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SELECTING A RITZ BASIS FOR THE REANALYSIS OF THE FREQUENCY RESPONSE FUNCTIONS OF MODIFIED STRUCTURES

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The exact reanalysis of a structure is time consuming and when repeated reanalyses are needed, it is often preferable to use an approximate Ritz technique. This approximate method consists in expressing the new frequency response function as a linear combination of vectors in a truncated modal basis. One notices that even though the convergence to the exact frequency response function is monotonic, it is generally irregular when the number of vectors introduced in the modal sub-basis increases. A solution to this consists of choosing the additional eigenvectors of the basis in such a way so as to give a best representation of the new frequency response function. This choice is not an easy task, particularly when the parametric modifications are very local and of large amplitudes. This article describes an original approximate reanalysis technique for accurately evaluating frequency response of a modified structure. It introduces new concepts for evaluating the static contribution of the neglected eigenvectors resulting in a set of additional vectors completing the original Ritz basis. This method will be appreciated by designers working on the optimization of a prototype. It can also be used during the iterations of a model updating procedure based on measured frequency response functions.

1. INTRODUCTION

The ever increasing demand for faster engineering analysis in the design process has resulted in substantial research and development on faster and more accurate approximate reanalysis methods. Indeed, the design engineer knows well how much time it takes to arrive at a final prototype: many potential prototypes and their respective models are developed before a choice is made. Too much time is devoted to analyze and compare each of their respective behaviours. Meanwhile, following a parametric modification of the prototype or the model, the structure is often very much like the initial one; its form and characteristics have simply been updated. It is therefore necessary to look for efficient and fast methods that speed up the design procedure. More attention has been devoted to looking for reanalysis techniques that use existing data as a basis to estimate the behaviour of design variants [1–10]. The difficulty of each of these procedures lies in how to complete

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the representation basis in order to reduce truncation effects. The latter may affect the accuracy of the predicted dynamic behaviour of the modified structure.

The purpose of the work described in this paper was to develop a method for constructing additional vectors by using the dynamic behaviour of the structure before modification. These vectors will complete the representation basis in order to obtain a better approximation of the frequency response for the structure after modification.

After presenting the details of the method, an application will be shown that illustrates its accuracy.

2. DESCRIPTION OF THE METHOD

The equation representing the behaviour of an N-degrees-of-freedom (d.o.f.) structure in its initial state, under a harmonic excitation, is expressed in matrix form as

$$\mathbf{Z}(\omega_i)\mathbf{y}(\omega_i) = \mathbf{f}(\omega_i), \tag{1}$$

where $\mathbf{Z}(\omega_i) \triangleq [\mathbf{K} + \mathrm{j}\omega_i\mathbf{B} - \omega_i^2\mathbf{M}] \in C^{N,1}$ is the dynamic stiffness matrix of the structure, $\mathbf{M}$, $\mathbf{K}$, $\mathbf{B} \in R^{N,N}$ are respectively the symmetric mass, stiffness and damping matrices of the structure, $\mathbf{M}$ and $\mathbf{K}$ being positive definite and $\mathbf{B}$ positive semi-definite and $\mathbf{f}(\omega_i)$, $\mathbf{y}(\omega_i) \in C^{N,1}$ represent the external force and response vectors, respectively. (A list of notation is given in the Appendix.)

The associated autonomous conservative system is given by

$$[\mathbf{K} - \omega_v^2\mathbf{M}]\mathbf{y}_v = 0, \qquad v = 1, 2, \ldots, N, \tag{2}$$

the solution of which gives the spectral matrix $\Lambda = \mathrm{Diag}\,\{\lambda_v = \omega_v^2\} \in R^{N,N}$ and the modal matrix $\mathbf{Y} = [\cdots \mathbf{y}_v \cdots] \in R^{N,N}$, which satisfy the orthonormality equations:

$$\mathbf{Y}^{\mathrm{T}}\mathbf{K}\mathbf{Y} = \Lambda, \qquad \mathbf{Y}^{\mathrm{T}}\mathbf{M}\mathbf{Y} = \mathbf{I}_N. \tag{3}$$

In order to express the approximate frequency response functions of the modified structure, the matrices $\mathbf{Y}$ and $\mathbf{\Lambda}$ are partitioned as

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{bmatrix}, \qquad \mathbf{Y} = [\mathbf{Y}_1 \mid \mathbf{Y}_2], \tag{4}$$

where $\mathbf{\Lambda}_1 = \mathrm{Diag}\{\lambda_v, v = 1\text{–}n\}$, $\mathbf{\Lambda}_2 = \mathrm{Diag}\{\lambda_v, v = (n+1)\text{–}N\}$, $\mathbf{Y}_1 \in R^{N,n}$, and $\mathbf{Y}_2 \in R^{N,N-n}$. The matrices $\mathbf{Y}_1 \in R^{N,n}$ and $\mathbf{\Lambda}_1 \in R^{n,n}$ are assumed to be known. The frequency response $\mathbf{y}(\omega_i) \in C^{N,1}$ is known for M discrete frequencies ω_i, $i = 1, 2, \ldots, M$, $0 \leqslant \omega_i \leqslant \omega_{\max}$ and has been obtained either by direct resolution of the linear system (1) of order N, or by a representation of $\mathbf{y}(\omega_i)$ in a large enough modal sub-basis $\mathbf{Y}_1' \in R^{N,n'}$, $n' > n$: i.e., containing all the eigenvectors corresponding to the eigenvalues included in the frequency band $[0; 3\text{–}4\omega_{\max}]$ and regrouped in the spectral sub-matrix $\mathbf{\Lambda}_1' \in R^{n',n'}$. More generally, this frequency band is chosen to include all eigenmodes that have an appreciable dynamic effect on the analysed frequency band: i.e., $[0, \omega_{\max}]$. The approximate solution of equation (1) is then written as a linear combination of the n' normal modes $\mathbf{Y}_1'$: i.e.,

$$\mathbf{y}(\omega_i) \cong \mathbf{Y}_1'\mathbf{d}(\omega_i), \tag{5}$$

where

$$\mathbf{d}(\omega_i) = [\mathbf{\Lambda}_1' + \mathrm{j}\omega_i\beta_1' - \omega_i^2\mathbf{I}_n]^{-1}\mathbf{Y}_1'^{\mathrm{T}}\mathbf{f}(\omega_i) \in C^{n',1}$$

in which

$$\beta_1' = \mathbf{Y}_1'^{\mathrm{T}}\mathbf{B}\mathbf{Y}_1' \in R^{n',n'}. \tag{6}$$

