Fundamental Frequency of Rocket and Missile Upper Stages: Quick Preliminary Design Solution

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Nomenclature

A, B, C	=	defined in Eqs. (8), (9), and (10)
a, b	=	undetermined coefficients in Eq. (6)
E	=	Young's modulus, N/m^2
I	=	area moment of inertia, m^4
J	=	concentrated rotary inertia, kg · m^2
k	=	stiffness of the rotational spring, N/rad
L	=	length of the column, m
M	=	concentrated mass, kg
m	=	mass per unit length of the column, kg/m
W	=	lateral displacement, m
w	=	nondimensional lateral displacement
α_1–α_3	=	undetermined coefficients in Eq. (2)
γ	=	nondimensional rotational spring stiffness parameter, kL/EI
λ_f	=	nondimensional frequency parameter, $m\omega^2 L^4/EI$
λ_J	=	nondimensional concentrated rotary inertia parameter, J/mL^3
λ_M	=	nondimensional concentrated mass parameter, M/mL
ξ	=	nondimensional axial coordinates, x/L
ω	=	circular frequency, rad/s
$'$	=	differentiation with respect to ξ

Introduction

THE solid propellant upper stage of rockets and missiles are very optimally designed to enhance the payload capability. The fore end of these stages is connected to the payload through a payload adaptor, and the aft end is connected to the relatively stiff lower stage by an interstage structure that is rotationally flexible. It is often necessary to find the local vibration characteristics, in terms of fundamental frequencies, to estimate the transverse loads transmitted to the payload from the upper stage. Thus, an accurate prediction of the fundamental frequency of the upper stage is required. Mathematically, the upper stage can be idealized as a cantilever beam with a concentrated mass and rotary inertia at the free end (by proper simulation of the mass and mass moment of inertia of the payload and payload adaptor). The flexibility of the interstage structure at the other end can be simulated by a rotational spring with a specific rotational spring stiffness. Thus, the problem considered turns out to be a spring-hinged uniform cantilever beam, with a concentrated mass and rotary inertia at the free end, for which an accurate estimate of the fundamental frequency is required.

Although the versatile finite element method[1] can be effectively used to solve this problem, it is typically too laborious and time consuming to conduct a parametric study. The present problem is defined by three parameters, namely, the spring stiffness parameter, the concentrated mass parameter, and the concentrated rotary inertia parameter. Alternatively, a simple and accurate energy method approach with admissible functions representing the spring-hinged condition is more attractive both for designers and analysts.

Thus, in the present Note, an attempt is made to determine simple expressions for the fundamental frequency of the aforementioned rocket structures, modeled as spring-hinged cantilever beams, that are amenable to parametric studies. The admissible functions used in the analysis are derived following Elishakoff.[2] The theoretical formulation is briefly discussed subsequently.

Theoretical Analysis

Figure 1 shows a spring-hinged cantilever beam with a concentrated mass and rotary inertia at the free end. The Lagrangian function of the beam, executing harmonic oscillations, is given by

$$\prod = \left(\frac{EI}{2}\right)\int_0^L W''^2\,\mathrm{d}x + \left(\frac{1}{2}\right)kW'^2\Big|_{x=0} - \left(\frac{m\omega^2}{2}\right)\int_0^L W^2\,\mathrm{d}x - \left(\frac{\omega^2 M}{2}\right)W^2\Big|_{x=L} - \left(\frac{\omega^2 J}{2}\right)W'^2\Big|_{x=L} \tag{1}$$

Nondimensionalizing W and x with respect to L, the preceding expressions can be rewritten in terms of nondimensional lateral displacement $w(=W/L)$ and axial coordinate $\xi(=x/L)$.

