Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Product Decompositions for Certain Types of Coordinate Transformation

Terrence W. Barbee*
Space Applications Corporation, Santa Ana, California

Nomenclature

E^n = Euclidean n space
I = 3×3 identity matrix
A^t = transpose of A
$\{e_1, e_2, e_3\}$ = usual basis for E^3
G = $\{0, \pm\pi/2, \pm\pi\}$
$R[j, \pm\theta]$ = rotation about e_j by $\pm\theta$ ($+\theta$ indicating a counterclockwise rotation)

Introduction

ANALYSIS within a particular component of a large space system is usually performed with respect to an established coordinate reference frame, the aggregate of all such frames being related by certain coordinate transformations which are frequently the product of numerous orthogonal rotation matrices. It is often the case that the composite transformation is represented by a single matrix whose matrix elements involve complicated combinations of trigonometric functions. Previously, no efficient algorithm existed for decomposing the transformation into its component rotations which also accounted for the required order of the factors. The aim of this Note is the determination of a simple algorithm which yields a unique decomposition in a well-defined, iterative manner. The method employed finds use as either a check for the correct matrix product if the rotation factors are known, or, often it can be used to decompose a transformation into its constituent rotation factors when there is no a priori knowledge of the rotations involved (e.g., verification and validation efforts). This Note essentially establishes a procedure that complements the work of Pio,[1] which provides an alternative to the unpleasant task of multiplying numerous rotation matrices together.

Analysis

In applications, a coordinate transformation consists of a product of orthogonal rotations $R[j, \pm\theta]$. The property,

$$R[j,\theta]R[j,\phi] = R[j,\theta+\phi] \tag{1}$$

shows that two rotations about a common axis necessarily commute, which of course is false for rotations about different axes.

Lemma 1. Let $i,j \in \{1,2,3\}$. Then $R[i,\theta]$ and $R[j,\phi]$ commute if, and only if, $i=j$. ■

Equation (1) also suggests that if two or more consecutive rotations in a product occur about a common axis then they can be combined as one rotation about the same axis by a new variable. This gives rise to the following definition.

Received Sept. 8, 1982; revision received Feb. 14, 1983.

*Member Technical Staff.

Definition 1: Let $\theta_1,\ldots,\theta_n$ be variables in E^1, q an assignment of $\{1,2,\ldots,n\}$ to $\{1,2,3\}$ which satisfies $q(i) \neq q(i+1)$, and let p be a permutation of $\{1,2,\ldots,n\}$. An ordered product of rotations (OP) is an expression of the form

$$P(\theta_1,\ldots,\theta_n) = \prod_{i=1}^{n} R[q(i),\theta_{p(i)}] \tag{2}$$

This definition makes it possible for one to prove the following uniqueness theorem.

Theorem 1 (Uniqueness Theorem for Ordered Products). Let $\{R_i : 1 \leq i \leq n\}$ be a sequence of rotations such that the product

$$P(\theta_1,\ldots,\theta_n) = \prod_{i=1}^{n} R_i(\theta_i)$$

is an OP. Let Q be a matrix product of rotations such that $Q=P$. Then the decomposition of Q is identical to that of P.

Proof: For each $i=1, 2,\ldots,n$, $Q(0,\ldots,0,\theta_i,0,\ldots,0) = P(0,\ldots,0,\theta_i,0,\ldots,0) = R_i(\theta_i)$. It follows that $R_i(\theta_i)$ appears once, and only once, in the expression for Q and that no other rotation factors appear. Thus there is a permutation σ of $\{1,2,\ldots,n\}$ such that

$$Q = \prod_{i=1}^{n} R_{\sigma(i)}(\theta_{\sigma(i)})$$

Now, because $Q=P$,

$$R_i(\theta_i)R_{i+1}(\theta_{i+1}) = P(\theta_1,\ldots,\theta_n)\,|_{\theta_j=0, j\neq i,i+1}$$
$$= Q(\theta_1,\ldots,\theta_n)\,|_{\theta_j=0, j\neq i,i+1}$$