One now can obtain a new structure (s) by introducing into the model of the initial structure some known parametric modifications that do not change the order of the system, represented by the dynamic stiffness matrix

$$\Delta\mathbf{Z}(\omega_i) = \Delta\mathbf{K} + \mathrm{j}\omega_i\Delta\mathbf{B} - \omega_i^2\Delta\mathbf{M}, \tag{7}$$

where $\Delta\mathbf{M}$, $\Delta\mathbf{K}$, $\Delta\mathbf{B} \in R^{N,N}$ are the symmetric mass, stiffness and damping matrices of the structural modification, respectively.

The equations of motion for this new structure are

$$\mathbf{Z}^{(s)}(\omega_i)\mathbf{y}^{(s)}(\omega_i) = [\mathbf{Z}(\omega_i) + \Delta\mathbf{Z}(\omega_i)]\mathbf{y}^{(s)}(\omega_i) = \mathbf{f}(\omega_i). \tag{8}$$

One would like to estimate the frequency responses $\mathbf{y}^{(s)}(\omega_i)$ of this new structure by an accurate method without recourse to an exact but costly reanalysis.

In the following section, the classical Ritz frequency response reanalysis method that uses a truncated modal basis of the initial structure is reviewed. Then, a new method is proposed that consists in enlarging the Ritz basis with the addition of judiciously selected static residual vectors.

2.1. REANALYSIS OF FREQUENCY RESPONSES BY THE CLASSICAL RAYLEIGH-RITZ METHOD

The classical Rayleigh-Ritz technique consists in expressing the new frequency responses on the modal sub-basis $\mathbf{Y}_1 \in R^{N,n}$ formed from eigenvectors contained in the frequency band $[0;\ 1{\cdot}5\text{–}2\omega_{max}]$:

$$\mathbf{y}^{(s)}(\omega_i) \cong \mathbf{Y}_1\hat{\mathbf{c}}_1(\omega_i). \tag{9}$$

Substituting expression (9) into equation (8), premultiplying by $\mathbf{Y}_1^{\mathrm{T}}$ and taking into account the orthonormality expressions, one obtains

$$[\mathbf{\Lambda}_1 + \mathrm{j}\omega_i\mathbf{Y}_1^{\mathrm{T}}\mathbf{B}\mathbf{Y}_1 - \omega_i^2\mathbf{I}_n + \mathbf{Y}_1^{\mathrm{T}}\Delta\mathbf{Z}(\omega_i)\mathbf{Y}_1]\hat{\mathbf{c}}_1(\omega_i) = \mathbf{Y}_1^{\mathrm{T}}\mathbf{f}(\omega_i). \tag{10}$$

In practice, convergence to the exact response is monotonic but generally slow when n increases. The accuracy of the frequency response functions $\mathbf{y}^{(s)}(\omega_i)$ is uncertain, especially in the frequency regions where the contribution of $\mathbf{Y}_2$ type modes is predominant (example: antiresonant frequencies corresponding to a given d.o.f.). To improve the accuracy of $\mathbf{y}^{(s)}(\omega_i)$, two solutions can be examined as follows.

(a) Enlarge the modal basis: how to select the most efficient additional eigenvectors of the modal basis in order to improve the representation of $\mathbf{y}^{(s)}(\omega_i)$? This choice is not obvious in the case where parametric modifications are very local and of large amplitude.

(b) Enlarge the Ritz basis with additional vectors representing the static contribution of the $N - n$ neglected eigenvectors: how to construct a minimal number of additional vectors when the introduced structural modifications affect a great number of d.o.f.?

In the following, the second solution is deliberately chosen and a new method of constructing a static residual basis is developed.

2.2. REANALYSIS OF FREQUENCY RESPONSE FUNCTIONS BY EXTENDING THE RITZ BASIS

This approach consists of completing the sub-basis $\mathbf{Y}_1 \in R^{N,n}$ with a set of static displacement vectors regrouped in matrix $\mathbf{RG} \in R^{N,c}$. The latter are formed by a linear combination of the $N - n$ eigenvectors of the unknown complementary sub-basis

$\mathbf{Y}_2 \in R^{N,N-n}$. The column space of $\mathbf{Y}_2$ is represented by the static flexibility matrix $\mathbf{R} \in R^{N,N}$, of rank $N - m$:

$$\mathbf{R} = \mathbf{K}^{-1} - \mathbf{Y}_1 \mathbf{\Lambda}_1^{-1} \mathbf{Y}_1^{\mathrm{T}}. \tag{11}$$

In expression (11), the stiffness matrix $\mathbf{K}$ is assumed to be regular; in the contrary case a preliminary reduction is necessary.

The matrix $\mathbf{R} = \mathbf{Y}_2 \mathbf{\Lambda}_2^{-1} \mathbf{Y}_2^{\mathrm{T}}$ represents the static flexibility matrix associated with the $N - n$ modes not included in $\mathbf{Y}_1$. One thus can take as a Ritz basis the matrix $\hat{\mathbf{P}} \in R^{N,n+c}$:

$$\hat{\mathbf{P}} = [\mathbf{Y}_1 | \mathbf{R}\mathbf{G}]. \tag{12}$$

Here c is the number of degrees of freedom affected by the structural modifications. The c static displacements vectors are constructed by successively applying unit forces vectors $\mathbf{g}_i$, $i = 1$–c, (i.e. along the c-d.o.f. concerned by the modifications) upon the flexibility matrix $\mathbf{R}$.