Following Elishakoff,[2] the lateral displacement distribution of the beam is assumed as

$$w = \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 \tag{2}$$

Equation (2) satisfies the geometric boundary condition

$$w(o) = 0 \tag{3}$$

The spring-hinged condition is

$$(EI/L)w''(o) = kw'(o) \tag{4}$$

Equations (2) and (4) give

$$\alpha_1 = (2/\gamma)\alpha_2 \tag{5}$$

Hence, for the present problem, the admissible displacement distribution w becomes

$$w = a(2\xi/\gamma + \xi^2) + b\xi^3 \tag{6}$$

Substituting Eq. (6) in Eq. (1), integrating, taking variations with respect to a and b, and solving the resulting two equations for a and b (which is a standard Rayleigh–Ritz procedure) one obtains the equation for the frequency parameter λ_f, in terms of the nondimensional parameters γ, λ_M and λ_J, as

$$A\lambda_f^2 + B\lambda_f + C = 0 \tag{7}$$

where

$$A = \left(\tfrac{1}{1260} - 58/84\gamma - 509/84\gamma^2\right) + \lambda_M\left(\tfrac{1}{105} - 86/21\gamma - 170/21\gamma^2\right) + \lambda_J\left(\tfrac{26}{70} - 360/14\gamma - 244/14\gamma^2\right) + \lambda_J\lambda_M(1 + 8/\gamma + 16/\gamma^2) \tag{8}$$

$$B = \left(-\tfrac{34}{35} + 122/7\gamma - 16/\gamma^2\right) - \gamma_M(4 + 28/\gamma + 48/\gamma^2) - \lambda_J(12 + 60/\gamma + 48/\gamma^2) \tag{9}$$

$$C = 12(1 + 4/\gamma) \tag{10}$$

The lowest root of Eq. (7) gives the fundamental frequency of the system for different values of γ, λ_M, and λ_J. For the special case

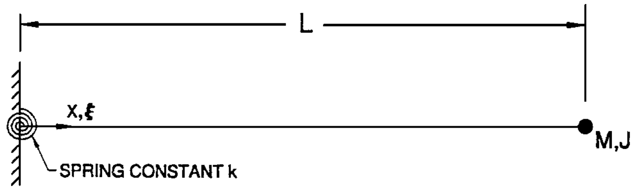

Fig. 1 Spring-hinged cantilever beam with concentrated mass and rotary inertia at the free end.

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Table 1 Values of frequency parameters, $\lambda_f^{1/4}$ of a spring-hinged cantilever beam

γ	λ_J	$\lambda_M = 0.1$	$\lambda_M = 1.0$	$\lambda_M = 10.0$	$\lambda_M = 100.0$
0.01	0.1	0.3700	0.2889	0.1758	0.0998
	1.0	0.2885	0.2556	0.1722	0.0996
	10.0	0.1755	0.1720	0.1487	0.0975
	100.0	0.0996	0.0994	0.0974	0.0839
0.1	0.1	0.6570	0.5121	0.3106	0.1762
	1.0	0.5059	0.4500	0.3040	0.1758
	10.0	0.3058	0.3001	0.2612	0.1720
	100.0	0.1734	0.1731	0.1698	0.1474
1.0	0.1	1.1216	0.8729	0.5202	0.2941
	1.0	0.8009	0.7320	0.5062	0.2933
	10.0	0.4704	0.4647	0.4194	0.2854
	100.0	0.2658	0.2654	0.2622	0.2363
10.0	0.1	1.4679	1.1505	0.6885	0.3895
	1.0	0.9546	0.9003	0.6605	0.3878
	10.0	0.5478	0.5442	0.5116	0.3721
	100.0	0.3087	0.3085	0.3064	0.2880
100.0	0.1	1.5093	1.1924	0.7269	0.4127
	1.0	0.9787	0.9284	0.6940	0.4106
	10.0	0.5599	0.5567	0.5274	0.3917
	100.0	0.3154	0.3152	0.3134	0.2969

of a cantilever beam (for which γ is very large) A, B, and C can be derived from the analysis presented here in by excluding the terms with γ.

