant. For geocentric orbits $c^2 = J_2(1 - J_3^2/4J_2^3)$, and $\delta = -J_3/2J_2$. The energy integral is

$$(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2 + V = (p_1\dot{\rho} + p_2\dot{\sigma} + p_3\dot{\alpha})/2 + V = \alpha_1 \quad (3)$$

where

$$p_1 = \dot{\rho}(\rho^2 + c^2\sigma^2)/(\rho^2 + c^2) = F(\rho)^{1/2}/(\rho^2 + c^2)$$
$$p_2 = \dot{\sigma}(\rho^2 + c^2\sigma^2)/(1 - \sigma^2) = G(\sigma)^{1/2}/(1 - \sigma^2)$$

and

$$p_3 = \dot{\alpha}(\rho^2 + c^2)(1 - \sigma^2) = \alpha_3$$

The dots indicate differentiation with respect to time, and p_1, p_2, and p_3 are the generalized momenta. In addition to the constants of integration α_1 and α_3, the separated Hamilton–Jacobi equation furnishes a third α_2, given by

$$\alpha_2^2 = 2\mu\rho + 2\alpha_1\rho^2 + [c^2\alpha_3^2 - F(\rho)]/(\rho^2 + c^2) = -2\mu\sigma\delta - 2\alpha_1c^2\sigma^2 + [\alpha_3^2 + G(\sigma)]/(1 - \sigma^2) \quad (4)$$

The equations of motion are then

$$t - T = R_1 + c^2N_1,\ \omega = N_2 - R_2 \quad (5)$$

and

$$\Omega = \alpha + c^2R_3 - N_3$$

where T, ω, and Ω are the remaining constants of integration

$$R_1 = \int_{\rho_1}^{\rho} \rho^2 d\rho/F(\rho)^{1/2},\ F(\rho)^{1/2}\dot{\rho} \geq 0$$

$$R_2 = \alpha_2 \int_{\rho_1}^{\rho} d\rho/F(\rho)^{1/2},\ \alpha_2 \geq 0$$

$$R_3 = \alpha_3 \int_{\rho_1}^{\rho} d\rho/(\rho^2 + c^2)\ F(\rho)^{1/2}$$

$$N_1 = \int_a^{\sigma} \sigma^2 d\sigma/G(\sigma)^{1/2},\ G(\sigma)\dot{\sigma} \geq 0$$

$$N_2 = \alpha_2 \int_a^{\sigma} d\sigma/G(\sigma)^{1/2}$$

$$N_3 = \alpha_3 \int_a^\sigma d\sigma/(1-\sigma^2)G(\sigma)^{1/2}$$

The limits ρ_1 and a are arbitrary, but specific values will be assigned later.

Factoring of the Integrands

Let $\gamma_0 = 2\alpha_1/\mu$, $p_0 = \alpha_2^2/\mu$, and $S_0 = 1 - \alpha_3^2/\alpha_2^2$. By rearrangement of Eq. (4),

$$F(\rho) = \mu[c^2p_0(1-S_0) + (\rho^2+c^2)(\gamma_0\rho^2 + 2\rho - p_0)] \quad (6)$$

and

$$G(\sigma) = \mu[-p_0(1-S_0) + (1-\sigma^2)(p_0 + 2\delta\sigma + c^2\gamma_0\sigma^2)] \quad (7)$$

