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# LETTER TO THE EDITOR

## ANALYTICAL SOLUTIONS IN THE TIME DOMAIN FOR VIBRATION PROBLEMS OF DISCRETE VISCOELASTIC SYSTEMS

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### 1. INTRODUCTION

The term "viscoelastic" material has quite a broad meaning. For example, in the literature there is a use of this term if an equation of motion includes a viscous damping term; i.e., the equation of motion is written in terms of current instant values of displacement, velocity, acceleration. In this study the equation of motion will include the integral term and history of strain (or stress) is required for its formulation. Materials yielding such a constitutive relation (requiring history) are also called viscoelastic, but the term "hereditary" materials will be used in this study to distinguish them.

Hereditary properties are present in any polymeric material. The definition of the hereditary medium will be given below. At this stage we just note that the class of viscoelastic materials includes as a subclass the hereditary materials; i.e., these two terms are not identical.

In this study, isotropic homogeneous hereditary materials are considered. The material is supposed to be in an isothermal state.

The principal equations to be used are (1) the constitutive relation between stress and strain of the deformable body, and (2) the equation of motion. Boundary and initial conditions are assumed to be known.

As to the constitutive relations, there are different models in use. The elementary viscoelastic models are shown in Figure 1. Different models of viscoelastic material are discussed in references [1] and [2].

In the theory of viscoelasticity, one of the models (see references [1, 3–7]) is a constitutive law of the form:

$$a_0\sigma + a_1\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \cdots + a_m\frac{\mathrm{d}^m\sigma}{\mathrm{d}t^m} = b_0 e + b_1\frac{\mathrm{d}e}{\mathrm{d}t} + \cdots + b_n\frac{\mathrm{d}^n e}{\mathrm{d}t^n}.$$

Taking the Laplace transformation of this equation (assuming that at $t = 0$ the material was unstressed and undeformed), one obtains

$$(a_0 + a_1 p + \cdots + a_m p^m)\bar{\sigma} = (b_0 + b_1 p + \cdots + b_n p^n)\bar{e}$$

or, in abbreviated form,

$$\bar{\sigma} = \frac{B(p)}{A(p)}\bar{e}, \qquad \bar{e} = \frac{A(p)}{B(p)}\bar{\sigma}. \tag{1}$$

Assuming that $m = n$, the fraction of two polynomials can be represented as a sum of fractions; i.e.,

$$\frac{B(p)}{A(p)} = E + \sum_{i=1}^{n}\frac{A_i}{p + \alpha_i}. \tag{2}$$

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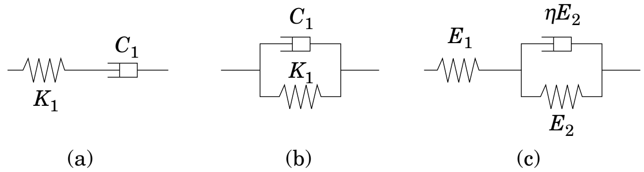


Figure 1. Elementary models: (a) the Maxwell model; (b) the Voigt model; (c) the Kelvin model.

If $m = n$ and the roots of polynomials $A$ and $B$ are real, distinct and negative, the material is called a hereditary-elastic medium [3]. Using the inverse Laplace transformation and the convolution theorem one obtains from equations (1) and (2):

$$\sigma = E\left(e - \int_0^t \Gamma(t-\tau)e(\tau)\,\mathrm{d}\tau\right)$$

and, analogously,

$$e = \frac{1}{E}\left(\sigma + \int_0^t K(t-\tau)\sigma(\tau)\,\mathrm{d}\tau\right),$$

where the functions

$$\Gamma(t-\tau) = \sum_{i=1}^{n} A_i \exp[-\alpha_i(t-\tau)], \qquad K(t-\tau) = \sum_{i=1}^{n} B_i \exp[-\beta_i(t-\tau)],$$

are called relaxation and creep kernels, respectively. As an example, for an elementary three-parameter model of a hereditary medium (Figure 1(c)) the relation between stress and strain will be

$$\dot{\sigma} + \gamma\sigma = E_1(\dot{e} + \mu e),$$

where $\mu = 1/\eta$, $\gamma = (E_1 + E_2)/\eta E_2$. This relation can be written as

$$\sigma = E_1\left(e - \int_0^t (\gamma - \mu)\exp[-\gamma(t-\tau)]e(\tau)\,\mathrm{d}\tau\right).$$