With $\mathbf{G} = [\cdots \mathbf{g}_i \cdots] \in R^{N,c}$, the new expression for the frequency response $\mathbf{y}^{(s)}(\omega_i)$ is

$$\mathbf{y}^{(s)}(\omega_i) = [\mathbf{Y}_1 | \mathbf{R}\mathbf{G}]\mathbf{w}(\omega_i). \tag{13}$$

The adjunction of static residual matrix $\mathbf{RG}$ generally yields a better approximation for the frequency responses with lower calculation times than with a modal basis of dimension $n + c$ (the static responses are less costly than the dynamic ones). However, many difficulties are hidden in this formulation. The first is from a practical point of view in that in parametric identification procedures, structural modifications may affect a significant percentage of the model d.o.f. The number c of affected d.o.f. may be of order 10^2–10^3. The second problem is from a theoretical point of view: the residual matrix $\mathbf{RG}$ is not necessarily of maximum rank and this procedure may lead to completely erroneous results.

In the following, an original method is proposed which allows one to construct a static residual basis of relatively reduced dimension from a certain number of combinations of the columns of the initial matrix $\mathbf{R}$. In this method, a new procedure has been developed for selecting a reduced set among the d.o.f. affected by the modifications (i.e. the $\mathbf{g}_i$).

2.3. ESTIMATION OF THE LINKING FORCES

Expanding equation (8) in the form

$$\mathbf{Z}^{(s)}(\omega_i)\mathbf{y}^{(s)}(\omega_i) = [\mathbf{Z}(\omega_i) + \Delta\mathbf{Z}(\omega_i)](\mathbf{y}(\omega_i) + \hat{\mathbf{y}}(\omega_i)) = \mathbf{f}(\omega_i)$$

one obtains $\Delta\mathbf{Z}(\omega_i)\mathbf{y}^{(s)}(\omega_i) + \mathbf{Z}(\omega_i)\hat{\mathbf{y}}(\omega_i) = 0$: that is,

$$\mathbf{g}^{ex}(\omega_i) \triangleq -\Delta\mathbf{Z}(\omega_i)\mathbf{y}^{(s)}(\omega_i) = \mathbf{Z}\hat{\mathbf{y}}(\omega_i), \qquad i = 1, 2, \ldots, M. \tag{14}$$

The vector $\hat{\mathbf{y}}(\omega_i)$ represents the particular solution of the model due to the unknown force vector $\mathbf{g}^{ex}(\omega_i)$. This latter represents the unknown exact linking forces vector on the c-active d.o.f. between the structural modification and the initial structure.

The objective is to replace the basis $\mathbf{G} \in R^{N,c}$ of unit forces applied on the c-modified d.o.f. by an average real basis, independent of ω_i, formed of a minimum number of orthogonal vectors and enabling accurate representation of the linking forces $\mathbf{g}^{ex}(\omega_i)$.

In the expression for the exact linking force $\mathbf{g}^{ex}(\omega_i)$, the unknown frequency response vector $\mathbf{y}^{(s)}(\omega_i)$ is replaced by the "neighbouring" vector $\mathbf{y}(\omega_i)$. This approximation is reasonable and well justified by the fact that one is not looking for an exact force vector but rather wants only to construct a reasonable column space to represent them.

With these linking forces and the exterior force $\mathbf{f}(\omega_i)$ applied on the structure, one can form the matrix $\mathbf{F} \in R^{N,q}$, given by

$$\mathbf{F} = [\cdots \| \, \mathrm{real}\,(\mathbf{g}(\omega_i)) \, \| \, \mathrm{imag}\,(\mathbf{g}(\omega_i)) \, \| \cdots | \cdots \| \, \mathrm{real}\,(\mathbf{f}_j(\omega_i)) \, \| \, \mathrm{imag}\,(\mathbf{f}_j(\omega_i)) \, \| \cdots], \quad (15)$$

where

$$\mathbf{g}(\omega_i) = \Delta\mathbf{Z}(\omega_i)\mathbf{y}(\omega_i), \qquad \mathbf{f}_j(\omega_i) = \mathbf{e}_j(\mathbf{f}(\omega_i)^{\mathrm{T}}\mathbf{e}_j) \in C^{N,1}, \qquad j = 1\text{–}s, \quad (16)$$

represents the non-zero components of the exterior force $\mathbf{f}(\omega_i)$ applied on the structure, and ω_i, $i = 1$–h are the frequencies.

In expression (15), the forces $\mathbf{g}(\omega_i)$ and $\mathbf{f}(\omega_i)$ are calculated for h discrete frequencies ω_i chosen among the M ones of the frequency band (e.g., two frequencies inside and two others outside the six dB band of each of the resonances included in the frequency band). This procedure leads to a total of q vectors in the matrix $\mathbf{F}$.

The columns of the matrix $\mathbf{F}$ are not necessarily linearly independent and in order to condense this basis one retains only the r principal vectors, using the SVD of the matrix $\mathbf{F}$, given by

$$\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathrm{T}} \cong \mathbf{U}_r\boldsymbol{\Sigma}_r\mathbf{V}_r^{\mathrm{T}}, \quad (17)$$

where $\mathbf{U}_r \in R^{N,r}$, in which $r = \mathrm{rank}\,(\mathbf{F})$ corresponds to the r singular values σ_i satisfying the inequality $\sigma_{\max}/\sigma_i \leqslant t$, with t designating a given tolerance.