The last product above is either $R_i(\theta_i)R_{i+1}(\theta_{i+1})$ or $R_{i+1}(\theta_{i+1})R_i(\theta_i)$. The latter possibility is impossible by lemma 1. Therefore, in the product for Q, $R_i(\theta_i)$ precedes $R_{i+1}(\theta_{i+1})$ for every i, and hence σ must be the identity. ■

Suppose now that P is an OP which has the decomposition in Eq. (2). Ultimately, P will assume the form of a 3×3 matrix,

$$P(\theta_1,\ldots,\theta_n) = [f_{jk}(\theta_1,\ldots,\theta_n)] \tag{3}$$

where each f_{jk} is a function of $\sin\theta_i$, $\cos\theta_i$, etc. If P is given in the expanded form of Eq. (3), then by theorem 1, there is only one ordering of the rotation factors of P which will produce the form in Eq. (3). The following lemma will aid in the determination of an algorithm (theorem 2) for finding the desired decomposition of an OP.

Lemma 2: Let P be an OP of n rotations in variables $\theta_1,\ldots,\theta_n$, which, in final 3×3 matrix form, is given by Eq. (3). Then the following statements are equivalent. (i) P is missing the variable $\theta_{p(1)}$ in row $q(1)$. (ii) P has the form

$$P(\theta_1,\ldots,\theta_n) = R[q(1),\theta_{p(1)}]M(\theta_1,\ldots,\theta_n)$$

where M is an OP of $n-1$ rotations in the variables $\theta_{p(j)}$, $j \neq 1$.

Proof: The proof of (i)→(ii) will be by contradiction; the converse will be shown directly. Assume that (i) holds. Then the following product is also missing the variable $\theta_{p(1)}$ in row

$q(1)$:

$$P(\theta_1,\ldots,\theta_n)\,|_{\theta_{p(j)}=0} \tag{4}$$

where j is in any subset of $\{2,3,\ldots,n\}$. Suppose that P had the form

$$P(\theta_1,\ldots,\theta_n)=R[q(i),\theta_{p(i)}]N(\theta_1,\ldots,\theta_n)$$

for some $i\neq 1$, where N is an OP in the $n-1$ variables $\theta_{p(j)}$, $j\neq i$. A contradiction will now be obtained. Let j be such that the rotation $R[q(j),\ \theta_{p(j)}]$ immediately precedes $R[q(1),\ \theta_{p(1)}]$ in the decomposition for P. Then, $R[q(j),\ \theta_{p(j)}]R[q(1),\ \theta_{p(1)}]$ is also missing $\theta_{p(1)}$ in row $q(1)$ since it has the form of Eq. (4). However, direct multiplication of this product shows that this can be true only if $q(j)=q(1)$. This would contradict the fact that P is an OP. Hence (ii) holds.

Next, suppose (ii) holds. Then

$$e^t_{q(1)}P(\theta_1,\ldots,\theta_n)=e^t_{q(1)}R[q(1),\theta_{p(1)}]M(\theta_1,\ldots,\theta_n)$$
$$=e^t_{q(1)}M(\theta_1,\ldots,\theta_n)$$

Thus row $q(1)$ of P coincides with row $q(1)$ of M. But M is missing the variable $\theta_{p(1)}$ in row $q(1)$ so that P must also be missing $\theta_{p(1)}$ in row $q(1)$. Therefore (i) holds and the proof is complete. ∎

Theorem 2 (Decomposition Theorem for Ordered Products). Let P be an OP in variables $\theta_1,\ldots,\theta_n$ which is given in the form of Eq. (3). Then P has the unique decomposition

$$P(\theta_1,\ldots,\theta_n)=\prod_{i=1}^{n}R[q(i),\theta_{p(i)}]$$
$$=\prod_{i=1}^{n}P(\theta_1,\ldots,\theta_n)\Big|_{\theta_{p(j)}=0,j\neq i} \tag{5}$$

where p and q are defined as follows:

(a) P is missing $\theta_{p(1)}$ in row $q(1)$, and, in general,

(b) $P(\theta_1,\ldots,\theta_n)\,|_{\theta_{p(1)}=\ldots=\theta_{p(j)}=0}$ is missing $\theta_{p(j+1)}$ in row $q(j+1)$ for $j=1,2,\ldots,n-1$.