The application of the finite element method (FEM) to elastic systems allows the formulation of dynamic problems in terms of mass, stiffness matrices, vectors of displacement (response) and force excitation for arbitrary dynamical systems. According to the correspondence (Volterra) principle, a solution of the viscoelastic problem can be obtained from a solution of the corresponding elastic problem by replacement of elasticity constants (Young's modulus and Poisson ratio) by their hereditary analogs (operators), but this works only in some cases when the elastic solution is found analytically. Application of the finite element method provides a *numerical* solution to the problem. Thus, in this case the correspondence principle is not applicable to the elastic solution. In this case its application is reduced to replacement of material constants $E$, $v$, or $\lambda$, $G$ (in the stiffness matrix) by their viscoelastic analogs (operators) $\tilde{E}$, $\tilde{v}$ or $\tilde{\lambda}$, $\tilde{G}$. As to the finite element formulation, in the literature it is usually described as a step-by-step numerical integration scheme (in time). Incremental schemes (step-by-step) were also utilized in a series of programs devoted to the description of viscoelastic behavior; e.g., one can see this in references [8] and [9].

In references [10] and [11] an introduction of "mini-oscillators" was shown to be incorporated into the finite element formulation, where a combination of certain exponential terms for the relaxation kernel was assumed in order to obtain symmetric matrices in the Laplace transformed equation of motion, written in some state form (additional dissipative co-ordinates were introduced). Each "mini-oscillator" must consist of two exponential terms

$$\Gamma(\xi) = a_1 \exp(-\alpha_1 \xi) + a_2 \exp(-\alpha_2 \xi),$$

where the coefficients in this expression must satisfy the condition

$$a_1 \alpha_1 = -a_2 \alpha_2.$$

This may restrict the behavior of viscoelastic materials. To illustrate this circumstance, it is sufficient to consider a material, the relaxation kernel of which is described by just one exponential term,

$$\Gamma(\xi) = a_1 \exp(-\alpha_1 \xi),$$

or consists of an odd number of exponential terms.

Models based on fractional derivatives were considered in references [3] and [12–16]. In such models the creep kernel is expressed as [12]

$$K(t-\tau) = A(\alpha)(t-\tau)^{-\alpha}, \qquad 0 < \alpha < 1.$$

In reference [16] this model was used, where the modulus of elasticity was replaced by the corresponding hereditary operator and the Poisson ratio was considered to be constant.

In this study the use of exponential terms for relaxation kernels is utilized, which allows a good fit to experimental data and, moreover, allows one to obtain analytical solutions for the equation of motion; i.e., there is no need to use incremental schemes.

## 2. MODELLING OF VISCOELASTIC DYNAMIC SYSTEMS

### 2.1. *Single-degree-of-freedom systems*

Consider a mass $m$ connected by a viscoelastic (massless) spring to the base, subjected to a force excitation $f(t)$. The constitutive relation between stress and strain is assumed to have the form

$$\sigma = E\left(e - \int_0^t \Gamma(t-\tau) e(\tau)\,\mathrm{d}\tau\right).$$

The equation of motion will be

$$m\ddot{x}(t) + k\left(x(t) - \int_0^t \Gamma(t-\tau) x(\tau)\,\mathrm{d}\tau\right) = f(t), \tag{3}$$

where the kernel of the relaxation operator is given by

$$\Gamma(t-\tau) = \sum_{i=1}^{n} a_i \exp[-\alpha_i (t-\tau)].$$

One can consider free vibrations first: $f(t) = 0$. Applying the Laplace transformation to equation (3), one obtains:

$$m(p^2\bar{x} - \dot{x}(0)) - px(0)) + k\bar{x} - k\bar{x}\sum_{i=1}^{n}\frac{a_i}{p+\alpha_i} = 0.$$

