On the Variety of Traveling Fronts in One-Variable Multistable Reaction−Diffusion Systems

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The still-open problem of the variety of asymptotic solutions to one-variable, one-dimensional infinite multistable reaction−diffusion systems is solved. We show that in such systems, besides monotonic traveling fronts, nonmonotonic traveling fronts can exist for appropriate initial conditions. The dependence of numbers of various types of traveling fronts on the number of stable stationary states also is given. Examples of traveling fronts for the chemical model describing two enzymatic (catalytic) reactions inhibited by an excess of their reactant is presented.

1. Introduction

The problem of the existence of traveling fronts in reaction−diffusion (RD) systems was investigated at the beginning of the previous century by Luther[1] and next by Fisher.[2] Exact results are available only for one-variable one-dimensional (1D) as well as two-dimensional and three-dimensional[3] infinite RD systems with two stationary states. Traveling fronts may propagate from an unstable stationary state to a stable stationary state or from a stable stationary state to another stable stationary state. In the first case, traveling fronts are unstable and the absolute values of their velocities are bounded from below and may depend on the initial conditions according to the Kolmogorov−Petrovsky−Piskunov theorem.[4] In bistable systems, traveling fronts connecting stable stationary states are attractive solutions for appropriate initial conditions and have unique velocity as follows from the Kanel[5] and Fife-McLeod[6] theorems. Traveling fronts have been observed in real chemical systems such as: the iodate-arsenate reaction for which exact results have been presented,[7] the Belousov−Zhabotinsky reaction,[8] and the oxidation ferrous ions (II) by nitric acid.[9]

Only partial results are known for one-variable 1D infinite tristable systems.[10] Fife and McLeod have obtained a relation for velocities of traveling fronts connecting external stable stationary states in a tristable system.[11] Czajkowski and Ebeling have studied motionless monotonic fronts in one-variable 1D infinite tristable systems.[12] Sheintuch and Nekhamkina have mentioned the existence of multiple monotonic fronts in a one-variable 1D tristable RD model.[13] Zemskov has studied solutions to a one-variable 1D infinite RD system with the piece-wise linear source term and analyzed their stability.[14] Vives et al. have considered multiple traveling fronts in the two-variable Sel'kov model with two stable states separated by a saddle point.[15] In all of these cases, the existence of nonmonotonic traveling fronts has not been mentioned. The existence of two impulses with different amplitudes has been shown in the two-variable model with two thresholds describing an enzymatic reaction allosterically activated by its product in an open system.[16]

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The existence of various possible types of traveling fronts as well as their numbers in multistable systems is still an open problem. In the present paper, we solve this problem. We show that in one-variable 1D infinite systems besides simple fronts and monotonic multiple fronts it is possible to observe also nonmonotonic fronts which to our best knowledge are never described in the literature. In Section 2, we show all possible types of traveling fronts that can be observed in 1D infinite RD systems with $n+1$ stable stationary states and calculate the number of possible fronts of each type. These general results are illustrated by two models in Section 3. In the last section, we discuss shortly the obtained results.

2. Results

The general problem of trigger fronts connecting different stable stationary states in a one-variable 1D infinite RD system is described by

$$\frac{\partial S(x,t)}{\partial t} - \frac{\partial^2 S(x,t)}{\partial x^2} = f(S) \qquad (1)$$

with appropriate initial conditions. We consider the case when the source term $f(S)$ has $2n+1$ ($n = 1, 2, ...$) positive ($0 < S_1 < S_2 < ... < S_{2n+1}$) stationary states among which $n+1$ odd ones are attractive and the remaining n even ones are repulsive. In chemical systems, $f(S)$ describes the kinetics of chemical reactions, and their roots correspond to homogeneous stationary states. The diffusion coefficient D is set equal to 1 due to the proper scaling of the spatial coordinate $x' = xD^{1/2}$ where x' is the physical coordinate.