2.4. REANALYSIS USING THE CONDENSED BASIS CONSTRUCTED FROM THE LINKING FORCES

One is finally led to the following basis of representation:

$$\mathbf{y}^{(s)}(\omega_i) = [\mathbf{Y}_1 | \mathbf{R}_r]\mathbf{c}(\omega_i) = \mathbf{P}\mathbf{c}(\omega_i). \quad (18)$$

Here

$$\mathbf{P} \in R^{N,n+r}, \qquad \mathbf{R}_r = \mathbf{R}\mathbf{U}_r, \qquad \mathbf{c}(\omega_i) = [\mathbf{c}_1(\omega_i)/\mathbf{c}_2(\omega_i)].$$

Premultiplying equation (8) by $[\mathbf{Y}_1|\mathbf{R}_r]^{\mathrm{T}}$ and using equation (18) one obtains

$$\left[\frac{\mathbf{Y}_1^{\mathrm{T}}}{\mathbf{R}_r^{\mathrm{T}}}\right][\mathbf{K} - \omega_i^2\mathbf{M} + \mathrm{j}\omega_i\mathbf{B} + \Delta\mathbf{Z}(\omega_i)][\mathbf{Y}_1|\mathbf{R}_r]\left[\frac{\mathbf{c}_1(\omega)}{\mathbf{c}_2(\omega_i)}\right] = \left[\frac{\mathbf{Y}_1^{\mathrm{T}}}{\mathbf{R}_r^{\mathrm{T}}}\right]\mathbf{f}(\omega_i).$$

Expanding this expression yields

$$\begin{bmatrix} \boldsymbol{\Lambda}_1 - \omega_i^2\mathbf{I}_n + \mathrm{j}\omega_i\beta_1 + \mathbf{Y}_1^{\mathrm{T}}\Delta\mathbf{Z}(\omega_i)\mathbf{Y}_1 & \mathbf{Y}_1^{\mathrm{T}}\Delta\mathbf{Z}(\omega_i)\mathbf{R}_r \\ \mathbf{R}_r^{\mathrm{T}}\Delta\mathbf{Z}(\omega_i)\mathbf{Y}_1 & \mathbf{R}_r^{\mathrm{T}}[\mathbf{Z}(\omega_i) + \Delta\mathbf{Z}(\omega_i)]\mathbf{R}_r \end{bmatrix}\left[\frac{\mathbf{c}_1(\omega_i)}{\mathbf{c}_2(\omega_i)}\right] = \left[\frac{\mathbf{Y}_1^{\mathrm{T}}\mathbf{f}(\omega_i)}{\mathbf{R}_r^{\mathrm{T}}\mathbf{f}(\omega_i)}\right]. \quad (19)$$

Solving this equation leads to the linear combination vector $\mathbf{c}(\omega_i)$. The approximate frequency response $\mathbf{y}^{(s)}(\omega_i)$ of the modified system is then obtained by using equation (18).

2.5. REMARKS

(1) A normalization (reconditioning) of the columns of the matrix $\mathbf{F}$ before applying the SVD helps to take into account the different frequency contributions in a balanced way.

(2) The frequency responses $\mathbf{y}^{(s)}(\omega_i)$ may be obtained more accurately by subdividing the frequency band $[0; \omega_{\max}]$ into p sub-bands and by evaluating successively the p matrices $\mathbf{U}_r^{(k)}$, $k = 1$–p, of principal vectors.

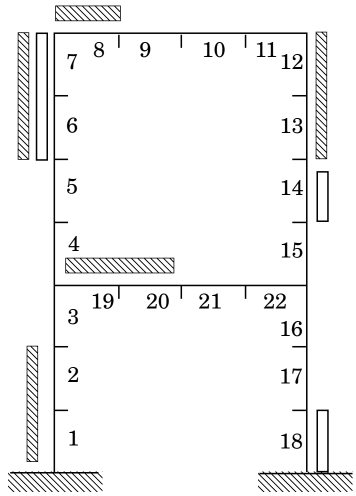

Figure 1. Stiffness and mass modifications of the initial model. ▧, Modified stiffness; ▭, modified mass.

(3) It is possible to evaluate the influence of the elementary sub-domains $\Delta \mathbf{Z}^{(j)}(\omega_i)$ in the modification matrix $\Delta \mathbf{Z}(\omega_i)$, given by

$$\Delta \mathbf{Z}(\omega_i) = \sum_{j}^{s} \Delta \mathbf{Z}^{(j)}(\omega_i). \tag{20}$$

Following the same principle as previously applied to the total structural modification, one proceeds successively with each modification of a sub-domain j to evaluate the corresponding residual forces

$$\mathbf{g}_i^{(j)}(\omega_i) = \Delta \mathbf{Z}^{(j)}(\omega_i)\mathbf{y}(\omega_i), \qquad i = 1, 2, \ldots, M, \tag{21}$$

which are regrouped in the matrix

$$\mathbf{F}^{(j)} = [\cdots \mathbf{g}_i^{(j)}(\omega_i) \cdots]. \tag{22}$$

TABLE 1

Model characteristics

	Position		
Damper number	Node number	Direction	Value (kg/s)
1	3	1	50
2	8	1	90
3	10	1	70
4	11	2	80
5	15	1	50
6	16	2	20
7	20	1	20
8	17	1	110
9	5	2	40
10	22	2	40

Young's modulus $E = 0{\cdot}21 \times 10^{12}$ N/m^2; beam section $S = 0{\cdot}50 \times 10^{-3}$ m^2; density $\rho = 7800$ kg/m^3; moment of inertia $I = 0{\cdot}417 \times 10^{-8}$ m^4

TABLE 2

Stiffness and mass modification

Element no.	$\frac{\text{Initial value}}{\text{Final value}}$ (ρ modification)	$\frac{\text{Initial value}}{\text{Final value}}$ (E modification)
1	1	0·5
2	1	0·5
6	2	0·5
7	2	0·5
8	1	0·5
12	1	2·0
13	1	2·0
14	0·5	1
18	0·5	1
19	1	2·0
20	1	2·0

TABLE 3

First five natural frequencies (Hz) *for the initial structure*

Mode number	Natural frequency
1	17·96
2	59·71
3	97·57
4	131·16
5	191·54

to which one applies the SVD in order to obtain the sub-basis $\mathbf{U}_r^{(j)}$. One then forms the matrix

$$\tilde{\mathbf{U}} = [\mathbf{U}_r^{(j)} \cdots \mathbf{U}_r^{(s)}], \tag{23}$$

where $\mathbf{U}_r^{(j)}$ contains a few principal vectors for each sub-domain. If the sub-domains are spatially uncoupled, $\tilde{\mathbf{U}}$ must be of maximum rank since it has a block diagonal form. By

