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Engineering Notes

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Time-Optimal Three-Dimensional Trajectories for Solar Photon Thruster Spacecraft

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Introduction

THE attractive idea of solar sail is that, when the momentum transfer of solar photons onto a large, lightweight and reflective surface is used, the sail is propelled without the need of an active propulsion system or of any propellent. Of course, solar sails are capable of experiencing very low thrust, with accelerations on the order of $0.1-1$ mm/s^2 at 1 astronomical unit (AU) solar distance. Nevertheless, sails operate continuously over indefinitely long periods, giving rise to energy changes greater than that are possible with conventional (either chemical or electrical) propulsion systems. Accordingly, solar sail technology enables several missions at a low development cost. This is particularly important for small-mass, high-energy interplanetary missions, as well as for asteroid rendezvous and solar polar missions.[1–3]

In this Note an ideal flat solar sail and a solar photon thruster (SPT) spacecraft are considered. In an SPT spacecraft, the functions of collecting and directing the solar radiations are separated.[4] A large sun-pointing reflector directs the incoming solar radiation onto a small mirror (the collimator). This mirror directs the beam of radiation onto a second, directing mirror (the director), which is used to control the orientation of the force acting on the SPT. Accordingly, the force law is different from a flat sail. In fact, the thrust for an SPT sail is proportional to $\cos\alpha$ (where α is the sail cone angle, that is, the angle between the sun line and the direction of the sail acceleration), whereas the thrust exerted by the solar radiation pressure for a flat sail is proportional to $\cos^2\alpha$. Although the concept of SPT spacecraft is attractive, a detailed engineering study of this solar sail has not yet been performed.

This Note deals with the problem of finding the time histories of the sail orientation angles, for both solar sail models, that fulfill a given mission in a prescribed optimal manner. In doing so we extend the results originally found by McInnes[5] with the assumption of coplanar circular orbits to a three-dimensional problem including inclined, elliptic orbits for the launch and target planets.

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Equations of Motion

The equations of motion for a spacecraft in a heliocentric inertial frame $\mathcal{T}_\odot(x, y, z)$ are

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \tag{1}$$

$$\dot{\boldsymbol{v}} = -\left(\mu_\odot / r^3\right)\boldsymbol{r} + \boldsymbol{a} \tag{2}$$

where $[\boldsymbol{r}]_{\mathcal{T}_\odot} = [r_x, r_y, r_z]^T$ and $[\boldsymbol{v}]_{\mathcal{T}_\odot} = [v_x, v_y, v_z]^T$ are the spacecraft position and velocity relative to $\mathcal{T}_\odot$, $\mu_\odot$ is the sun's gravitational parameter, and $\boldsymbol{a}$ is the acceleration due to the solar radiation pressure. When it is assumed that the solar sail is a perfectly reflecting body, the propelling acceleration $\boldsymbol{a}$ exerted by the solar radiation pressure P is given by

$$\boldsymbol{a} = (2PA/m)(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{a}})^p \hat{\boldsymbol{a}} \tag{3}$$

where A is the sail area, $\hat{\boldsymbol{r}} \triangleq \boldsymbol{r}/r$ is the unit vector in the direction of the incident radiation from the sun, and $\hat{\boldsymbol{a}} \triangleq \boldsymbol{a}/a$ is the unit vector in the direction of thrust. The index p is used to define the type of solar sail, with $p = 2$ corresponding to a flat solar sail and $p = 1$ corresponding to an SPT. Note that $\hat{\boldsymbol{a}}$ is normal to the sail as long as a flat sail model is assumed.

Under the hypothesis that the solar radiation pressure has an inverse square variation with the distance r from the sun, one has

$$\frac{PA}{m} = \frac{\beta}{2}\frac{\mu_\odot}{r^2} = \frac{a_c \sigma^* \mu_\odot}{4P_\oplus r^2} \tag{4}$$

where m is the spacecraft mass; $\beta \triangleq \sigma^*/\sigma$ is the solar sail lightness number, that is, the ratio between the critical solar sail loading parameter $\sigma^* \triangleq 1.53$ g/m^2 and the sail loading $\sigma \triangleq m/A$; a_c is the characteristic acceleration of the solar sail; and $P_\oplus \cong 4.557 \times 10^{-6}$ N/m^2 is the solar radiation pressure at 1 AU.