Proof: If P is given by Eq. (3), then by lemma 2, P is missing some variable $\theta_{p(1)}$ in row $q(1)$ so that

$$P(\theta_1,\ldots,\theta_n)=R[q(1),\theta_{p(1)}]M(\theta_1,\ldots,\theta_n)$$

where M is as in the statement of lemma 2. Applying the above argument to M shows that there are integers $q(2)$ and $p(2)$, and an OP, M_1, consisting of $n-2$ rotations in the variables $\theta_{p(j)}$, $j\neq 1,2$ such that

$$M(\theta_1,\ldots,\theta_n)=R[q(2),\theta_{p(2)}]M_1(\theta_1,\ldots,\theta_n)$$

Thus,

$$P(\theta_1,\ldots,\theta_n)=\prod_{i=1}^{2}R[q(i),\theta_{p(i)}]M_1(\theta_1,\ldots,\theta_n)$$
$$=\left[\prod_{i=1}^{2}P(\theta_1,\ldots,\theta_n)\,|_{\theta_{p(j)}=0,j\neq i}\right]M_1(\theta_1,\ldots,\theta_n)$$

The full statement of the theorem now follows in a routine manner by induction. By theorem 1, the functions p and q are well defined and q satisfies $q(i)\neq q(i+1)$ since P is an OP. This concludes the proof. ∎

The transpose of an OP, P^t, is also an OP. Applying the result of theorem 2 to P^t leads to the following corollary.

Corollary 1: Under the hypothesis of theorem 2, P has the decomposition in Eq. (5), where

1) P is missing $\theta_{p(n)}$ in column $q(n)$, and, in general

2) $P(\theta_1,\ldots,\theta_n)\,|_{\theta_{p(n)}=\ldots=\theta_{p(n-j)}=0}$ is missing $\theta_{p(n-j-1)}$ in column $q(n-j-1)$ for $j=1,2,\ldots,n-2$. ∎

Any rotation factor appearing in the decomposition of an OP in n variables can be isolated by simply setting a particular set of $n-1$ variables equal to zero. This fact was used in the proofs of theorems 1 and 2 above. However, many products occurring in applications are not rotations by angles $\theta_1,\ \theta_2,\ldots$, but rather are rotations by $\pi/2-\theta_1$, $\pi+\theta_2$, etc. Certain identities like $\sin(\pi/2-\theta_1)=\cos\theta_1$, $\sin(\pi+\theta_2)=-\sin\theta_2$, etc., are then used to simplify each rotation so that the final form [Eq. (3)] is in terms of the variables $\theta_1,\theta_2,\ldots$. Of course such a product may not be an OP in the variables $\theta_1,\theta_2,\ldots$, and, by inspection, it is not at all clear which variables could be used to write P as an OP. In the sequel it will be shown that such products do have (unique) product decompositions and these decompositions are rather easily obtained through an application of theorem 2. At this point, the permutation p in definition 1 will be dispensed with for the sake of conciseness.

Definition 2: Let $P(\theta_1,\ldots,\theta_n)$ be an OP. A generalized ordered product (GOP) in the variables $\theta_1,\ldots,\theta_n$ is an expression of the form

$$T(\theta_1,\ldots,\theta_n)=P(\theta_1+\theta_1^0,\ldots,\theta_n+\theta_n^0)$$

where $\theta_1^0,\ldots,\theta_n^0\in G=\{0,\ \pm\pi/2,\ \pm\pi\}$.

The following two lemmas will be used in the proof of theorem 3 which provides the desired decomposition algorithm for GOP's.