Thus

$$\bar{x} = \frac{m(\dot{x}(0) + px(0))}{p^2 m + k - k\sum_{i=1}^{n}[a_i/(p+\alpha_i)]} = \frac{A(p)}{B(p)},$$

where $A(p)$ is a polynomial of order $n+1$, and $B(p)$ is of order $n+2$. Knowing the roots $p_i$ of polynomial $B(p)$ (and if they are simple), one can restore the original (solution) in the form (see the theorem in reference [17]):

$$x(t) = \sum_{i=1}^{n+2}\frac{A(p_i)}{B'(p_i)}e^{p_i t}. \tag{4}$$

Thus the free vibration solution is given by equation (4).

For forced vibrations, $f(t) \neq 0$, and the image $\bar{x}$ is determined as

$$\bar{x} = \frac{m(\dot{x}(0) + px(0)) + \bar{f}(p)}{p^2 m + k - k\sum_{i=1}^{n}[a_i/(p+\alpha_i)]}.$$

The time domain response will now depend on the form of the function $\bar{f}(p)$.

As a next step, the extension of the above procedure to a multi-degree-of-freedom system is considered.

### 2.2. *Multi-degree-of-freedom systems*

#### 2.2.1. Free vibration response

The finite element method appication yields a mass matrix of the system $M$ and a stiffness matrix $K$. The derivation of the system stiffness matrix starts with an element stiffness matrix, where the constitutive law (Hooke's law) is involved:

$$[\sigma] = [E][e], \tag{5}$$

where

$$[\sigma] = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{13} \quad \sigma_{23}]^T, \qquad [e] = [e_{11} \quad e_{22} \quad e_{33} \quad e_{12} \quad e_{13} \quad e_{23}]^T,$$

$$[E] = \begin{bmatrix} \lambda+2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2G & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix}, \tag{6}$$

and where

$$\lambda = \frac{vE}{(1+v)(1-2v)}, \qquad 2G = \frac{E}{1+v},$$

in which $E =$ Young's modulus and $v$ is the Poisson ratio. The hereditary analog of this matrix is obtained by replacement of $E$ and $v$ by corresponding operators, i.e., $E \to \bar{E}$ and $v \to \tilde{v}$, or by another replacement, $\lambda \to \tilde{\lambda}$ and $G \to \tilde{G}$.

Assume that the kernel for operator $\tilde{E}$ is known. Consider

$$\tilde{E} = E(1 - \Gamma^*),$$

where the operator $\Gamma^*$ is defined by

$$\Gamma^* y = \int_0^t \Gamma(t-\tau) y(\tau)\, \mathrm{d}\tau, \tag{7}$$

and the kernel $\Gamma(\xi)$ is assumed to be known for a given material from the experimental data. Here it is supposed that an exponential form exists for it:

$$\Gamma(t-\tau) = \sum_{i=1}^{n} a_i \exp[-\alpha_i(t-\tau)]. \tag{8}$$

For the sake of simplicity, one can put $\tilde{v} = v =$ constant, as is proposed in reference [13] and work only with operator $\tilde{E}$. The assumption that the Poisson ratio is constant is not supported by experiment [3]. In this study the operator $\tilde{v}$ will not be assumed to be constant. One possible technique, mentioned in reference [3], is to assume that the modulus of compression is constant. Then an expression for the Poisson ratio operator is

$$\tilde{v} = v\left(1 + \frac{1-2v}{2v}\Gamma^*\right),$$

where the operator $\Gamma^*$ is defined by equation (7).