Let $\mathcal{T}_{\text{orb}}(x_{\text{orb}}, y_{\text{orb}}, z_{\text{orb}})$ be an orbital frame whose unit vectors are $\hat{\boldsymbol{i}}_{\text{orb}} \equiv \hat{\boldsymbol{r}}, \hat{\boldsymbol{j}}_{\text{orb}}$, and $\hat{\boldsymbol{k}}_{\text{orb}}$. Assume that the plane $z_{\text{orb}} = 0$ contains the axis z of the $\mathcal{T}_\odot$ frame and y_{orb} points toward the ecliptic pole. It is useful to express the components of $\hat{\boldsymbol{a}}$ in the $\mathcal{T}_{\text{orb}}$ frame as a function of the thrust cone angle $\alpha \triangleq \arccos(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{a}}) \in [0, \pi/2]$ and of the thrust clock angle $\delta \in [-\pi, \pi]$ (Fig. 1). One has

$$[\hat{\boldsymbol{a}}]_{\mathcal{T}_{\text{orb}}} = [\cos\alpha, \sin\alpha\cos\delta, \sin\alpha\sin\delta]^T \tag{5}$$

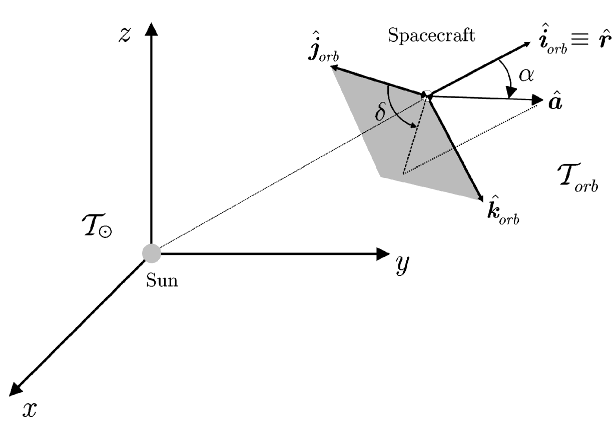

Fig. 1 Reference frames.

From Eqs. (3) and (4), it is clear that, r and β being equal, an SPT sail gives a propelling acceleration $\|\boldsymbol{a}_{\mathrm{SPT}}\|$ greater (or equal when $\alpha = 0$ or $\alpha = \pi/2$) than the corresponding acceleration of a flat sail $\|\boldsymbol{a}_{\mathrm{flat}}\|$. The maximum difference between $\|\boldsymbol{a}_{\mathrm{SPT}}\|$ and $\|\boldsymbol{a}_{\mathrm{flat}}\|$ (equal to 25% of $\|\boldsymbol{a}_{\mathrm{flat}}\|$) is obtained when $\alpha = \pi/3$.

Trajectory Optimization

The problem addressed here is that of finding the optimal control law $\boldsymbol{u}(t) = [\alpha(t), \delta(t)]^T$ (where $t \in [0, t_f]$) that minimizes the time t_f necessary to transfer the spacecraft from an initial $(\boldsymbol{r}_0, \boldsymbol{v}_0)$ to a final $(\boldsymbol{r}_f, \boldsymbol{v}_f)$ prescribed state. This amounts to maximizing the performance index:

$$J = -t_f \tag{6}$$

From Eqs. (1) and (2), the Hamiltonian of the system is

$$H = \boldsymbol{\lambda}_r \cdot \boldsymbol{v} - \left(\mu_\odot / r^3\right) \boldsymbol{\lambda}_v \cdot \boldsymbol{r} + \boldsymbol{\lambda}_v \cdot \boldsymbol{a} \tag{7}$$

where $\boldsymbol{\lambda}_r \triangleq [\lambda_{r_x}, \lambda_{r_y}, \lambda_{r_z}]^T$ and $\boldsymbol{\lambda}_v \triangleq [\lambda_{v_x}, \lambda_{v_y}, \lambda_{v_z}]^T$ are the vectors adjoint to the position and the velocity, respectively. Their time derivatives are provided by the Euler–Lagrange equations:

$$\dot{\boldsymbol{\lambda}}_r = -\frac{\partial H}{\partial \boldsymbol{r}} = \frac{\mu_\odot}{r^3}\boldsymbol{\lambda}_v - \frac{3\mu_\odot \boldsymbol{r}(\boldsymbol{\lambda}_v \cdot \boldsymbol{r})}{r^5} - \beta\mu_\odot(\boldsymbol{\lambda}_v \cdot \hat{\boldsymbol{a}}) \frac{p(\boldsymbol{r}\cdot\hat{\boldsymbol{a}})^{(p-1)}\hat{\boldsymbol{a}} - (p+2)(\boldsymbol{r}\cdot\hat{\boldsymbol{a}})^p r^{(p-3)}\boldsymbol{r}}{r^{(p+2)}} \tag{8}$$

$$\dot{\boldsymbol{\lambda}}_v = -\frac{\partial H}{\partial \boldsymbol{v}} = -\boldsymbol{\lambda}_r \tag{9}$$

Recall that $\boldsymbol{\lambda}_v$ is referred to as the primer vector.[6] In this paper, the boundary conditions are constrained by the planetary ephemerides.[7] Accordingly, the tranversality condition is given by[8,9]

$$H(t_f) = 1 + \boldsymbol{\lambda}_r(t_f) \cdot \boldsymbol{v}_p(t_f) + \boldsymbol{\lambda}_v(t_f) \cdot \left.\frac{\partial \boldsymbol{v}_p}{\partial t}\right|_{t=t_f} \tag{10}$$

where $\boldsymbol{v}_p = \boldsymbol{v}_p(t)$ is the velocity of the target planet.

From the Pontryagin's maximum principle the optimal control law $\boldsymbol{u}(t)$, to be selected in the domain of feasible controls $\mathcal{U}$, is such that, at any time instant, the Hamiltonian is an absolute maximum. This amounts to maximizing the function H' that coincides with that portion of the Hamiltonian H that explicitly depends on the control vector, namely, with $H' \triangleq \boldsymbol{\lambda}_v \cdot \boldsymbol{a}$,

$$\boldsymbol{u} = \arg\max_{u\in\mathcal{U}} H \equiv \arg\max_{u\in\mathcal{U}} H' \tag{11}$$

The components of the primer vector $\boldsymbol{\lambda}_v$ in the $\mathcal{T}_{\mathrm{orb}}$ frame are more suitably expressed in the form

$$[\boldsymbol{\lambda}_v]_{\mathcal{T}_{\mathrm{orb}}} = \lambda_v[\cos\tilde{\theta}, \sin\tilde{\theta}\cos\tilde{\delta}, \sin\tilde{\theta}\sin\tilde{\delta}]^T, \qquad \tilde{\theta} \in [0, \pi] \tag{12}$$

where $\lambda_v = \|\boldsymbol{\lambda}_v\|$ and $\tilde{\theta}$ and $\tilde{\delta}$ are the primer vector cone and clock angle, respectively.

Substituting Eq. (5) into Eq. (7) one has

$$H' = \beta\left(\mu_\odot / r^2\right)\lambda_v \cos^p\alpha(\cos\alpha\cos\tilde{\theta} + \sin\alpha\cos\delta\sin\tilde{\theta}\cos\tilde{\delta} + \sin\alpha\sin\delta\sin\tilde{\theta}\sin\tilde{\delta}) \tag{13}$$

Checking for stationary values of H', that is, letting $\partial H'/\partial \boldsymbol{u} = 0$, we find that, for $\tilde{\theta} \in [0, \pi]$,

$$\tan\alpha = \frac{-(1+p)\cos\tilde{\theta} + \sqrt{(1+p)^2\cos^2\tilde{\theta} + 4p\sin^2\tilde{\theta}}}{2p\sin\tilde{\theta}} \tag{14}$$

$$\tan\delta = \tan\tilde{\delta} \tag{15}$$

As long as a flat sail model is assumed, Eq. (14) gives the well-known optimal control law that agrees with Sauer.[9] On the other hand, assuming an SPT model ($p = 1$), for $\tilde{\theta} \in [0, \pi]$, Eq. (14) reduces to the following relationship:

$$\alpha_{\mathrm{SPT}} = \tilde{\theta}/2 \tag{16}$$

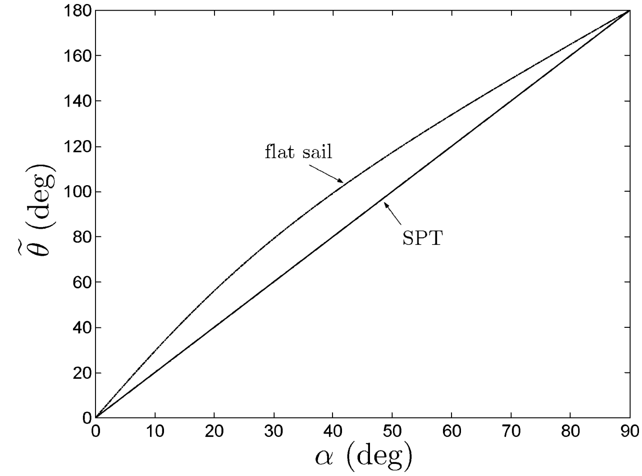

Fig. 2 Optimal sail cone angles for flat solar sail and SPT.

The optimal control law for an SPT sail is impressively simple. Equation (16) extends the result found by McInnes[5] under the assumption of coplanar, circular orbits. Figure 2 shows a comparison of the optimal sail cone angles for flat solar sail and SPT spacecraft. Note that the steering law (16) also may be derived through simple geometrical considerations, when it is observed that the optimal α is the angle that, according to Eq. (11), maximizes the projection of the vector $\boldsymbol{a}$ on the direction of the primer vector $\boldsymbol{\lambda}_v$.

As a consequence of Eq. (15), the unit vectors $\hat{\boldsymbol{a}}, \hat{\boldsymbol{r}}$, and $\hat{\boldsymbol{\lambda}}_v \triangleq \boldsymbol{\lambda}_v/\lambda_v$ are coplanar. This allows one to remove the dependence on $\hat{\boldsymbol{a}}$ in the equations of motion (2) and (3) and in the Euler–Lagrange equations. The result is

$$\hat{\boldsymbol{a}} = \begin{cases} \dfrac{\sin(\tilde{\theta}-\alpha)}{\sin\tilde{\theta}}\hat{\boldsymbol{r}} + \dfrac{\sin\alpha}{\sin\tilde{\theta}}\hat{\boldsymbol{\lambda}}_v & \text{for} \quad \tilde{\theta} \in (0, \pi) \\ \hat{\boldsymbol{r}} & \text{for} \quad \tilde{\theta} = 0 \end{cases} \tag{17}$$

where

$$\cos\tilde{\theta} = \hat{\boldsymbol{\lambda}}_v \cdot \hat{\boldsymbol{r}}, \qquad \sin\tilde{\theta} = |\hat{\boldsymbol{\lambda}}_v \times \hat{\boldsymbol{r}}| \tag{18}$$

Observe that $\tilde{\theta} = \pi$ corresponds to $\alpha = \pi/2$. [See Eq. (14) and Fig. 2.] Accordingly, in this case, all of the terms containing $\hat{\boldsymbol{a}}$ vanish both in the equations of motion (because $\boldsymbol{a} = 0$) and in the Euler–Lagrange equations. Finally, from Eqs. (16) and (17), the acceleration unit vector for an SPT model is given by

$$\hat{\boldsymbol{a}}_{\mathrm{SPT}} = \sqrt{1/[(2+\cos\tilde{\theta})]}(\hat{\boldsymbol{r}} + \hat{\boldsymbol{\lambda}}_v) \qquad \text{for} \quad \tilde{\theta} \in [0, \pi) \tag{19}$$