Lemma 3: Let $\{j_1,j_2,j_3\}$ be a permutation of $\{1,2,3\}$, and let $\theta_0\in\{\pm\pi/2,\pm\pi\}$. Then

$$R[j_1,\theta]R[j_2,\theta_0]=R[j_2,\theta_0]R[j_3,\pm\theta],\ \text{if}\ |\theta_0|=\pi/2$$
$$=R[j_2,\theta_0]R[j_1,-\theta],\ \text{if}\ |\theta_0|=\pi \quad ∎$$

The proof of the lemma is clear from the geometry involved. For example, if $\theta_0=|\pi/2|$, the lemma states that rotation by θ about e_{j1}, is equivalent to first rotating about e_{j_3} by θ_0, then rotating about e_{j_2} by either θ or $-\theta$, and then rotating back about e_{j2} by $-\theta_0$. ∎

Lemma 4: Let

$$P(\theta_1,\ldots,\theta_n)=\prod_{i=1}^{n}R[q(i),\theta_i]$$

be an OP. Define T by

$$T(\theta_1,\ldots,\theta_n)=P(\theta_1,\ldots,\theta_n)R[q,\theta_0]$$

where $q\in\{1,2,3\}$ and $\theta_0\in G$. Then there is an ordered product S such that

$$T(\theta_1,\ldots,\theta_n)=R[q,\theta_0]S(\theta_1,\ldots,\theta_n)$$

Note that, in general, $S\neq P$ except in trivial cases.

Proof: If $\theta_0=0$ the result is trivial. If $\theta_0\neq 0$, the proof will consist of essentially "factoring" $R[q,\theta_0]$ to the left, one step at a time in such a way that, at any point in the process, T has the form

$$T=\Pi_1R[q,\theta_0]\Pi_2 \tag{6}$$

where Π_1, Π_2 are OP's. Note that Π_1 will always be an OP since it is an undisturbed ordered subproduct of P. Thus, we will only need to show that Π_2 is an OP at each iterative stage of the factoring process. First note that from lemma 3, if $|\theta_0|=\pi$, then the result easily follows with $S(\theta_1,\ldots,\theta_n)=P(-\theta_1,\ldots,-\theta_n)$. Next, let $\theta_0=|\pi/2|$. If $q=q(n)$ then Eq. (1) can be used to write T on the first iteration of the process as

$$T=\Pi_1R[q,\theta_0]R[q(n),\theta_n]$$

which has the form of Eq. (6) above. On the other hand, if $q \neq q(n)$, lemma 3 can be invoked to write T on the first iteration of the process as

Iteration 1:

$$T = \Pi_l R[q,\theta_0] R[s(n), \pm\theta_n]$$

where $s(n) \neq q$ and $s(n) \neq q(n)$. Thus T once again has the required form of Eq. (6). The first iteration is therefore complete. Suppose that this factoring process has been carried out successfully through the $(n-1)$st iteration so that

Iteration n − 1:

$$T = R[q(1),\theta_1] R[q,\theta_0] R[s(2), \pm\theta_2] \ldots$$

Beginning with $R[q(2),\theta_2^0]$, factor each constant matrix to the left using lemma 4. The result is

$$T(\theta_1,\ldots,\theta_n) = AP(\theta_1,\ldots,\theta_n)$$

where P is an OP. Noting that $A = T(0,\ldots,0)$ completes the proof. ∎

Applying theorem 3 to T^t yields the following corollary.

Corollary: With the assumptions in theorem 3, there is an ordered product P such that $T(\theta_1,\ldots,\theta_n) = P(\theta_1,\ldots,\theta_n) T(0,0,\ldots,0)$.

The Note will be concluded with an example which illustrates the usefulness of theorem 3.

Example: Let $T(\theta_1,\theta_2,\theta_3)$ be given by the matrix,

$$\begin{bmatrix} -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\sin\theta_3 \\ \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\sin\theta_3 \\ \sin\theta_1\cos\theta_3 & \sin\theta_3 & -\cos\theta_1\cos\theta_3 \end{bmatrix}$$

where $q \neq q(1)$, $q \neq s(2)$. Suppose the process failed on iteration n so that

Iteration n:

$$T = R[q,\theta_0] R[s(1), \pm\theta_1] R[s(2), \pm\theta_2] \ldots$$

where $s(1) = s(2)$. A contradiction will be found.