Some of the following processes diverge if p_0 is very small. The series of the text have been carried to enough terms to give accuracy to the order of J_2^3 when $p_0 > 1$. Assume now that

$$F(\rho) = \mu\gamma_1(\gamma\rho^2 + 2\rho - p)(\rho^2 - 2A_1\rho + B_1) \quad (8)$$

$$G(\sigma) = \mu S_1 p_0(S + 2P\sigma - \sigma^2)(1 + P_1\sigma - Q_1\sigma^2) \quad (9)$$

with $\gamma_1\gamma = \gamma_0$, and $S_1S = S_0$. Equating coefficients in Eqs. (5) and (7) one obtains

$$\gamma_1 = 1 + \gamma_0 A_1 \quad (10)$$

$$p\gamma_1 = p_0 - c^2\gamma_0 + B_1\gamma_0 - 4A_1\gamma_1 \quad (11)$$

$$B_1 = c^2S_0p_0/p\gamma_1 \quad (12)$$

$$A_1 = (c^2 - \gamma_1 B_1)/p\gamma_1 \quad (13)$$

By starting with $A_1 = B_1 = 0$ in Eqs. (10) and (11), a few cyclic interations will give arbitrarily accurate solutions for the constants of Eq. (8). A similar treatment of Eqs. (7) and (9), with initial values of $P = 0$ and $S_1 = 1$ results in

$$Q_1 = -c^2\gamma_0/p_0S_1 \quad (14)$$

$$P_1 = (2\delta/p_0S_1) - 2Q_1P \quad (15)$$

$$P = (\delta/p_0S_1) - S_0P_1/2S_1 \quad (16)$$

$$S_1 = (p_0 - c^2\gamma_0 - S_0p_0Q_1)/(1 - 2PP_1)p_0 \quad (17)$$

Evaluation of the R Integrals

The universal variables[2] are functions of $\hat{X}$ and γ, defined by

$$\hat{C}(\hat{X}) = \hat{C}(\hat{X},\gamma) = \hat{X}^2(1/2! + \gamma\hat{X}^2/4! + \gamma^2\hat{X}^4/6! +++)$$
$$\hat{U}(\hat{X}) = \hat{U}(\hat{X},\gamma) = \hat{X}^3(1/3! + \gamma\hat{X}^2/5! + \gamma^2\hat{X}^4/7! +++)$$
$$\hat{S}(\hat{X}) = \hat{S}(\hat{X},\gamma) = \hat{X} + \gamma\hat{U}(\hat{X})$$

They satisfy the relationships $\hat{S}^2 = 2\hat{C} + \gamma\hat{C}^2$, $\hat{S}' = 1 + \gamma\hat{C}$, $\hat{C}' = \hat{S}$, and $\hat{U}' = \hat{C}$, where primes represent differentiation with respect to $\hat{X}$. Since A_1 and B_1 are small quantities one can write

$$(\rho^2 - 2A_1\rho + B_1)^{-1/2} = \sum_{\alpha=0}^{6} A_\alpha/\rho^{\alpha+1} \quad (18)$$

where

$A_0 = 1$ $\qquad A_4 = 3(B_1^2 - 10A_1^2B_1)/8$
$A_2 = (3A_1^2 - B_1)/2$ $\qquad A_5 = 15A_1B_1^2/8$
$A_3 = (5A_1^3 - 3A_1B_1)/2$ $\qquad A_6 = -5B_1^3/16$

Then R_1 can be expressed as the sum of two integrals

$$R_4 = (\mu\gamma_1)^{-1/2}\int_{\rho_1}^{\rho}(\gamma\rho^2 + 2\rho - p)^{-1/2}(\rho + A_1)d\rho$$

$$R_5 = (\mu\gamma_1)^{-1/2}\int_{\rho_1}^{\rho}(\gamma\rho^2 + 2\rho - p)^{-1/2}\left(\sum_{k=2}^{6} A_k/\rho^{k-1}\right)d\rho$$

Now introduce the independent variable $\hat{X}$ defined by

$$\rho = \rho_1 + e\hat{C}(\hat{X}), \quad \hat{S}(\hat{X})\dot{\rho} \geq 0 \quad (19)$$

where $e = (1 + p\gamma)^{1/2}$ and $\rho_1 = p/(1+e)$. Since $\dot{\rho}$ and $[F(\rho)]^{1/2}$ have the same sign the inequality can also be written $\hat{S}(\hat{X})\,[F(\rho)]^{1/2} \geq 0$. The auxiliary variable W, corresponding to the true anomaly, satisfies $\rho = p/(1 + e\cos W)$, $\dot{\rho}\sin W \geq 0$.