We consider the family of initial condition problems (Cauchy problems) for all values of parameters, which specify $f(S)$ and initial conditions in the form

$$0 < S(x, 0), \limsup_{x\to-\infty} S(x, 0) < S_{2i} < \liminf_{x\to\infty} S(x, 0) < \infty \qquad (2)$$

for at least one value of i ($i = 1, 2, ..., n$). Such initial conditions ensure the existence of at least one traveling front joining two different stable stationary states as an asymptotic solution to eq 1 because S_{2i} are repulsive stationary states. A traveling front

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$S(z)$ where $z = x - ct$ and c is its velocity is a solution to the system of two ordinary differential equations

$$\frac{dS}{dz} = P(z) \tag{3a}$$

$$\frac{dP}{dz} = -cP(z) - f(S) \tag{3b}$$

System 3 has $2n+1$ stationary states $(0, S_i)$ for $i = 1, 2, ..., 2n+1$. They are nodes if the zeros of $f(S)$ S_i are repulsive or saddle points if S_i are attracting. The uniqueness of the velocity of the traveling front connecting the attracting zeros is the result of the structural instability of the heteroclinic orbit joining the corresponding saddle points.[17] Let us mention that system 3 is invariable with respect to transformation

$$z \rightarrow -z + z_0,\ c \rightarrow -c,\ z_0 \in \mathbf{R} \tag{4}$$

therefore, only initial conditions (eq 2) are considered below. The symmetry (eq 4) means that if a traveling front exists for given initial conditions, then also its mirror reflection exists for symmetrical initial conditions.

2.1. Monotonic Traveling Fronts. Traveling fronts of the simplest type connect two stable stationary states. They can join two adjacent stable stationary states or two stable stationary states separated by some number of stable stationary states. If they connect adjacent stable stationary states S_{2i-1} and S_{2i+1} ($i = 1, 2, ..., n$), one can use directly the Kanel and Fife−McLeod theorem to show the existence and the stability of n traveling fronts $S_{2i-1,2i+1}$. There are n such fronts running to the right. For appropriately chosen values of parameters of $f(S)$, fronts connecting states S_{2i-1} and S_{2i+1} can run to the left. Taking into account the mirror symmetry condition (eq 4) and fronts running to the left, we obtain the total number of traveling fronts connecting adjacent stable stationary states in the multistable system with $2n+1$ stationary states equal to $4n$.

For appropriate choice of the parameter values in $f(S)$, there are possible traveling fronts connecting nonadjacent stable stationary states S_{2i-1} and S_{2j+1} ($i = 1, 2, ..., j-1$, $j = i+1, i+2, ..., n$). Such fronts appear if initial conditions are such that $S(x, 0)$ belongs on the half-line $-\infty < x \leq x_1$ to the basin of attraction of S_{2i-1} and on the other half-line $x_1 \leq x_2 < x < \infty$ is in the basin of attraction of S_{2j+1}. $S(x, 0)$ must be bounded from below by S_{2i-1} and from above by S_{2j+1} for $x_1 < x \leq x_2$.

$$S_{2i-2} < S(x, 0) < S_{2i} \quad \text{for } x \in (-\infty, x_1] \tag{5a}$$

$$S_{2i-2} < S(x, 0) < S_{2j+2} \quad \text{for } x \in (x_1, x_2] \tag{5b}$$

$$S_{2j} < S(x, 0) < S_{2j+2} \quad \text{for } x \in (x_2, \infty] \tag{5c}$$

Integration of system 3 gives the general formula for the velocity of the traveling front $S_{2i-1,2j+1}$

$$c_{2i-1,2j+1} = -\frac{\sum_{k=2i-1}^{k=2j-1} \int_{S_k}^{S_{k+2}} f(S)\mathrm{d}S}{\int_{-\infty}^{\infty} P^2_{2i-1,2j+1}(z)\mathrm{d}z} \tag{6}$$

The direction of propagation of the front is determined by the propagation direction of the fronts connecting subsequent adjacent stable stationary states between S_{2i-1} and S_{2j+1}. All traveling fronts considered in this paper can be presented as trajectories on the phase plane $(S, P_{2i-1,2j+1})$. The increasing traveling front joining S_{2i-1} and S_{2j+1} is presented as a directed curve positioned above $P(S) = 0$. Examples of such trajectories are shown in Figure 1 together with schematic pictures of initial conditions and their asymptotic profiles. Taking into account the mirror symmetry property (see eq 4), we obtain the total number of single traveling fronts connecting nonadjacent stable stationary states equal to $2n(n-1)$. In tristable systems, four such fronts are possible, connecting stationary states S_1 and S_5 (Figure 1). According to the Kanel lemma,[6] $c_{13} > c_{15} > c_{35}$ for the fronts shown in Figure 1.