Case Study

The described control laws have been applied to simulate minimum-time trajectories from Earth to Mars using ideal flat sail and SPT spacecraft. Both positions and velocities of the planets at departure ($t = 0$) and arrival ($t = t_f$) are calculated by means of ephemeris data. The starting mission date is 21 January 2016. This date corresponds to the optimal launch date for a flat sail. A set of canonical units, distance unit $(\mathrm{DU}_\odot) = 1$ AU and time unit $(\mathrm{TU}_\odot) = 58.132440906$ solar days,[10] has been used in the integration of the differential equations to reduce their numerical sensitivity. The differential equations were integrated in double precision using a Runge–Kutta-fifth-order scheme with absolute and relative errors of 10^{-10}. The boundary value problem (BVP) associated to the variational problem is constituted by the equations of motion (1) and (2) and the Euler–Lagrange equations (8) and (9). Also, the adjoint vectors $\boldsymbol{\lambda}_r$ and $\boldsymbol{\lambda}_v$ should satisfy the transversality condition (10). The BVP has been solved by means of a hybrid numerical technique that combines the use of genetic algorithms,[11] to obtain a rough estimate of the adjoint variables, with gradient-based and direct methods[12,13] to refine the solution.

Table 1 shows the mission time for a number of values of the characteristic acceleration a_c both for flat sail and SPT spacecraft.

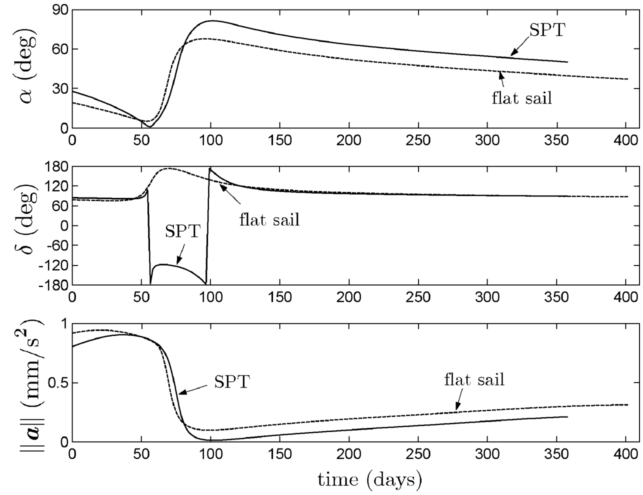

Fig. 3 Time histories of control angles and propulsive acceleration for flat ideal sail and SPT spacecraft with a_c = 1 mm/s^2 (starting date 21 January 2016).

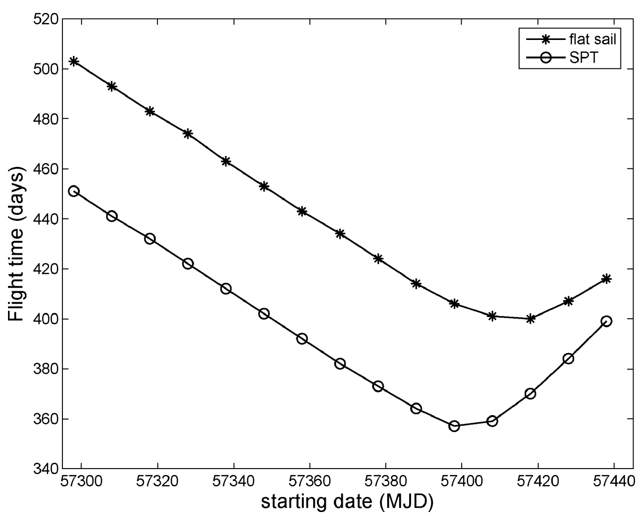

Fig. 4 Flight times for a Mars mission with ideal flat sail and SPT spacecraft (a_c = 1 mm/s^2).

As expected, there is a significant improvement of system performance, that is, a reduction of mission times, when a SPT spacecraft model is employed. Note that the trip time saving is greater when low values of characteristic acceleration are considered. This behavior is in accordance with the results found by McInnes,[5] who assumed coplanar circular orbits and did not take into account the planetary ephemeris data. Figure 3 show the time histories of control angles α and δ for a flat ideal sail and an SPT spacecraft, when a characteristic acceleration $a_c = 1$ mm/s^2 ($\beta = 0.1679$) is assumed. Note that, for the flat sail model, both the control histories and the mission time (401 days) are in agreement with other data found in the literature.[9,14]

Finally, a number of trajectories have been investigated in the time interval ranging from October 2015 to February 2016. A comparison of mission times as a function of the modified Julian date (MJD) is shown in Fig. 4. Simulations reveal that ideal flat sails require a considerable increase of flight time (when compared to an SPT spacecraft), ranging from a minimum of 4% to a maximum of 12%.