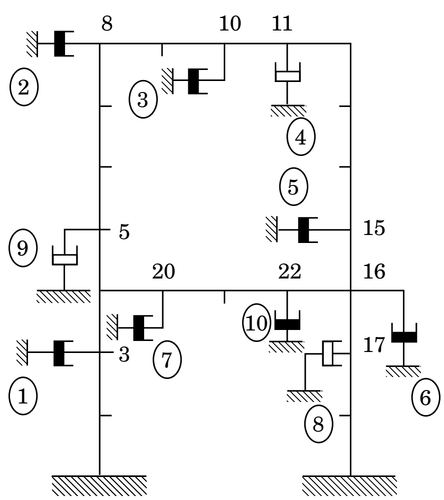

Figure 2. Damping modifications of the initial model. —■⊏, Modified damper.

TABLE 4

Damping modification

Damper number	Position: Node	Position: Direction	Initial value / Final value
1	3	1	1·800
2	8	1	0·444
3	10	1	1·571
5	15	1	1·400
6	16	2	2·000
7	20	1	3·500
10	22	2	1·500

replacing $\mathbf{U}_r$ by $\tilde{\mathbf{U}}$, the precision of reanalysis is again improved. In addition to this, applying QR decomposition with pivoting to the matrix $\tilde{\mathbf{U}}$, one can obtain an insight concerning the respective contribution of each elementary sub-domain j.

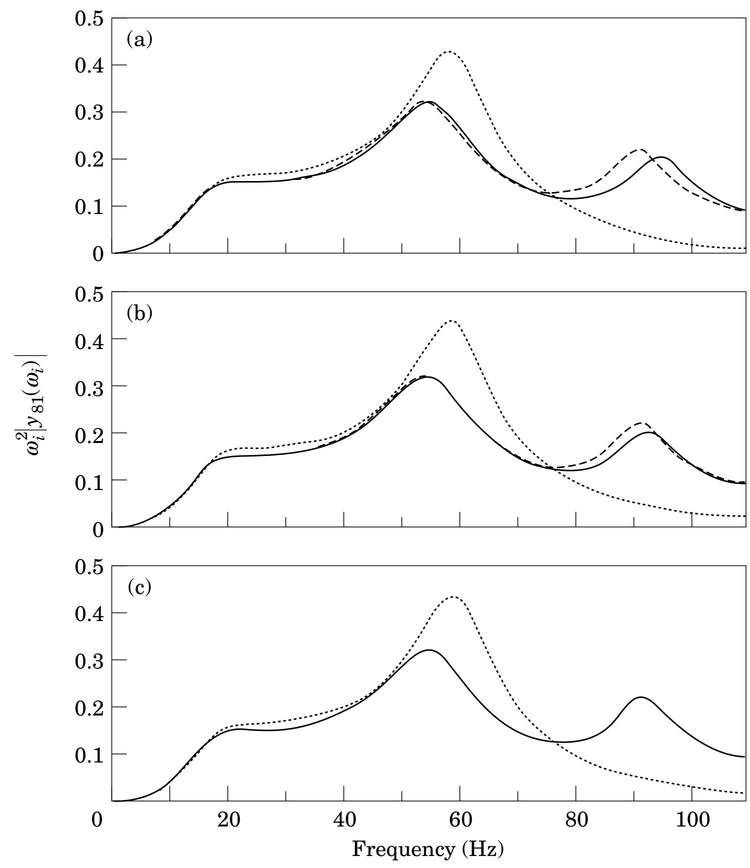

Figure 3. Acceleration amplitude $\omega_i^2|y_{81}(\omega_i)|$; $n = 7$ vectors in $\mathbf{Y}_1$ and (a) case 1, $m = 0$ static residual vectors; (b) case 2, $m = 3$; (c) case 3, $m = 9$. · · · , exact (initial structure); – –, exact (modified structure); —, approximate (modified structure).

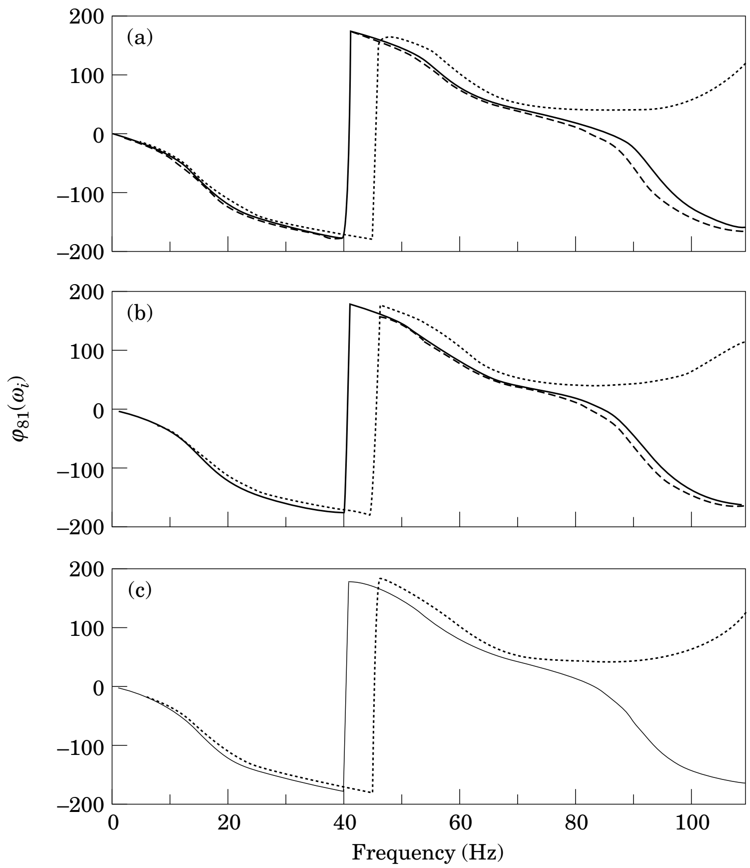

Figure 4. Phase $\varphi_{81}(\omega_i)$; $n = 7$ vectors in $\mathbf{Y}_1$ and (a) case 1, $m = 0$ static residual vectors; (b) case 2, $m = 3$; (c) case 3, $m = 9$. · · · exact (initial structure); – –, exact (modified structure); —, approximate (modified structure).