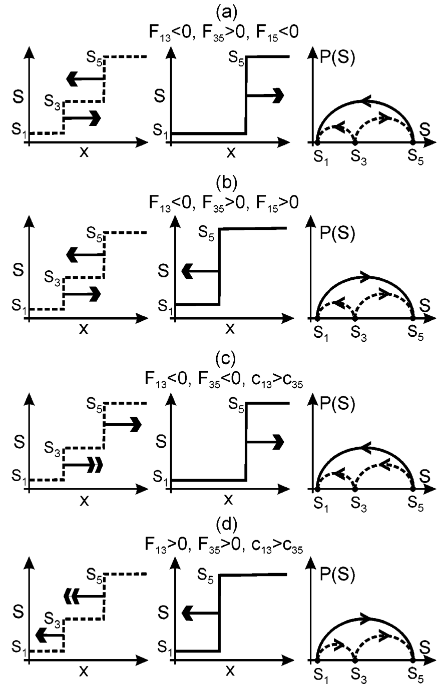

Figure 1. Single travelling fronts (solid) connecting nonadjacent stable stationary states S_1 and S_5 in the tristable system, examples of initial conditions (dashed), and corresponding trajectories are shown schematically. Numbers denote stable stationary states. $F_{i,j} = \int_{S_i}^{S_j} f(S)\mathrm{d}S$ and $c_{i,j}$ is the velocity of the front $S_{i,j}$. Faster fronts have two arrowheads.

One can choose the values of the parameters of $f(S)$ in such a way that monotonic multiple traveling fronts appear for appropriate initial conditions (eq 5). Such fronts are monotonic compositions of single traveling fronts. In the simplest case of a tristable system for $F_{13} > 0$ ($c_{13} < 0$) and $F_{35} < 0$ ($c_{35} > 0$) and initial conditions

$$\limsup_{x\to-\infty} S(x, 0) < S_2, S_4 < \liminf_{x\to\infty} S(x, 0) \tag{7}$$

the multiple front $S_{1,3,5}$ exists. $F_{i,j}$ denotes $\int_{S_i}^{S_j} f(S)\mathrm{d}S$. In this case, two single fronts propagate in the opposite directions (Figure 2a). The multiple fronts may be presented on one plane (S, P) with vertical axes $P(S)$ corresponding to the constituent fronts. Such a "compound phase plane" is just the graphical composition of phase planes corresponding to the constituent fronts. The trajectory corresponding to the multiple front shown schematically in Figure 2a consists of two trajectories above the half-axis $0S$. They begin at stable stationary states S_1 and S_5, and end at S_3.

For $F_{13} < 0$ and $F_{35} < 0$ and $c_{13} < c_{35}$ (Figure 2b) or $F_{13} > 0$ and $F_{35} > 0$ (Figure 2c), another type of the multiple front exists. In this case, the constituent fronts run in the same direction. On "the compound phase plane," the multiple front shown in Figure 2b corresponds to the trajectory situated above the half axis $0S$, which begins at S_5, passes through S_3 and ends

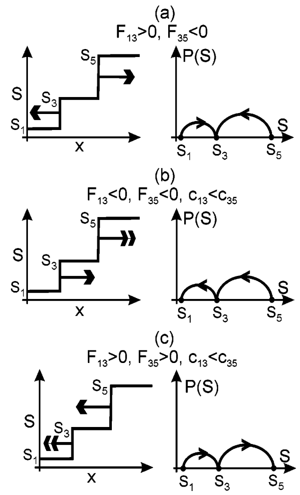

Figure 2. The multiple fronts and corresponding trajectories on "the compound phase plane" (S,P) for the tristable system are shown schematically.

at S_1. The multiple-front propagating in the opposite direction (Figure 2c) corresponds to the opposite-directed trajectory. Let us mention that if two trajectories start at the same stable stationary state, then no multiple fronts passing through it can exist. In this case, a single front is formed. This conclusion helps us to construct all possible multiple fronts. Taking into account the mirror symmetry and the possibility of change of the direction of propagation for appropriate chosen values of $f(S)$, we obtain the total number of multiple fronts whose constituent fronts run in the same direction in the multistable system equal to

$$2^{n+3} - 2(n+1)(n+2) - 4 \tag{8}$$

Similarly, one can obtain the number of all multiple fronts whose constituent fronts run in the opposite directions equal to

$$2((n-3)2^n + n + 3) \tag{9}$$

2.2. Nonmonotonic Traveling Fronts. Besides the monotonic fronts described above, there are also possible nonmonotonic compositions of monotonic fronts propagating in the same direction in multistable systems. Such fronts are impossible in bistable systems. Their existence has never been discussed in the literature.