One can see that relation (5) can be presented in a two-part form,

$$[\sigma] = [\lambda I_1 + 2GI][e],$$

where

$$I_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $I$ is a unit matrix. Then the global system stiffness matrix, due to the previous form for an element matrix, will have the form

$$K = \lambda K_1 + 2GK_2. \tag{9}$$

Note that the matrices $K_1$ and $K_2$ will not depend on the material's constants. A method, which is proposed in this study, to find the expressions for operators $\tilde{\lambda}$ and $\tilde{G}$ is based on the use of the Laplace transformation. As mentioned above, it will be assumed that the compression modulus is constant; i.e.,

$$\frac{\tilde{E}}{1-2\tilde{v}} = \frac{E}{1-2v} = \text{constant}.$$

Therefore, what is required is to determine the operators $\tilde{v}/(1+\tilde{v})$ and $\tilde{E}/(1+\tilde{v})$. Having determined them, the operators $\tilde{\lambda}$ and $\tilde{G}$ in expression (6) can be found.

Consider the expression for the operator $1/(1+\tilde{v})$, denote it by $\tilde{T}$. The expressions

$$y = \frac{1}{1+\tilde{v}}x = \tilde{T}x = a_0 x + \int_0^t T(t-\tau)x(\tau)\,\mathrm{d}\tau \tag{10}$$

and

$$x = (1+\tilde{v})y = \left(1 + v + \frac{1-2v}{2}\Gamma^*\right)y \tag{11}$$

are equivalent. Note that $a_0$ and $T(t-\tau)$ in equation (10) need to be determined. Applying the Laplace transformation to equation (11) and using the convolution theorem, one obtains

$$\bar{x} = \left(1 + v + \frac{1-2v}{2}\bar{\Gamma}\right)\bar{y}.$$

Thus,

$$\bar{y} = \left[1\Big/\left(1 + v + \frac{1-2v}{2}\bar{\Gamma}\right)\right]\bar{x}. \tag{12}$$

Analogously, applying the Laplace transformation to equation (10), one obtains

$$\bar{y} = a_0\bar{x} + \bar{T}\bar{x} = (a_0 + \bar{T})\bar{x}. \tag{13}$$

Comparing equations (12) and (13), one can conclude that

$$a_0 + \bar{T} = 1\Big/\left(1 + v + \frac{1-2v}{2}\bar{\Gamma}\right)\Bigg].$$

Using equation (8), one obtains

$$a_0 + \bar{T} = 1\Bigg/\left(r + \sum_{i=1}^{n}\frac{d_i}{\alpha_i + p}\right) = \frac{\prod_{i=1}^{n}(\alpha_i + p)}{r\prod_{i=1}^{n}(\alpha_i + p) + \sum_{i=1}^{n} d_i \prod_{k\neq i}^{n}(\alpha_k + p)},$$

where $r = 1 + v$ and $d_i = (1 - 2v)/2a_i$. The fraction of these two polynomials can be presented as

$$a_0 + \bar{T} = r_0 + \sum_{i=1}^{n} \frac{r_i}{p + \gamma_i}.$$

Hence, one obtains

$$a_0 = r_0, \qquad \bar{T} = \sum_{i=1}^{n} \frac{r_i}{p + \gamma_i}.$$

Applying now the inverse Laplace transformation to $\bar{T}$, the original form of the operator $\tilde{T}$ is

$$\tilde{T}x = r_0 x(t) + \int_0^t \sum_{i=1}^{n} r_i \exp[-\gamma_i(t - \tau)]x(\tau)\,\mathrm{d}\tau. \tag{14}$$

As far as operators $\tilde{v}/(1 + \tilde{v})$ and $\tilde{E}/(1 + \tilde{v})$ are concerned, they are determined as

$$\tilde{E}\frac{1}{1 + \tilde{v}} = [E(1 - \Gamma^*)] \Big/ \left(1 + v + \frac{1 - 2v}{2}\Gamma^*\right) = A + B \Big/ \left(1 + v + \frac{1 - 2v}{2}\Gamma^*\right) = A + B\tilde{T},$$

where $A = 2E/(2v - 1)$ and $B = 3E/(1 - 2v)$. Analogously,

$$\frac{\tilde{v}}{1 + \tilde{v}} = 1 - \frac{1}{1 + \tilde{v}} = 1 - \tilde{T},$$

where $\tilde{T}$ was determined in equation (14). Thus the operators $\tilde{\lambda}$ and $\tilde{G}$ can be determined.