When $\gamma \geq 0$

$$W = \tan^{-1}[p^{1/2}\hat{S}/(\rho_1 - \hat{C})],\ (\rho_1 - \hat{C})\cos W \geq 0 \qquad -\pi < W < \pi \quad (20)$$

If $\gamma < 0$ let $E = \beta^*\hat{X}$, where $\beta^* = (-\gamma)^{1/2}$. Then one can show that

$$W = E + 2\tan^{-1}[e\sin E/\{1 + (1-e^2)^{1/2} - e\cos E\}] \quad (21)$$

with $-\pi < W - E < \pi$. Also

$$(\gamma\rho^2 + 2\rho - p)^{-1/2}d\rho = d\hat{X} = p^{-1/2}\rho dW \quad (22)$$

giving

$$R_1 = (\mu\gamma_1)^{-1/2}[(\rho_1 + A_1)\hat{X} + e\hat{U}(\hat{X})] + (\mu\gamma_1 p)^{-1/2}\sum_{k=2}^{6}\int_0^W A_k\left[\frac{1 + e\cos W}{p}\right]^{k-2}dW \quad (23)$$

$$R_2 = \left(\frac{p_0}{\gamma_1 p}\right)^{1/2}\sum_{k=0}^{6}\int_0^W\left(\frac{A_k}{\rho^k}\right)dW \quad (24)$$

and

$$R_3 = \alpha_3(\mu p\gamma_1)^{-1/2}\int_{\rho_1}^{\rho}\left(1 - \frac{c^2}{\rho^2} + \frac{c^4}{\rho^4}\right)\left(\sum_{k=0}^{6}\frac{A_k}{\rho^{k+2}}\right)dW \quad (25)$$

If $V_0 = W$, $V_1 = \sin W$, and

$$V_k = [V_1\cos^{k-1}W + (k-1)V_{k-2}]/k,\ k = 2, 3, \ldots 6$$

and if $W_0 = W$, $W_1 = (W + eV_1)/p$, and

$$W_k = \left[W + \sum_{\alpha=0}^{k}\binom{k}{\alpha}e^\alpha V_\alpha\right]/p_k$$

the R integrals can be written

$$R_1 = (\mu\gamma_1)^{-1/2}[(\rho_1 + A_1)\hat{X} + e\hat{U}(\hat{X}) + p^{-1/2}\sum_{k=0}^{4}A_{k+2}W_k]\ R_2 = (p_0/\gamma_1 p)^{1/2}\sum_{k=0}^{6}A_kW_k \quad (26)$$

$$R_3 = \alpha_3(\mu p\gamma_1)^{-1/2}[W_2 + A_1W_3 + (A_2 - c^2)W_4 + (A_3 - A_1c^2)W_5 + (A_4 - A_2c^2 + c^4)W_6]$$

Evaluation of the N Integrals

Let

$$Q = (P^2 + S)^{1/2},$$
$$\beta = (P_1 + 2Q_1P)/(1 + P_1P - Q_1P^2 - Q_1Q^2)$$
$$g = -\beta/[1 + (1 - \beta^2Q^2)^{1/2}],\ a = P + gQ^2$$
$$b = 1 + gP,\ D_5 = 1 + P_1a - Q_1a^2,\ k_1 = mQ^2$$
$$D_1 = [(1 - g^2Q^2)/S_1D_5]^{1/2},\ m = (Q_1b^2 - P_1bg - g^2)/D_5$$