In the tristable system for which $F_{13} < 0$, $F_{35} > 0$, $F_{15} = F_{13} + F_{35} < 0$ and $c_{1,5} < c_{3,5}$, the nonmonotonic front $S_{1,5,3}$ running to the right (Figure 3a) exists for initial conditions

$$\lim_{x\to-\infty} \sup S(x, 0) < S_2 \tag{10a}$$

$$S_4 < S(x, 0) < \infty,\ x \in [x_1,x_2] \tag{10b}$$

$$S_2 < \lim_{x\to\infty} \sup S(x, 0) < S_4 \tag{10c}$$

This front connects stable stationary states S_1, S_5, and S_3. [x_1, x_2] is the interval on which $S(x, t)$ belongs to the basin of attraction of S_5 such that for $t > T$, its length grows.

Nonmonotonic fronts consist of alternately increasing and decreasing monotonic fronts whose amplitudes (velocities) decrease (increase) in the direction of their propagations. These fronts can be single or multiple fronts whose constituent fronts run in the same direction. Let us consider the nonmonotonic front consisting of an odd number $l+1$ of single fronts ($l = 2$, 4, ..., $j-i$, $i = 1, 2, ..., n-1$, $j = i+1, i+2, ..., n$) propagating to the right. Assume that $l/2+1$ of them are increasing

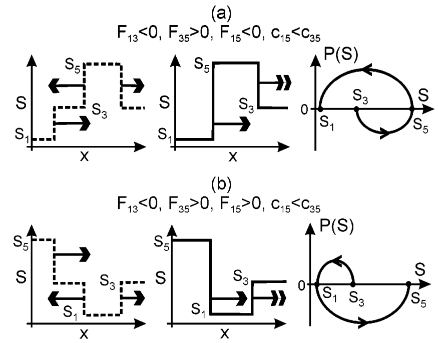

Figure 3. Nonmonotonic fronts (solid) in the tristable system, examples of initial conditions (dashed), and corresponding trajectories on "the compound phase plane" are shown schematically.

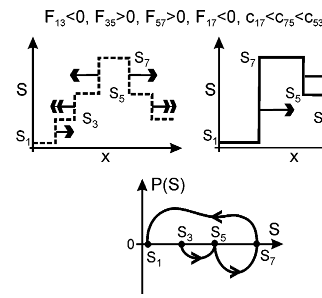

Figure 4. One (solid) of 28 possible nonmonotonic fronts in the multistable system with four stable stationary states ($n = 3$), the example of initial conditions (dashed), and the corresponding trajectory. All objects are drawn schematically.

$$S_{2i-1,2j+1}, S_{2k_1+1,2k_2+1,} S_{2k_3+1,2k_4+1}, ..., S_{2k_{l-1}+1,2k_l+1} \tag{11a}$$

and $l/2$ are decreasing

$$S_{2j+1,2k_1} + 1, S_{2k_2+1,2k_3+1}, ..., S_{2k_{l-2}+1,2k_{l-1}+1} \tag{11b}$$

where: $k_1 = i, i+1, ..., j-1$, $k_2 = k_1+1, k_1+2, ..., j-1, ..., k_l = k_{l-1}+1, k_{l-1}+2, ..., k_{l-2}-1$. The first increasing front has the largest amplitude and connects stable stationary states S_{2i-1} and S_{2j+1}. The last increasing front has the smallest amplitude and connects states $S_{2k_{l-1}+1}$ and S_{2k_l+1}. Such nonmonotonic front we denote as:

$$S_{2i-1,\ 2j+1,\ 2k_1+1,\ 2k_2+1,\ ...,\ 2k_{l-1}+1,\ 2k_l+1} \tag{12}$$

If the velocities of the constituent monotonic fronts fulfill the condition

$$0 < c_{2i-1,\ 2j+1} < c_{2j+1,\ 2k1+1} < ... < c_{2k_{l-1}+1,\ 2k_l+1}$$

the nonmonotonic front (eq 12) exists for appropriate initial conditions. Examples of such initial conditions are shown schematically in Figures 3 and 4 (dashed line). Each nonmonotonic front corresponds to a trajectory that has the shape of "a spiral" on "the compound phase plane" (see Figures 3 and 4). The whole trajectory is directed to the state that the system approaches after the last monotonic sequence has passed. The arms of "the spiral" correspond to subsequent monotonic fronts. For appropriate values of parameters of $f(S)$, the monotonic fronts can be multiple fronts. Each multiple front can join *m+2* different stable stationary states. The example of such multiple constituent fronts joining S_7, S_5, and S_3 is shown in Figure 4. The number of "spirals" with the first increasing front connecting states S_{2i-1} and S_{2j+1} is given by the following formula

$$\sum_{l=1}^{j-i}\binom{j-i}{l}\sum_{m=0}^{j-i-l}\binom{j-i-l}{m} \tag{13}$$

where: $l+1$ is the number of monotonic fronts for a given nonmonotonic front. Let us mention that for each nonmonotonic front moving in the one direction, one can change parameters of $f(S)$ in such a way that the nonmonotonic front connecting the same stable stationary states in a different order is possible. Examples of such nonmonotonic fronts for the tristable system are shown in Figure 3. Taking into account the mirror symmetry and the possibility of changes in the direction propagation of fronts, we obtain finally

$$3^{n+1} - 2^{n+3} + 2n + 5 \tag{14}$$

as the number of all nonmonotonic fronts possible in the considered multistable system.

3. Models

3.1. The Generalized Schlögl Model. To obtain a generic model of a multistable chemical system, one can formally generalize the Schlögl model.[18] The model consists of $2n+1$ reactions

$$A + iS \underset{k_{-i}}{\overset{k_i}{\rightleftharpoons}} (i+1)S \quad i = 2, ..., 2n \tag{15a}$$

$$S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B \tag{15b}$$

In this case, $f(S)$ is the polynomial of degree $2n+1$

$$f(S) = -k_{-2n}\,S^{2n+1} + k_{2n}\,A\,S^{2n} - ... - k_1\,S + k_{-1}B \tag{16}$$

where concentrations of A and B are assumed to be constant. From the signs of eq 16, we can easily see that the necessary condition for the existence of $2n+1$ roots of $f(S)$ stemming from the Descartes theorem is fulfilled. Figures 5 and 6 show the nonmonotonic fronts to eq 1 and corresponding trajectories on "the compound phase plane" for the source term $f(S) = a(S - S_1)(S - S_2)...(S - S_7)$. The solutions to eq 1 with appropriate initial conditions are obtained numerically on the interval $[0, L]$ by an algorithm using the Cranck−Nicolson[19] scheme for the diffusion term and the fourth order Runge−Kutta method for the source term. Zero-flux (Neumann) boundary conditions are used.

$$\frac{\partial S}{\partial x}(0, t) = \frac{\partial S}{\partial x}(L, t) = 0 \tag{17}$$

Because for traveling fronts $\lim_{x\to\infty} \partial S/\partial x\ (x, t) = \lim_{x\to-\infty} \partial S/\partial x\ (x, t) = 0$, if the constituent fronts are sufficiently far from the boundaries, then the Cauchy problem can be approximated by the Fourier problem with the zero-flux boundary conditions (eq 17). In bounded systems, all traveling fronts attend the boundaries after sufficiently long time and disappear. Therefore, asymptotic solutions in bounded systems have the form of homogeneous distributions with values equal to the most stable stationary state. Figure 7 shows compositions of multiple fronts whose middle front is almost motionless. In the case of a motionless front connecting stable stationary states S_{2i-1} and S_{2j+1}, the analytic solution for the trajectory can be easily obtained

$$P(S) = \sqrt{2|F_{2i-1}(S)|} \tag{18}$$

where $F_{2i-1}(S) = \int_{S_{2i-1}}^{S} f(\sigma)\,d\sigma$. Figure 8 shows the trajectory corresponding to the motionless front connecting S_1 and S_7. Let us notice that the Schlögl model contains nonelementary (higher than the second order and autocatalytic) reactions. However, using the Korzoohin algorithm,[20] one can construct a scheme composed of elementary reactions only, which can be reduced to the system described by an arbitrary polynomial.