Now the equation of free vibrations can be written as

$$M\ddot{X} + \tilde{K}X = 0, \tag{15}$$

where operator $\tilde{K} = \tilde{\lambda}K_1 + 2\tilde{G}K_2$. Note that one can add a term $C\dot{X}$ (linear viscous damping term) into equation (15) as well, i.e.,

$$M\ddot{X} + C\dot{X} + \tilde{K}X = 0,$$

which, for example, can take into account the presence of some dashpot elements (lumped or consistent) arbitrary (in general) distributed over the system. For now it will be assumed that $C = 0$, but case $C \neq 0$ can be easily incorporated into the derivations below.

Applying the Laplace transformation to equation (15) one obtains (intermediate manipulations are omitted):

$$\left[p^2 M + K + K_1 \sum_{i=1}^{n} \frac{g_i}{p + \gamma_i} + K_2 \sum_{i=1}^{n} \frac{h_i}{p + \gamma_i}\right]\bar{X} = M(\dot{X}(0) + pX(0)),$$

where the $\gamma_i$ are mentioned in equation (14), and the coefficients $g_i$ and $h_i$ (expressions for which are omitted here) depend on $a_i$ and $\alpha_i$, which are mentioned in equation (8).

In abbreviated form, denoting the matrix coefficient at $\bar{X}$ as $S$:

$$S(p)\bar{X} = M(\dot{X}(0) + pX(0)), \tag{16}$$

where matrix $S(p)$ can be written as

$$S(p) = \frac{1}{\prod_{i=1}^{n} (p + \gamma_i)} D(p),$$

in which the elements of matrix $D(p)$ are polynomials of degree $n + 2$. Thus,

$$S^{-1}(p) = \prod_{i=1}^{n} (p + \gamma_i) D^{-1}(p).$$

Introducing the adjoint $A(p) = D(p)^{-1} \det(D(p))$, equation (16) can be rewritten as

$$\bar{X} = \prod_{i=1}^{n} (p + \gamma_i) \frac{A(p)}{\det(D(p))} [M(\dot{X}(0) + pX(0))].$$

The roots of the characteristic equation $\det(D(p)) = 0$ can be determined numerically. We denote them as $p_k$, $k = 1, 2, \ldots, m \times (n + 2)$, where $m$ is the order of the matrices $M$, $K_1$ and $K_2$ (the number of degrees of freedom); $n$ is the number of exponential terms in the relaxation kernel representation. Hence, the determinant can be expressed as

$$\det(D(p)) = c_0 \prod_{k=1}^{L} (p - p_k),$$

where $L = (n + 2) \times m$. Assuming that the roots are simple, then one can express the solution of the free vibration problem in a form analogous to equation (4), i.e.,

$$X(t) = \sum_{i=1}^{L} \prod_{j=1}^{n} (p_i + \gamma_j) \frac{A(p_i)}{c_0 \prod_{k=1, k \neq i}^{L} (p_i - p_k)} [M(\dot{X}(0) + p_i X(0))] \, \mathrm{e}^{p_i t}, \tag{17}$$

where $L = (n + 2) \times m$ (note that $L \geqslant 2m$, where $m$ is the number of degrees of freedom of the system). The proof of this formula is obtained by a substitution of expression (17) into the equation of motion (15) (for the sake of brevity it is not demonstrated here). It should be said that a formula analogous to expression (17), but for an elastic undamped system, is mentioned in reference [18].

### 2.2.2. Steady state response

Consider the equation of motion of a discrete system under some periodic disturbing function, which can be presented as

$$F(t) = D_1 \sin \omega t + D_2 \cos \omega t,$$

where $F, D_1, D_2 \subset R^m$. Note that a general periodic function can be represented as a Fourier series and for each term a particular solution (steady state response) can be determined. Then the individual responses can be summed up to obtain a total steady state response. For the sake of brevity, just one term in the Fourier series is considered.