3. ACADEMIC EXAMPLE

3.1. INITIAL STRUCTURE

An illustration of this procedure is provided by the 2D frame represented in Figure 1, discretized into 22 finite beam elements. The beam characteristics are reported in Table 1. The energy dissipation is modelled by 10 discrete viscous dampers (see Table 1) and by Rayleigh damping ($\alpha_i \mathbf{M}_i + \beta_i \mathbf{K}_i$, $\alpha_i = 49{\cdot}3$, $\beta_i = 0$, i = 1, . . . , 22) for the 22 beam elements. With this damping, the first three modal damping ratios are respectively equal to 0·3974, 0·1155 and 0·0622.

3.2. MODIFIED STRUCTURE

3.2.1. *Modification of the conservative part*

One can modify separately, or simultaneously, the stiffness (variation of Young's modulus) and the mass (variation of mass density) of a set of finite beam elements. These stiffness and mass modifications are indicated in Figure 1. The characteristics of the modified finite beam elements are reported in Table 2, in columns 2 and 3 respectively.

The first seven frequencies of the initial model and the structure are reported in Table 3.

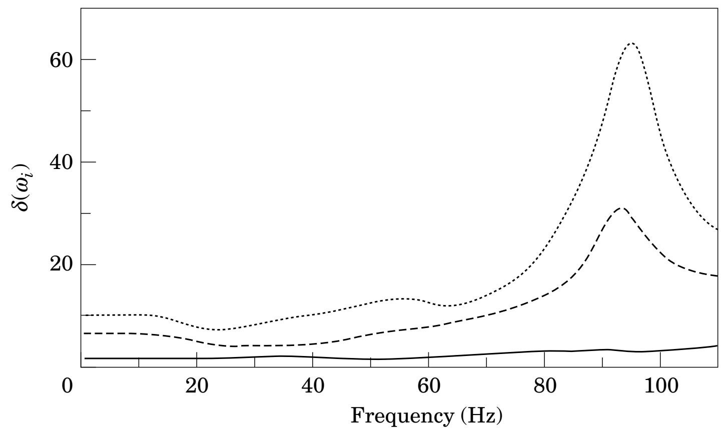

Figure 5. Relative amplitude representation error $\delta(\omega_i)$, for various numbers m of static residual vectors. Key as Figure 4.

3.2.2. *Modification of the dissipative part*

The damping modifications concern only seven of the discrete dampers. These are represented in Figure 2. The damping modifications are reported in Table 4, in column 3.

3.3. FREQUENCY BAND AND EXCITATION

The frequency band analyzed (0 to 110 Hz) contains the first three eigenmodes. The external excitation is independent of frequency and is applied at node 15, direction 1.

3.4. FREQUENCY RESPONSES

The accuracy of the frequency responses estimated by this new method is reported in Figures 3–9. The calculated frequency response converges more rapidly towards the exact frequency response than that calculated by using the classical Ritz method. The improvement in the frequency response of the structure is measured by the reduction of

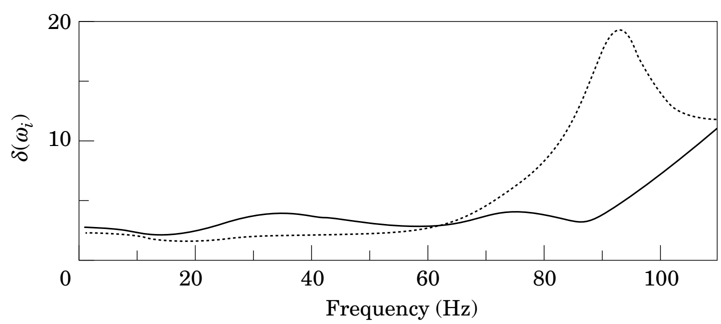

Figure 6. Relative amplitude representation error $\delta(\omega_i)$ for two cases of combinations $n + m = 12$. $\cdots$, $n = 7$, $m = 5$; —, $n = 5$, $m = 7$.

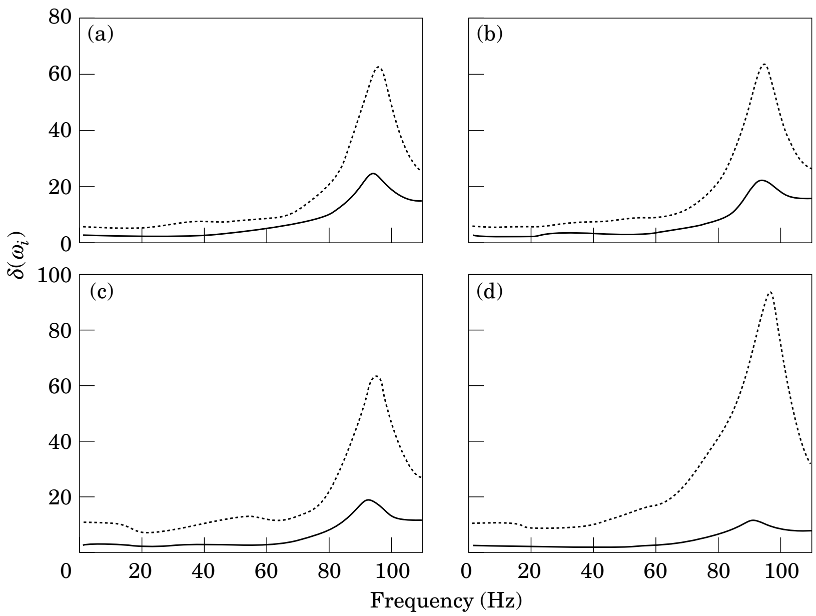

Figure 7. Relative amplitude representation error $\delta(\omega_i)$ for four cases of combinations $n + m = 12$. (a) $n = 9$, $\cdots$, $m = 0$; —, $m = 3$; (b) $n = 8$, $\cdots$, $m = 0$; —, $m = 4$; (c) $n = 7$, $\cdots$, $m = 0$; —, $m = 5$; (d) $n = 6$, $\cdots$, $m = 0$; —, $m = 6$.