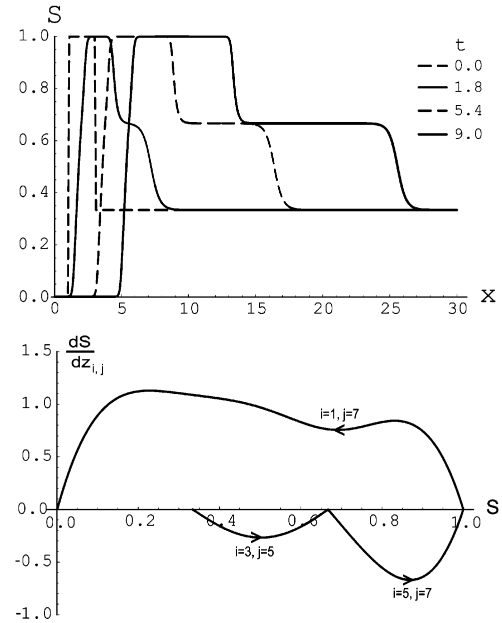

Figure 5. Nonmonotonic solutions to eq 1 with $f(S) = -10^4(S - 0.001)(S - 0.32)(S - 0.334)(S - 0.345)(S - 0.667)(S - 0.846)(S - 1)$. The initial condition is drawn as a dashed black line. The trajectory corresponding to the solution for time $t = 9$ (solid) is presented below on "the compound phase plane".

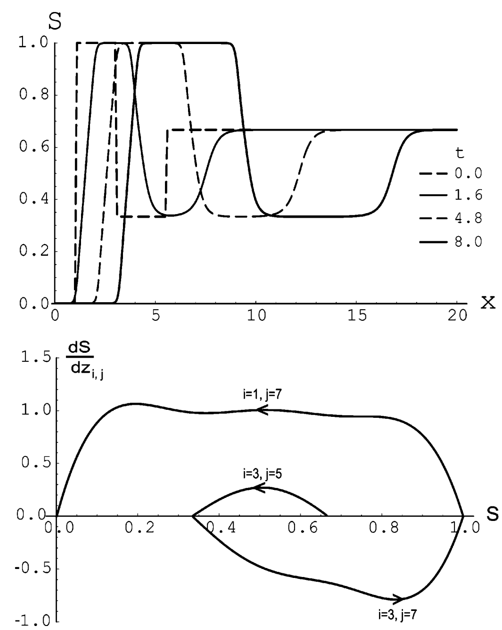

Figure 6. Nonmonotonic solutions to eq 1 with $f(S) = -10^4(S - 0.001)(S - 0.21)(S - 0.334)(S - 0.57)(S - 0.667)(S - 0.8)(S - 1)$. The trajectory corresponding to the solution for time $t = 8$ (solid) is presented below on "the compound phase plane".

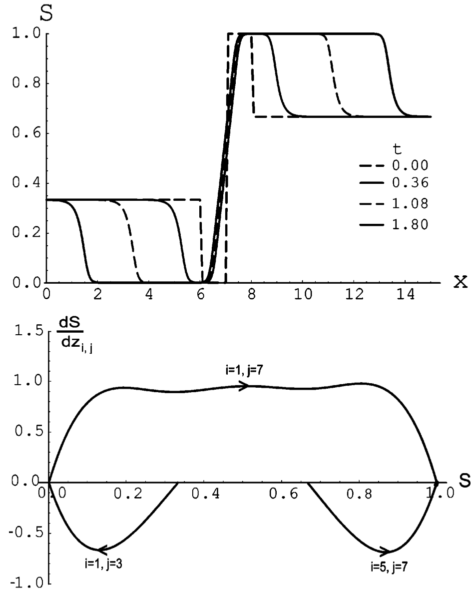

Figure 7. Nonmonotonic solutions to eq 1 with $f(S) = -10^4(S - 0.001)(S - 0.19)(S - 0.334)(S - 0.5)(S - 0.667)(S - 0.8)(S - 1)$. The trajectory corresponding to the solution for time $t = 1.8$ (solid) is presented below on “the compound phase plane”. The middle front $S_{1,7}$ is almost motionless.

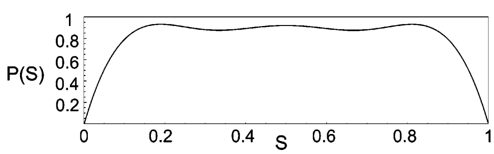

Figure 8. Trajectory $P(S)$ computed from eq 18 for the front connecting S_1 and S_7.