Introduction of the variable u defined by

$$\sigma = (a + bQ\sin u)/(1 + gQ\sin u),\ \dot{\sigma}\cos u \geq 0 \quad (27)$$

reduces N_2 to

$$N_2 = D_1\int_o^u(1 - k_1\sin^2 u)^{-1/2}du \quad (28)$$

Since a, g, and k_1 are small, N_1 and N_2 can be expanded in

terms of the form,

$$T_k = \int_0^u \sin^k u \, du, \text{ or } T_0 = u, \; T_1 = 1 - \cos u$$

$$T_k = [(k - 1)T_{k-2} - \cos u \sin^{k-1} u]/k, \; k = 2, 3, \ldots 6$$

Letting

$$\bar{C}_0 = a^2, \; \bar{C}_1 = 2ab, \; \bar{C}_2 = b(b - 4ag)$$

$$\bar{C}_3 = b(ma - 2bg), \; \bar{C}_4 = b^2(3g^2 + m/2),$$

$$\bar{C}_5 = -mb^2g, \; \bar{C}_6 = 3m^2b^2/8$$

the N_1 and N_2 integrals become

$$N_1 = \frac{D_1}{\alpha_2} \sum_{k=0}^{6} \bar{C}_k Q^k T_k \tag{29}$$

$$N_2 = D_1(u + k_1T_2/2 + 3k_1^2T_4/8 + 5k_1^3T_6/16)$$

To evaluate N_3 compute

$$\beta_1 = (b - g)/(1 - a), \; \beta_2 = -(b + g)/(1 + a)$$

$$M_k = \pm(1 - \beta_k^2Q^2)^{1/2}$$

with

$$M_k\alpha_3 \geq 0, \; k = 1, 2, \; S^* = \sin u/2, \; C^* = \cos u/2$$

Then introduce

$$\psi_k = \tan^{-1}[M_kS^*/(C^* - \beta_kQS^*)],$$
$$(C^* - \beta_kQS^*) \cos\psi_k \geq 0. \quad k = 1, 2 \tag{30}$$

giving

$$Q^\alpha \sin^\alpha u \, du/(1 - \beta_kQ \sin u) = 2d\psi_k/M_k\beta_k^\alpha - du\beta_k^\alpha \sum_{j=0}^{\alpha-1} (\beta_kQ \sin u)^j \tag{31}$$

Then

$$N_3 = \frac{\alpha_3}{2} \int_a^\sigma \frac{d\sigma}{G(\sigma)^{1/2}} \left[\frac{1}{1 - \sigma} + \frac{1}{1 + \sigma}\right]$$

can be expanded into series of terms with integrands like those of Eq. (31) giving

$$N_3 = d_{10}\psi_1 + d_{20}\psi_2 + \alpha_3N_4 \tag{32}$$

where

$$d_{10} = [(b - ag)/(b - g)][(1 - D_3)/D_5(1 - n_1)((1 - 2D_2)]^{1/2}$$

$$d_{20} = [(b - ag)/(b + g)][(1 - D_3)/D_5(1 - n_2)(1 + 2D_2)]^{1/2}$$

and N_4 consists of terms in T_k, $k = 0, 1, 2\ldots5$. In the expressions for d_{10} and d_{20}, $n_k = m/\beta_k^2$, $D_2 = \delta/p_0(S_1 - S_0Q_1)$, and $D_3 = Q_1 + 2P_1D_2$. Factors like α_3/α_2M_1, which become indeterminate for polar orbits, have been avoided by making use of $P = D_2(1 - S)$, $1 - Q^2 = (1 - PD_2)(1 - S)$, and $1 - S_0 = S_1(1 - D_3)(1 - S)$, so that $1 - S$ can be cancelled out of the denominator. Finally,

$$N_3 = d_{10}\psi_1 + d_{20}\psi_2 - \frac{D_4}{(1 - a)} \sum_{k=0}^{5} C_{1k}(\beta_1Q)^kT_k - \frac{D_4}{(1 + a)} \sum_{k=0}^{5} C_{2k}(\beta_2Q)^kT_k \tag{33}$$

where

$$D_4 = D_1\alpha_3/2\alpha_2, \; d_1 = g, \; d_2 = m/2, \; d_3 = d_2g$$

$$d_4 = 3m^2/8, \; d_5 = d_4g, \; d_6 = 5m^3/16$$

$$C_{1k} = \sum_{\alpha=k+1}^{6} \frac{d_\alpha}{\beta_1^\alpha} \text{ and } C_{2k} = \sum_{\alpha=k+1}^{6} \frac{d_\alpha}{\beta_2^\alpha}$$