the relative representation errors $\delta(\omega_i)$ for the amplitudes and $\theta(\omega_i)$ for the phases defined respectively by

$$\delta(\omega_i) = \frac{\| \mathbf{y}^{(s)}_{approx}(\omega_i) - \mathbf{y}^{(s)}_{exact}(\omega_i) \|}{\| \mathbf{y}^{(s)}_{exact}(\omega_i) \|} \, 100, \tag{24}$$

$$\theta(\omega_i) = \left(1 - \frac{\| \overline{\mathbf{y}^{(s)}_{exact}(\omega_i)^{\mathrm{T}}} \mathbf{y}^{(s)}_{approx}(\omega_i) \|}{\| \mathbf{y}^{(s)}_{exact}(\omega_i) \| \, \| \mathbf{y}^{(s)}_{approx}(\omega_i) \|} \right) 100. \tag{25}$$

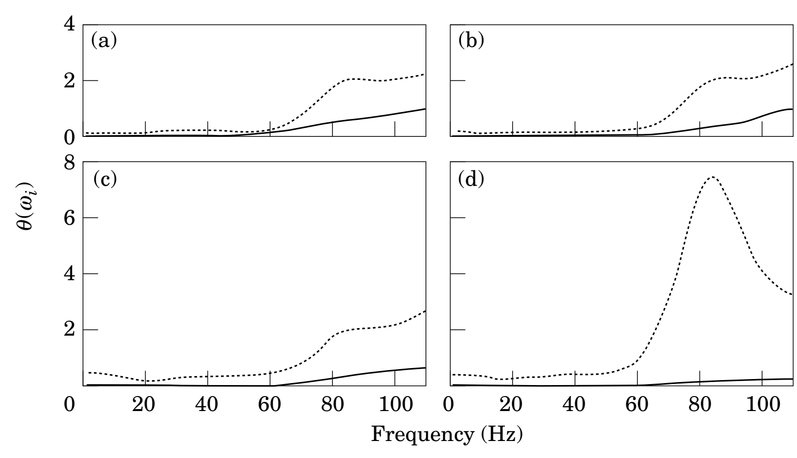

Figure 8. Relative phase representation error $\theta(\omega_i)$ for four cases of combinations $n + m = 12$. (a) $n = 9$, $\cdots$, $m = 0$; —, $m = 3$; (b) $n = 8$, $\cdots$, $m = 0$; —, $m = 4$; (c) $n = 7$, $\cdots$, $m = 0$; —, $m = 5$; (d) $n = 6$, $\cdots$, $m = 0$; —, $m = 6$.

In the following, n and m are respectively the number of vectors in the sub-basis $\mathbf{Y}_1$ and the reduced basis $\mathbf{R}_r$ forming the representation basis. n is greater or equal to the number $n_0 = 3$ of eigenvectors included in the frequency band $[0, \omega_{max} = 110$ Hz$]$.

In Figure 3 are shown the amplitude of the exact frequency responses (for a sensor on node 8, direction 1) before and after modification and the approximate one when using a representation basis constituted of a truncated modal basis $\mathbf{Y}_1 \in R^{N,7}$ and case (1) without any additional residual vector, case (2) three static residual vectors and case (3) nine static residual vectors. The corresponding results for the phase are shown in Figure 4. The convergence of the approximate frequency response towards the exact response is clearly improved when the number of static residual vectors in the representation basis is increased. In Figure 5 are shown the relative errors with respect to frequency, for $n = 7$ vectors in $\mathbf{Y}_1$ and respectively $m = 0, 3, 9$ static residual vectors in $\mathbf{R}_r$. The error diminishes as m increases.

In Figure 6, the representation errors are compared for two cases with $n = 7$, $m = 5$ and $n = 5$, $m = 7$. For $m > n$, the error is clearly smaller. This observation is again confirmed in Figure 7 and Figure 8 where the relative amplitude and phase representation errors are shown for four cases of combinations (n, m), with $n + m = 12$. The resulting accuracy for the frequency response is always better when $n < m$. This justifies the utility of this method.

Illustration of the method is completed by an analysis of cases with and without the external applied force when constructing the matrix $\mathbf{F}$. In each case, a selection of a number m of vectors of $\mathbf{F}$ is undertaken, with use of the QR decomposition with pivoting. Then the additional basis $\mathbf{R}_r$ is formed and the corresponding representation basis is constructed. In Figure 9, the accuracies of the resulting frequency responses as obtained by using each of the representation bases are shown. For m greater or equal to eight, the accuracy is better when the force is taken into account when constructing the representation basis. For example, for $\omega_i = 110$ Hz, the relative error was previously of the order 10–15% and it decreases to 4% when the force is taken into account when constructing $\mathbf{R}_r$.