3.2. The Catalytic Model of the Tristable System. The model describing two parallel reactions catalyzed by two catalysts (enzymes) E and E′, which are inhibited by an excess of their reactant S, is a more realistic system exhibiting tristability. The model is based on the Langmuir−Hinshelwood (Michaelis−Menten) scheme. The system is open only for the reactant due to the reaction 19a.

$$S_0 \underset{k_{-0}}{\overset{k_0}{\rightleftharpoons}} S \tag{19a}$$

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} SE \tag{19b}$$

$$SE \xrightarrow{k_2} P + E \tag{19c}$$

$$S + SE \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} S_2E' \tag{19d}$$

$$S + E' \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} SE' \tag{19e}$$

$$SE' \xrightarrow{k'_2} P' + E' \tag{19f}$$

$$S + SE' \underset{k'_{-3}}{\overset{k'_3}{\rightleftharpoons}} S_2E' \tag{19g}$$

It should be emphasized that the model consists of only elementary reactions. Below, capital letters denote reagents and their concentrations as well, because this notation does not introduce misunderstandings. Reactions of formation products P and P′ are irreversible so their concentrations do not enter into the kinetic description of the system. E and E′ are

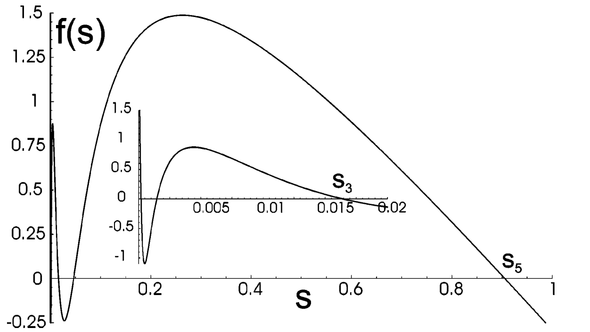

Figure 9. Graph of $f(s)$ (eq 20) for $k_0S_0 = 3.4$, $k_{-0} = 10^{-3}$, $k_2E_0 = 10$, $K_m = 100$, $K_3 = 0.01$, $k'_2E'_0 = 13$, $K'_m = 10^{0.2}$, and $K'_3 = 10^{-0.2}$. Values of s are normalized $s = k_{-0}/(k_0S_0)S$. The inset shows $f(s)$ for small s.

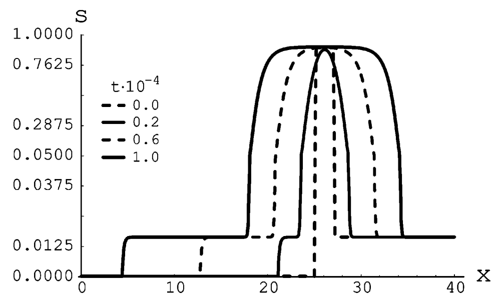

Figure 10. Composition of the multiple monotonic front $S_{1,3,5}$ and the single front $S_{3,5}$ being the solution to eq 1 with $f(S)$ (eq 20) where $k_0S_0 = 3.4$, $k_{-0} = 10^{-3}$, $k_2E_0 = 10$, $K_m = 100$, $K_3 = 0.01$, $k'_2E'_0 = 13$, $K'_m = 10^{0.2}$, and $K'_3 = 10^{-0.2}$. Values of s are normalized $s = k_{-0}/(k_0S_0)S$, and the vertical axis has scale changed at $s = 0.05$.

isoenzymes when P and P′ are the same chemical compound. The system is closed to catalysts and their complexes. Hence, the total concentrations of catalysts E_0 and E'_0 are the first integrals of kinetic equations, which allows one to eliminate the concentrations of SE and SE′ from considerations. The concentrations of catalysts (enzymes) are usually a few orders of magnitude smaller than concentrations of reactants. In such conditions E, S_2E, E′, and S_2E' are fast variables. Let us mention that kinetic equations for fast variables are linear, and each of them has only one quasi-stationary state. Therefore, the subsystem of fast variables approaches its quasi-stationary state for all values of the reactant concentration. In the slow time scale the dynamics of the system can be reduced to the equation describing changes of the reactant concentration according to the Tikhonov theorem.[21]

$$\frac{dS}{dt} = k_0S_0 - k_{-0}S - \frac{k_2E_0S}{K_m + S + K_3S^2} - \frac{k'_2E'_0S}{K'_m + S + K'_3S^2} \equiv f(S) \tag{20}$$