Analysis

Let x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, and $\dot{z}_0$ be position and velocity components at time t_0. The spheroidal coordinates ρ, σ, and α can be found from Eq. (1). Then, omitting subscripts, $\alpha_3 = x\dot{y} - y\dot{x}$ and $r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z}$. Also $(F)^{1/2} = \rho r\dot{r} + (c^2\sigma + \delta\rho)\dot{z}$, giving $\alpha_2 > 0$ with the help of Eq. (4). With α_1 from Eq. (3) the polynomials can now be factored and all necessary constants evaluated. The quadrant of u is found from

$$[(\rho - \delta\sigma)\dot{z} - \sigma r\dot{r}] \cos u \geq 0 \tag{34}$$

when $t = t_0$. A singularity at $e = 0$, or $Q = 0$ can be resolved, when $t = t_0$, by setting $\hat{X} = W = 0$, or $u = 0$.

The parameters p, γ, e, I, ω, Ω, and T, where $I = \sin^{-1}Q$ and $\alpha_3 \cos I \geq 0$, reduce to the corresponding Kepler elements when c^2 and δ are zero. When γ is negative the semimajor axis is $-1/\gamma$. Equations (10–17) can be rearranged to recover, by iteration, the constants of integration α_1, α_2, and α_3 when p, γ, and I are known.

The problem of finding position and velocity when the time is given can be solved when $\hat{X}$ is known. As a first approximation one can solve Kepler's equation

$$\Phi_0(\hat{X}) = \rho_1\hat{X} + e\hat{U}(\hat{X}) - (t_1 - T) = 0 \tag{35}$$

$\Phi'_0(\hat{X}) = \rho_1 + e\hat{C}(\hat{X})$. By Newton's method, $\hat{X}_{k+1} = \hat{X}_k - \Phi_0(\hat{X}_k)/\Phi'_0(\hat{X}_k)$, to an accuracy of 10^{-3} in $\hat{X}$. The iteration is then continued with

$$\Phi(\hat{X}) = R_1 + c^2N_1 + T - t_1 = 0, \; \Phi'(X) \simeq \Phi'_0(\hat{X}) \tag{36}$$

To compute $\Phi(\hat{X})$ start with R_2 and set

$$z^* = \frac{\omega + R_2}{D_1} = \int_0^u du(1 - k_1 \sin^2 u)^{1/2}$$

which can be inverted to find $u = am(z^*)$. Suitable formulas are 901.00, 900.04, and 908.00 of Byrd and Friedman[3]

$$q = k_1/16 + k_1^2/32 + 21k_1^3/1024$$

$$\frac{2K}{\pi} = 1 + 4 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \simeq (1 + 2q)^2$$

and

$$am(z^*) = y^* + 2 \sum_{m=0}^{2} \frac{q^{m+1}}{(m + 1)(1 + q^{2[u+1]})} \times \sin[2(m + 1)y^*]$$

where $y^* = \pi u/2K$. With $\hat{X}$, W, and u determined, one can evaluate R_1 and N_1. After convergence of the Newton iteration compute R_3, N_3, α, ρ, and σ, and, finally, the inertial position vector. If velocity components are required, compute $\dot{\rho} = e\hat{S}[\gamma_1(\rho^2 - 2A_1\rho + \beta_1)^{1/2}/(\rho^2 + c^2\sigma^2)]$, and $G(\sigma)$ from Eq. (9). Then $\dot{\sigma} = \pm G(\sigma)^{1/2}/(\rho^2 + c^2\sigma^2)$, with $\dot{\sigma} \cos u \geq 0$. Since