The frame structure analyzed has no local modes. In a case where the initial structure has local modes inside the frequency band of the analysis, the modified structure may or

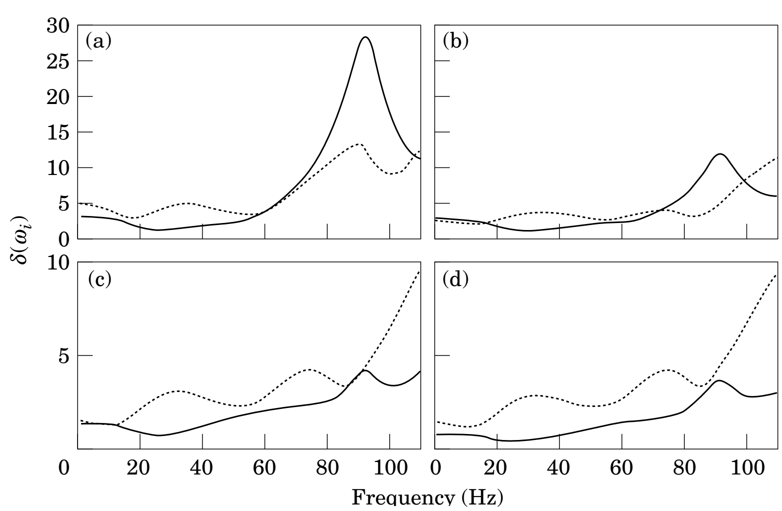

Figure 9. Effect of external applied force on relative amplitude representation error $\delta(\omega_i)$. (a) $n = 5$, $m = 6$; (b) $n = 5$, $m = 7$; (c) $n = 5$, $m = 8$; (d) $n = 5$, $m = 9$. · · ·, without force vector; —, with force vector.

may not have such local modes, depending on the modification. In the latter case, the previous analysis still applies while in the first case some care should be taken when carrying out the analysis. The only solution of this problem is to include more vectors of the modal sub-basis as well as residual vectors in the representation basis.

4. DISCUSSION-CONCLUSION

This technique provides an interesting alternative for calculating the new frequency responses of modified structures. An illustrative numerical example validates the utility of the proposed method.

In order to obtain a better precision on the frequency responses of the structure after modification, this reanalysis procedure allows different combinations of the vectors in the sub-basis $\mathbf{Y}_1$ and the static residual matrix $\mathbf{R}_r$ to construct an average representation basis. The results suggest that, for a given number t of vectors in the representation basis, $t = n + m$, it is better to increase the number of static residual vectors than the number of vectors in the truncated modal sub-basis $\mathbf{Y}_1$. Obviously, the maximum accuracy is attained with the maximum number of eigenvectors in $\mathbf{Y}_1$ and the maximum number ($r = \text{rank } \mathbf{F}$) of static residual vectors. In practice, the appropriate number n of eigenvectors in $\mathbf{Y}_1$ is between one and about twice the number n_0 of vectors contained in the frequency band $[0, \omega_{\max}]$.

The test case has shown that the proposed method satisfies the three requirements for the development of reanalysis methods for structures: namely, a method allowing the determination of the modal characteristics of a mechanical structure after modification on the basis of known data; an efficient method from the point of view of CPU times, a method allowing the most accurate results to be obtained. It can significantly contribute to reducing time and cost when developing a prototype and when updating models with measured frequency response data.

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APPENDIX: NOTATION

N	number of degrees of freedom of the discrete structural model
$\mathbf{K}$; $\mathbf{M}$; $\mathbf{B} \in R^{N,N}$	respectively stiffness, mass and damping matrices of the discrete model of the initial structure; $\mathbf{K}$ and $\mathbf{M}$ are symmetric and positive definite while $\mathbf{B}$ is symmetric and positive semi-definite
$\mathbf{Z} \in C^{N,N}$	symmetric dynamic stiffness matrix of the initial structure
$\mathbf{Y}$; $\mathbf{\Lambda} \in R^{N,N}$	modal and spectral matrices of the autonomous conservative system associated to the initial structure
$\omega_v \in R$	vth eigenvalue of the autonomous system associated with the initial structure
$\mathbf{y}_v \in R^{N,1}$	vth eigenvector of the initial structure
$\mathbf{Y}_1 \in R^{N,n} \mathbf{\Lambda}_1 \in R^{n,n}$	known modal sub-basis and spectral sub-matrix; $\mathbf{Y}_1$ constitutes the classical Ritz representation basis
$\mathbf{Y}_2 \in R^{N,N-n}$	unknown modal sub-basis
$\mathbf{\Lambda}_2 \in R^{N-n,N-n}$	unknown spectral submatrix
$\mathbf{R}_r \in R^{N,m}$	matrix of additional vectors completing the representation basis
$\Delta\mathbf{K}$, $\Delta\mathbf{M}$, $\Delta\mathbf{B} \in R^{N,N}$	respectively symmetric stiffness, mass and damping modifications matrices
$\Delta\mathbf{Z} \in C^{N,N}$	symmetric dynamic stiffness modification matrix
c	number of modified degrees of freedom
ω_i, $i = 1, \ldots, M$	subdivisions of the frequency band
$\omega_{\max}$	right hand limit of the frequency band
$\mathbf{f}(\omega_i) \in C^{N,1}$	external applied force
$\mathbf{y}(\omega_i) \in C^{N,1}$	frequency responses of the initial structure
$\mathbf{y}^{(s)}(\omega_i) \in C^{N,1}$	frequency responses of the modified structure
n_0	number of modes included in the frequency band
n	number of modes in the truncated modal basis
m	number of additional vectors in the representation basis
t	total number of vectors in the representation basis
$\mathbf{R}$	static flexibility matrix
$\mathbf{R}_r$	reduced static flexibility matrix
$\mathbf{P}$	improved Ritz basis, $\mathbf{P} = [\mathbf{Y}_1 \| \mathbf{R}_r]$
$\mathbf{c}(\omega_i)$	linear combination vector, $\mathbf{y}^{(s)}(\omega_i) = \mathbf{P}\mathbf{c}(\omega_i)$
$\triangleq$	"is defined as"
real()	real part of ()
imag()	imaginary part of ()
$()^{\mathrm{T}}$	transpose of ()
$\overline{()}$	complex conjugate of ()
$\\|\xi\\|$	$\\|\xi\\| \triangleq (\bar{\xi}^{\mathrm{T}} \cdot \xi)^{1/2}$