Numerical Results

When Eq. (7) derived here is used, the fundamental frequencies of spring-hinged cantilever beams with concentrated mass and rotary inertia at the free end can be evaluated for several cases of interest. The numerical results for the frequency parameter $\lambda_f^{1/4}$ are presented in Table 1 for different values of λ_M, λ_J, and rotational spring stiffness parameter γ. For a cantilever beam with very large values of γ and without concentrated mass and rotary inertia, Eq. (7) yields 1.879 for the frequency parameter, which is in very good agreement ($<1\%$ difference) with the published result $\lambda_f^{1/4} = 1.875$) (Ref. 3). The present results given in Table 1, when compared to corresponding results obtained by very accurate finite element method,[4] are in excellent agreement with a four significant figures accuracy for most of the cases. Somewhat less agreement was found for very low values of combined γ, λ_M, and λ_J, where the accuracy is up to three significant figures, even though the difference is less than 1%. The finite element results were compared with results of Lee,[5] and excellent agreement was reported for those presented. The present results are also in excellent agreement with those of Lee, except for a few combinations of γ, λ_J, and λ_M. (For example, for $\gamma = 10$, $\lambda_J = \lambda_M = 0.01$, the difference is about 9.5%, and for $\gamma = 1.0$, $\lambda_J = \lambda_M = 0.01$, the difference is about 5%.) Inasmuchas all other results are in very good agreement, the authors feel the present results are more accurate.

Conclusions

A simple formula for the free vibration behavior of the upper stage of a rocket or missile has been presented. This problem is treated mathematically as a spring-hinged cantilever beam with a concentrated mass and rotary inertia at the free end. The Rayleigh–Ritz method was used to obtain the fundamental frequency from suitably derived admissible functions that represent the spring-hinged boundary condition. Note that the fundamental frequencies obtained in the present study agree very well with those obtained by using a very accurate finite element method. Thus, the present formula can be reliably used to evaluate the fundamental frequency, which is an important design parameter, of a spring-hinged cantilever beam with end concentrated mass and rotary inertia, forming a mathematical model of the upper stage of a rocket/missile. The formula presented should be very useful for design engineers, who need simple, but accurate, closed-form solutions during the design phase.

References

[1]Zienkiewicz, O. C., *The Finite Element Method*, 3rd ed., McGraw–Hill, London, 1977, Chaps. 1–3.

[2]Elishakoff, I., "Apparently First Closed-Form Solution for Frequency of Beam with Rotational Spring," *AIAA Journal*, Vol. 39, No. 1, 2001, pp. 183–186.

[3]Hurty, W. C., and Rubinstein, M. F., *Dynamics of Structures*, Eastern Economy ed., Prentice–Hall of India, New Delhi, India, 1967, p. 203.

[4]Venkateswara Rao, G., and Kanaka Raju, K., "A Galerkin Finite Element Analysis of a Uniform Beam Carrying a Concentrated Mass and Rotary Inertia with a Spring-Hinge," *Journal of Sound and Vibration*, Vol. 37, No. 4, 1974, pp. 567–569.

[5]Lee, T. W., "Vibration Frequencies for a Uniform Beam with One End Spring-Hinged and Carrying a Mass at the Other Free End," *Journal of Applied Mechanics*, Vol. 40, No. 3, 1973, pp. 813–815.

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Fiber-Optic Sensors for the Study of Spacecraft-Thruster Interactions: Ion Sputtering

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Introduction

THE interaction between thruster effluents and spacecraft surfaces has received considerable attention recently. Historically, thruster interaction concerns have focused on self-contamination from nondirect and high-angle (measured from the thruster centerline) plume impingement. The growing popularity of distributed networks of cooperative, coorbiting satellite clusters has brought about an additional need to address direct plume impingement or cross contamination. Typically, quartz crystal microbalances (QCMs) are used to investigate spacecraft-thruster interactions where the major contamination mechanism is the adsorption of molecular species on critical surfaces.[1] New methods are required to investigate the complex nature of plume impingement from advanced ion electric thrusters where the major interaction is the sputtering of critical surfaces. Additionally, QCMs are limited in that they only provide interaction data at a single point; however, the plume characteristics of a typical ion thruster can vary by several orders of magnitude over short distances. This study focuses on the proof-of-principle demonstration of a fiber-optic contamination sensor (FOCS), which can provide a complete interaction map for ion thrusters as an alternative to QCMs. The FOCS measures the depletion of light transmitted through the fiber as the cladding material is removed (sputtered) by energetic plume ions. Although this work is primarily concerned with assessing the FOCS for highly energetic ion interactions that induce material sputtering, the sensor might also be appropriate as an adsorption monitor for molecular contaminants.[2]

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