The equation of motion of a viscoelastic system is

$$M\ddot{X} + KX - K_1 \int_{-\infty}^{t} \sum_{i=1}^{n} a_i \exp[-\gamma_i(t-\tau)]X(\tau)\,\mathrm{d}\tau - K_2 \int_{-\infty}^{t} \sum_{i=1}^{n} b_i \exp[-\gamma_i(t-\tau)]X(\tau)\,\mathrm{d}\tau$$
$$= D_1 \sin \omega t + D_2 \cos \omega t, \tag{18}$$

where $M$ and $K$ are mass and stiffness matrices respectively; $K_1$ and $K_2$ are mentioned in equation (9). The kernels of operators $\tilde{\lambda}$ and $\tilde{G}$ are presented by a series of exponential terms in equation (18), where the compression modulus is assumed to be constant and the expressions for operators $\tilde{\lambda}$ and $\tilde{G}$ were obtained in the previous section.

The solution of equation (18) is sought in the form

$$X(t) = X_1 \sin \omega t + X_2 \cos \omega t. \tag{19}$$

Substituting equation (19) in equation (18) and equating terms in $\sin \omega t$ and $\cos \omega t$ one obtains a linear system with respect to the unknown vectors $X_1$ and $X_2$:

$$D_1 = -\omega^2 M X_1 + KX_1 - K_1 \sum_{i=1}^{n} \frac{a_i\gamma_i}{\gamma_i^2+\omega^2} X_1 - K_2 \sum_{i=1}^{n} \frac{b_i\gamma_i}{\gamma_i^2+\omega^2} X_1 - K_1 \sum_{i=1}^{n} \frac{a_i\omega}{\gamma_i^2+\omega^2} X_2$$
$$- K_2 \sum_{i=1}^{n} \frac{b_i\omega}{\gamma_i^2+\omega^2} X_2, \tag{20}$$

$$D_2 = -\omega^2 M X_2 + KX_2 + K_1 \sum_{i=1}^{n} \frac{a_i\omega}{\gamma_i^2+\omega^2} X_1 + K_2 \sum_{i=1}^{n} \frac{b_i\omega}{\gamma_i^2+\omega^2} X_1 - K_1 \sum_{i=1}^{n} \frac{a_i\gamma_i}{\gamma_i^2+\omega^2} X_2$$
$$- K_2 \sum_{i=1}^{n} \frac{b_i\gamma_i}{\gamma_i^2+\omega^2} X_2. \tag{21}$$

Here, when taking the integral terms in equation (18), an assumption of fading memory was used, i.e., $\gamma_i > 0$ for all $i$.

## 3. NUMERICAL EXAMPLES

The results of calculations of steady state responses according to the formulae of the previous sections are presented below.

A clamped beam (Figure 2) was symmetrically loaded by a vertical periodic force $F(t)$, which is graphically presented in Figure 3 for one period $t \subset [0, T]$. The first (lowest)

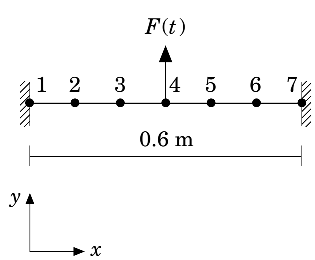


Figure 2. Clamped beam, 2-D problem.