Table 1 Mission time (days) as a function of the characteristic acceleration a_c for flat sail and SPT spacecraft

a_c, mm/s^2	Flat sail	SPT
0.7	484	433
0.93	409	367
1	401	358
1.5	363	326
2	344	304
2.5	332	299

[a]Start date 21 January 2016.

Conclusions

Minimum-time trajectories of flat and SPT solar sails for interplanetary missions have been investigated using an indirect approach. The real shape and space orientation of both the departure and arrival planetary orbits is considered. The optimal control law for a SPT sail is impressively simple, and it extends the result found in the literature under the assumption of coplanar, circular orbits to a three-dimensional problem. When the results obtained with both sail models for a Earth to Mars minimum-time transfer are compared it is shown that a SPT sail offers significant performance improvements over conventional flat sails. However, although an SPT is more efficient, it would likely be heavier and more complex than a flat sail, and so an exact assessment of the improvement of system performance needs a more detailed study.

References

[1]Morrow, E., Scheeres, D. J., and Lubin, D., "Solar Sail Orbit Operations at Asteroids," *Journal of Spacecraft and Rockets*, Vol. 38, No. 2, 2001, pp. 279–286.

[2]Sauer, C. G., Jr., "Solar Sail Trajectories for Solar Polar and Interstellar Probe Missions," AAS/AIAA Astrodynamics Specialist Conf., AAS Paper 99-336, Aug. 1999.

[3]Leipold, M., Seboldt, W., Lingner, S., Borg, E., Herrmann, A., Pabsch, A., Wagner, O., and Brückner, J., "Mercury Sun-Syncronous Polar Orbiter with a Solar Sail," *Acta Astronautica*, Vol. 39, No. 1–4, 1996, pp. 143–151.

[4]Forward, R. L., "Solar Photon Thrustor," *Journal of Spacecraft and Rockets*, Vol. 27, No. 4, 1990, pp. 411–416.

[5]McInnes, C. R., "Payload Mass Fractions for Minimum-Time Trajectories of Flat and Compound Solar Sails," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 6, 2000, pp. 1076–1078.

[6]Lawden, D. F., *Optimal Trajectories for Space Navigation*, Butterworths, London, 1963, pp. 54–68.

[7]Meeus, J., *Astronomical Algorithms*, Willmann Bell, Richmond, VA, 1991, pp. 202–204.

[8]Lewis, F. L., *Optimal Control*, Wiley, New York, 1986, pp. 235, 236.

[9]Sauer, C. G., Jr., "Optimum Solar-Sail Interplanetary Trajectories," AIAA Paper 76-792, Aug. 1976.

[10]Seidelmann, P. K., *Explanatory Supplement to the Astronomical Almanac*, Univ. Science Books, Mill Valley, CA, 1992, pp. 696–697.

[11]Goldberg, D. E., *Genetic Algorithms in Search, Optimization, and Machine Learning*, Addison–Wesley, New York, 1989, pp. 27–86.

[12]Coleman, T. F., and Li, Y., "An Interior, Trust Region Approach for Nonlinear Minimization Subject to Bounds," *SIAM Journal on Optimization*, Vol. 6, No. 2, 1996, pp. 418–445.

[13]Lagarias, J. C., Reeds, J. A., Wright, M. H., and Wright, P. E., "Convergence Properties of the Nelder–Mead Simplex Method in Low Dimensions," *SIAM Journal of Optimization*, Vol. 9, No. 1, 1998, pp. 112–147.

[14]Wood, L. J., Bauer, T. P., and Zondervan, K. P., "Comment on 'Time-Optimal Orbit Transfer Trajectory for Solar Sail Spacecraft,'" *Journal of Guidance, Control, and Dynamics*, Vol. 5, No. 2, 1982, pp. 221–224.

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