Now

$$T(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Thus, by theorem 3, $P = [p_{ij}] = T(0,0,0)^t T$ is an OP. Performing the multiplication $T(0,0,0)^t T$ yields P as

$$\begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\sin\theta_3 & \sin\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\sin\theta_3 \\ -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\sin\theta_3 \\ -\sin\theta_1\cos\theta_3 & -\sin\theta_3 & \cos\theta_1\cos\theta_3 \end{bmatrix}$$

Case 1: $q = s(1)$. Then $q = s(2)$. This would be possible only if T had the following form on iteration $n-2$:

$$T = R[s(1),\theta_1] R[s(2),\theta_2] R[q,\theta_0] \ldots$$

This would show that the product to the left of $R[q,\theta_0]$ is not an OP—a contradiction.

Case 2: $q \neq s(1)$. Then $\sigma_1 = \{q, q(1), s(1)\}$ would be a permutation of $\{1,2,3\}$. Now also $q \neq s(2)$ since $s(1) = s(2)$. Going back one more iteration we would have

Iteration n − 2:

$$T = R[q(1),\theta_1] R[q(2),\theta_2] R[q,\theta_0] \ldots$$

where $q(2) \neq s(2)$, $q \neq q(2)$, and $q \neq s(2)$. Then $\sigma_2 = \{q, s(2), q(2)\} = \{q, s(1), q(2)\}$ would also be a permutation of $\{1,2,3\}$ so that $\sigma_1 = \sigma_2$. Thus, $q(1) = q(2)$. But then the product to the left of $R[q,\theta_0]$ on iteration $n-2$ could not have been an OP. This contradiction proves the theorem. ∎

Theorem 3 (Decomposition Theorem for Generalized Ordered Products). Let T be a generalized ordered product in the variables $\theta_1,\ldots,\theta_n$. Then there is a unique ordered product P such that $T(\theta_1,\ldots,\theta_n) = T(0,0,\ldots,0)\, P(\theta_1,\ldots,\theta_n)$.

Proof: By assumption, there is an OP, say Q, and constants $\theta_1^0,\ldots,\theta_n^0 \in G$ such that

$$T(\theta_1,\ldots,\theta_n) = Q(\theta_1 + \theta_1^0,\ldots,\theta_n + \theta_n^0)$$

$$= R[q(1),\theta_1^0] R[q(1),\theta_1] R[q(2),\theta_2^0] R[q(2),\theta_2] \ldots$$

P is missing θ_2 in row 3. Setting $\theta_1 = \theta_3 = 0$ yields $R[3, -\theta_2]$. Set $\theta_2 = 0$ to obtain the reduced set,

$$\begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ -\sin\theta_1\sin\theta_3 & \cos\theta_3 & \cos\theta_1\sin\theta_3 \\ -\sin\theta_1\cos\theta_3 & -\sin\theta_3 & \cos\theta_1\cos\theta_3 \end{bmatrix}$$

Row 1 is missing θ_3. Set $\theta_1 = 0$ to get $R[1, -\theta_3]$. Finally, set $\theta_3 = 0$ to get $R[2, -\theta_1]$. Thus, $T = T(0,0,0) R[3, -\theta_2] R[1, -\theta_3] R[2, -\theta_1]$.

Acknowledgments

The author wishes to express his sincere thanks to N. Pixley for her help in the preparation of this Note. Acknowledgment is also given to Space Applications Corporation which provided the necessary funding for the publication of this Note.

Reference

[1] Pio, L.R., "Symbolic Representations of Coordinate Transformations," *IEEE Transactions on Aerospace and Navigational Electronics,* Vol. ANE-11, No. 2, June 1964, pp. 128-134.