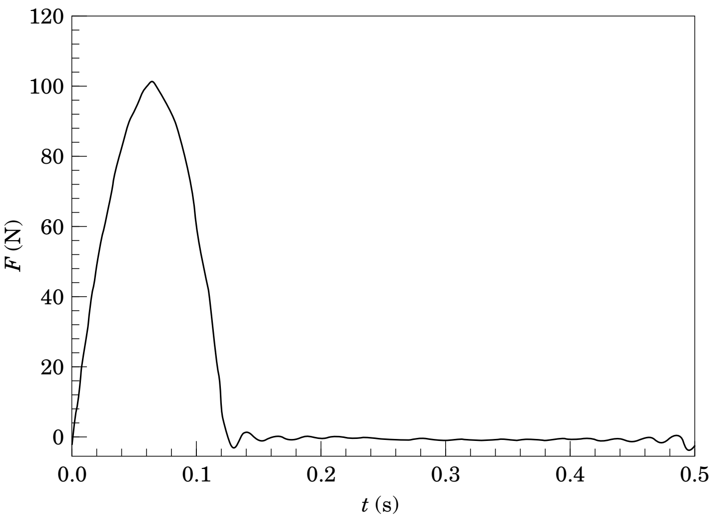


Figure 3. Periodic loading, $T = 0{\cdot}5$ s.

natural frequency of this beam (elastic model) is 146 Hz. Therefore, this specific loading (Figure 3) with a period $T = 0{\cdot}5$ s is rather a low frequency excitation.

The instantaneous constants of the material were $E = 0{\cdot}207 \times 10^6$ (MPa), $\nu = 0{\cdot}3$ and density of material $= 0{\cdot}784 \times 10^4$ (kg/m$^3$). The parameters of the beam's cross-section were $0{\cdot}01 \times 0{\cdot}01$ m.

Two hereditary material models were considered: in the first model the compression modulus was considered as constant and the appropriate expressions for the hereditary operators $\tilde{\lambda}$ and $\tilde{G}$ were taken (we denote this model as "A"). The second model had the Young's modulus replaced by a hereditary operator and the Poisson ratio was constant (we denote this model as "B").

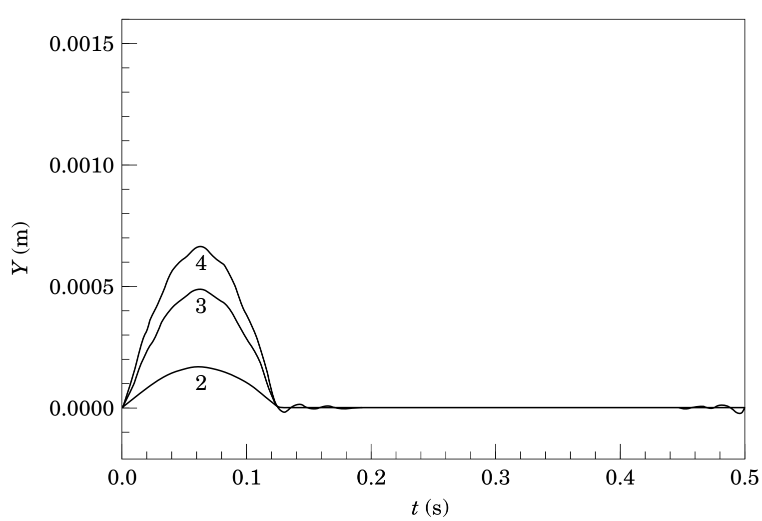


Figure 4. The steady state response, elastic beam: $a = 0$ (numbers 2, 3 and 4 denote the node numbers; see Figure 2).

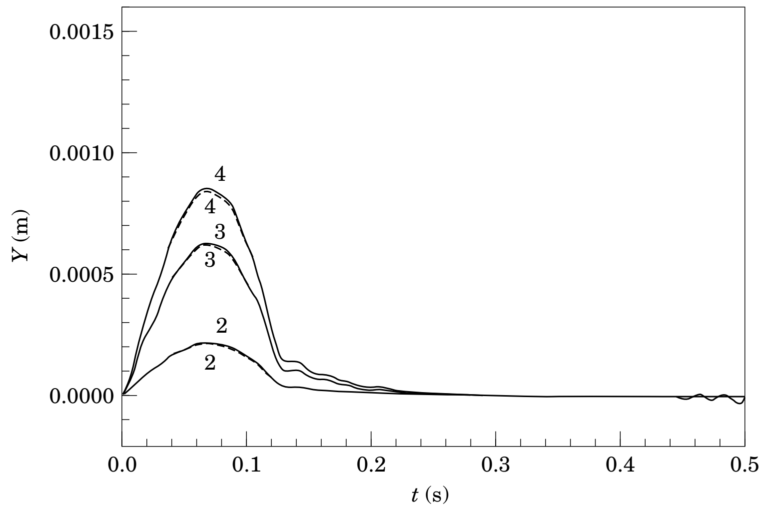


Figure 5. The steady state response with parameters of the relaxation kernel $a = 10$, $\alpha = 30$. —— Model "A" (compression modulus is constant): – – –, model "B" (the Poisson ratio is constant).