$$\dot{D} = [(1 - \sigma^2)\rho\dot{\rho} - (\rho^2 + c^2)\sigma\dot{\sigma}]/D \text{ and } \dot{\alpha} = \alpha_3/D_2$$

one can calculate velocity from

$$\dot{x} = \dot{D} \cos\alpha - y\dot{\alpha}, \; \dot{y} = \dot{D} \sin\alpha + x\dot{\alpha}, \; \dot{z} = \sigma\dot{\rho} + \rho\dot{\sigma} \tag{37}$$

Numerical Tests

The formulas have been tested for a variety of cases with the help of N. L. Bonavito and H. Walden of NASA Goddard Space Flight Center, Greenbelt, Maryland, by using numerical integration with $J_4 = -J_2^2 + J_3^2/J_2$ and all higher harmonics zero. The value of J_4 gives the best approximation to the Vinti potential and not to the Earth's. This value was chosen to test how well the equations do what

they are supposed to do, not how closely the result approximated an actual trajectory. Over 24 hr (~16 revolutions for a close Earth satellite), the position differences between the two programs remained less than 5 m. The small differences are due to neglect of the higher harmonics in the numerical integration and to truncation errors in the Vinti formulas. The following example illustrates the use of the method of the text.

Given: Distance unit = 6378.165 km, time unit = 806.8155985 sec, $\mu = 1$, $J_2 = 0.0010823$, $J_3 = -0.0000023$, $x_0 = 0.672$, $\dot{x}_0 = -0.01$, $y_0 = 0.896$, $\dot{y}_0 = 0$, $z_0 = 0$, $\dot{z}_0 = 1.4$, and $t_0 = 0$.

These values give: $\gamma = 0.1736148573$, $p = 2.4566030946$, $I = 89.67247152°$, $\Omega = 53.12964809°$, $T = 0.004899070359$, and $e = 1.194362925$.

At $t = 180$ min

$$x = -3.7629073618, \ \dot{x} = -0.3012763526$$
$$x_n = -3.7629073660$$
$$y = -4.9448301842, \ \dot{y} = -0.3982878817$$
$$y_n = -4.9448301895$$
$$z = 7.5913610280, \ \dot{z} = 0.3578871053$$
$$z_n = 7.5913611041$$

where x_n, y_n, z_n are given by accurate numerical integration

$\hat{X} = 3.4972483479$, $\hat{S} = 4.8731998933$, $\hat{C} = 7.2770772026$
$\hat{U} = 7.9253098881$, $W = 2.2492893566$, $u = 2.2563694937$
$R_1 = 13.3801123374$, $R_2 = 2.2488090913$, $R_3 = 0.0050762232$
$N_1 = 0.8758514470$, $N_2 = 2.2563173767$, $N_3 = 3.1346170450$
$\rho = 9.8109774995$, $\sigma = 0.7738702469$, $\alpha = 4.0618988456$

Conclusion

In addition to the use of universal variables in the evaluation of R_1, the principal differences between this treatment and other published versions are in the method of factoring the polynomials, and in the handling of indeterminate forms with special values of the eccentricity and inclination. For comparison with other methods of computation, using perturbation theory, or numerical integration including higher harmonics and other terms in the potential, one cannot get close agreement by equating initial position and velocity vectors as in the example. One should generate an ephemeris with the comparison method and fit the Vinti process by least squares. The principal difference will be due to neglect of the sectorial harmonic $J_{2,2}$ in the Earth's potential. For circular orbits this results in a cyclic error with a period of about 12 hr, which can be reduced by adding a sine term to the true anomaly. The amplitude depends upon the inclination of the orbit and the size of the semimajor axis. The rms error in position is typically of the order of 1 or 2 naut miles.

References

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