where: $K_m = (k_{-1}+k_2)/k_1$, $K_m' = (k'_{-1}+k'_2)/k'_1$, $K_3 = k_3/k_{-3}$, and $K'_3 = k'_3/k'_{-3}$. Let us mention that $f(S) = 0$ (eq 20) has only positive roots. For properly chosen values of the parameters, $f(S)$ may have five roots (see Figure 9). In this case, the first root is stable, so the $f(S)$ has properties required in the previous section. If the diffusion coefficients for E, E′, and their complexes can be neglected, then the behavior of the system is described by eq 1 with $f(S)$ given by eq 20. Figures 10 and 11 show the monotonic and nonmonotonic fronts, respectively, for the above-defined catalytic system.

4. Conclusions

In this paper, we solve the hitherto open general problem of types and numbers (see Table 1) of possible traveling fronts

TABLE 1: Types of Travelling Fronts for the Multistable System with $n+1$ Stable Stationary States

type of the solution	number of solutions	number of stationary states ($2n+1$) 3	5	7	9
single fronts connecting adjacent states	$4n$	4	8	12	16
single fronts connecting nonadjacent states	$2n(n-1)$	0	4	12	24
multiple fronts travelling in the same direction	$2^{n+3} - 2(n+1)(n+2) - 4$	0	4	20	64
multiple fronts travelling in the opposite directions	$2((n-3)2^n + n + 3)$	0	2	12	46
nonmonotonic solutions	$3^{n+1} - 2^{n+3} + 2n + 5$	0	4	28	128

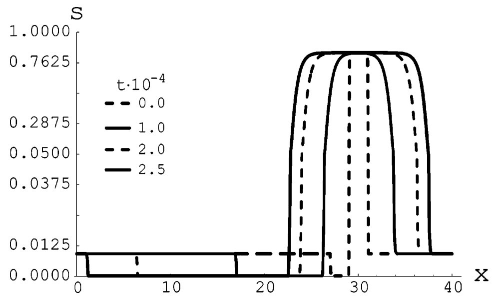

Figure 11. Composition of the nonmonotonic front $S_{3,1,5}$ and the single front $S_{3,5}$ being the solution to eq 1 with $f(S)$ (eq 20) where $k_0S_0 = 2.7$. Values of the remaining parameters are the same as in the previous figure.

connecting attracting stationary states in multistable systems. Besides well-known fronts connecting adjacent stable in bistable systems, the following types of traveling fronts are possible in multistable systems:

a. Single fronts connecting nonadjacent stable stationary states.

b. Monotonic multiple fronts composed of single fronts running with increasing velocities in the same direction.

c. Monotonic multiple fronts composed of two constituent fronts running in opposite directions.

d. Nonmonotonic compositions of fronts consisting of alternate increasing and decreasing monotonic fronts running in the same direction.

In the literature, only the fronts connecting nonadjacent stable stationary states in tristable systems have been mentioned.[10,11,13] The nonmonotonic fronts have never been reported earlier. We have found only one paper in which a nonmonotonic front is described.[22] However, this front appears in a bistable system. It joins successively two stable stationary states and an unstable stationary state. The part of the front joining two stable stationary states is motionless, but the part joining the lower stable stationary state with the unstable stationary state is unstable.

The fronts described in this paper may appear in all dynamic systems whose dynamics can be reduced to eq 1, provided that the kinetic term has an appropriate number of stable stationary states. The inhibition by an excess of a reactant described in Section 3.2. is a well-known phenomenon not only in biochemistry but also in microbiology[23] and heterogeneous catalysis. Enzymes inhibited by an excess of their reactant, such as ATCase inhibited by aspartate,[24] acetylcholinesterase inhibited by acetylcholine,[25] inorganic pyrophosphatase inhibited by pyrophosphate, and galaktokinase inhibited by galactose[26] immobilized on a solid support or in a CFUR (continuous-flow unstirred reactor) reactor, can give the traveling fronts presented above. Moreover, the fronts shown in this paper may be observed in multistable systems described by more than one variable like arsenate−iodate and chlorite−iodite systems, which exhibit tristability.[27] Our results seem to be useful also in experimental investigations of various waves in the bromate−cyclohexanedione−catalyst system in which no gas bubbles appear.[28]

References and Notes

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