The parameters of the relaxation kernel (one exponential term was retained),

$$\Gamma(\xi) = a \exp[-\alpha\xi],$$

were (1) $a = 0$ (elastic system), (2) $a = 10$, $\alpha = 30$ and (3) $a = 25$, $\alpha = 30$. Note that this model corresponds to a three-parameter model in Figure 1(c).

The displacements (in the $y$-direction) as functions of time are presented for nodes 2, 3 and 4 of the beam in Figure 4 for the elastic beam and in Figures 5 and 6 for both viscoelastic models "A" and "B".

Note that ratio of the long-time modulus $E_\infty$ to the instantaneous modulus $E$ for these three cases will be as follows [12]: (1) $E_\infty/E = 1$, (2) $E_\infty/E = 1 - a/\alpha = 2/3$ and (3) $E_\infty/E = 1 - a/\alpha = 1/6$. One can see that the difference between two models is more

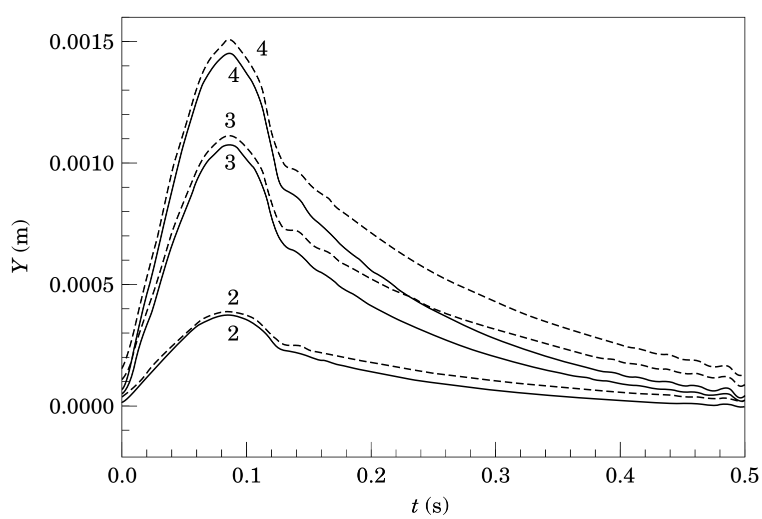


Figure 6. The steady state response with parameters of the relaxation kernel $a = 25$, $\alpha = 30$. —— Model "A" (compression modulus is constant): – – –, model "B" (the Poisson ratio is constant).

noticeable for case (3). In this case the maximum relative difference in the displacements obtained for these two models "A" and "B" reaches about 70 percent.

## 4. CONCLUSIONS

Application of the finite element method to viscoelastic problems is usually shown in the literature as an incremental (step-by-step) integration in time. In this study the analytical solution in the time domain for discrete dynamic problems (the free vibration problem and the steady state response problem) have been presented; i.e., there is no need to apply incremental schemes.

The constitutive relation (stress–strain) was used in a form of a hereditary law with the relaxation kernel represented by a series of exponential terms.

Two isotropic materials' constants were replaced by corresponding hereditary operators. The expressions for these operators $\tilde{\lambda}$ and $\tilde{G}$ have been derived. The use of the finite element procedure is virtually reduced to providing the usual mass and stiffness (elastic) matrices of the system.

Numerical experiments for a problem of beam flexural vibrations have been conducted. Comparison of obtained results with ones obtained for a model in which just one hereditary operator (the Young's modulus operator) is introduced has shown that there is a noticeable difference in the responses.

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