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REFLECTION OF FLEXURAL WAVES IN GEOMETRICALLY PERIODIC BEAMS

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The paper focuses on the reflection characteristics of elastic beams having a periodically varying cross-sectional area. Assuming a weak sinusoidal variation of the beam cross-section along its axis, perturbation methods are employed to determine flexural resonance conditions and analyze the resonant destructive interaction of flexural waves with the periodic beam. This interaction is represented in the form of coupled-wave equations, which are solved analytically, together with relevant boundary conditions, at the ends of the periodic section of the beam. The reflection coefficient is then calculated for beams having different types of periodicity. This study is intended to provide guidelines to control passively the flexural vibration in beams.

1. INTRODUCTION

Periodic beams have been a subject of special interest due to their common usage in several engineering applications. Based on the type of periodicity, they can be classified under the following two general categories: (1) Beams with geometric/material periodicity, (2) beams resting on equispaced supports. Most of the efforts so far to model dynamically periodic beams has been spent on the characterization of the second category.

As a result, several studies have been published on wave motion in continuous beams over equispaced supports. Heckl [1] used the receptance method to study the transmission of bending and torsional waves on beams with periodic discontinuities. Ungar [2] derived expressions that describe the steady-state behavior of infinitely long beams with uniformly spaced attached impedances. Lin and McDaniel [3] applied the transfer matrix method on a periodic Bernoulli-Euler beam on many elastic supports. Mead [4] looked at the free vibratory harmonic motion of an undamped infinite beam on regularly spaced identical supports as a group of sinusoidal waves travelling in different directions at different speeds. Mead and Pujara [5] have obtained the response of periodically stiffened beams due to spatial and temporal harmonic pressure in the form of a series of space harmonics, evolved from considerations of progressive wave propagation. Abrahamson [6] presented formulations for deriving normal modes in infinite periodic structures, by introducing wave propagation constants via "propagation selection conditions". Orris and Petyt [7] used a finite element technique to evaluate the frequency variation of the imaginary part of the propagation constant of a periodically supported infinite beam. Mead and Mallik [8] used an approximate assumed mode method to predict the space-averaged response of a periodically supported beam subject to convected loading. Mead and Markus [9] studied the interaction between longitudinal and flexural wave motion in beams, to which offset spring-mounted masses were attached at regular intervals. Mead [10] found a receptance function for a periodic Timoshenko beam, which is subjected to an array of harmonic

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forces or moments, by dealing with them as phased arrays. Zhang and Zhang [11] studied the energy flow in a periodically supported beam and reached the conclusion that the power transmitted along the periodic beam in both directions is equal. Mead *et al*. [12] provided a proof of this phenomenon. Mukherjee and Parthan [13] applied the wave approach, using deflection functions which satisfy geometric and force boundary conditions, to analyze the free vibration of rotationally restrained infinite periodic beams on rigid supports.

Comparatively speaking, beams with periodic geometry or material properties have received less attention. Periodically segmented beams, where material properties vary in a piecewise fashion, have been considered. Tassilly [14] used Floquet theory to obtain the dispersion relation and analyzed Brillouin zones for bending waves in a two-segment periodic beam. Each segment was modeled as a Bernoulli-Euler beam with damping properties and resting on an elastic foundation. Lee *et al*. [15] considered flexural wave propagation in a periodically segmented beam, with each unit cell made up of two materials, using Floquet theory. They investigated the decoupling of the dispersion relation at the end points of the Brillouin zone. Nayfeh and Hawwa [16] employed the transfer matrix method to reach the dispersion relation of flexural waves in multi-segmented periodic beams, by giving each segment the possibility of having material viscosity, elastic foundation, and axial force.

More scarce has been the study of wave motion in periodic beams with continuously varying material or geometric properties. Lee and Ke [17] used the Floquet theory to study flexural waves in a periodic beam with continuously varying mechanical impedance. They presented the banded structure of the dispersion curves and showed that the dispersion relation is uncoupled into two simpler ones at the end points of Brillouin zones as in reference [15].

The present paper gives special attention to the problem of wave propagation in beams with continuous geometric periodicity, emphasizing the reflection characteristics of such beams. Using perturbation techniques, an analysis is given for the interaction of flexural waves with an elastic beam having a sinusoidally varying cross-sectional area. The straightforward asymptotic expansion is found to break down when the wavenumber of the flexural wave is half of that of the beam periodicity, corresponding to the Bragg resonance in the area of solid-state physics. Uniform expansion near resonance is obtained by using the method of multiple scales, leading to the coupled-wave equations describing a stopband interaction. Numerical illustrations are given in terms of the wave reflection coefficient.

2. PROBLEM FORMULATION

Figure 1 represents a periodic section of an infinitely-extended elastic beam. The periodic segment extends from $\hat{x} = 0$ to $\hat{x} = L$ and has a sinusoidally varying thickness which is described as $\hat{h}(\hat{x}) = \hat{h}_0\{1 + \varepsilon[\delta \sin(\hat{k}_e\hat{x} + \theta) - \sin(\hat{k}_e\hat{x})]\}$, where $\hat{h}_0$ is the average thickness of the beam, k_e is the wavenumber of the periodic surfaces, ε is a small dimensionless parameter much smaller than unity and equals to the ratio of the amplitude of the sinusoidal variation of the surface to $\hat{h}_0$, δ is a parameter allowing for different amplitudes of the periodic surfaces, and θ is a phase angle.

The governing equation of time-harmonic flexural waves in terms of deflection in the y-direction ($\hat{V}$) is given by

$$E\frac{\partial^2}{\partial \hat{x}^2}\left[\hat{I}(\hat{x})\frac{\partial^2 \hat{V}}{\partial \hat{x}^2}\right] - \rho\omega^2\hat{A}(\hat{x})\hat{V} = 0, \tag{1}$$

where $\hat{A}$ is the cross-sectional area, $\hat{I}$ is the moment of inertia, ω is the frequency of oscillation, and ρ is the material density. Dimensionless quantities (without the carets) are introduced by using the average thickness of the beam, $\hat{h}_0$, as the characteristic length. The following dimensionless governing equation is obtained:

$$\frac{\mathrm{d}^4V}{\mathrm{d}x^4}+\left(\frac{2}{I}\frac{\mathrm{d}I}{\mathrm{d}x}\right)\frac{\mathrm{d}^3V}{\mathrm{d}x^3}+\left(\frac{1}{I}\frac{\mathrm{d}^2I}{\mathrm{d}x^2}\right)\frac{\mathrm{d}^2V}{\mathrm{d}x^2}-\left(\frac{\rho\omega^2}{E}\hat{h}_0^2\frac{A}{I}\right)V=0. \tag{2}$$

Using a power series expansion of $[I(x)]^{-1}$ and defining the flexural wavenumber $k=(\rho A_0/E\hat{I}_0)^{1/4}\omega^{1/2}\hat{h}_0$, equation (2) can be written in the following Hill-type form:

$$\begin{aligned}&\mathrm{d}^4V/\mathrm{d}x^4+\varepsilon\{[\delta\cos(k_ex+\theta)-\cos(k_ex)](6k_e\,\mathrm{d}^3V/\mathrm{d}x^3)+[\delta\sin(k_ex+\theta)\\&\quad-\sin(k_ex)](2k^4V-3k_e^2\,\mathrm{d}^2V/\mathrm{d}x^2)\}-k^4V+H.O.T.=0.\end{aligned} \tag{3}$$

The flexural wave is assumed to emanate from $x=-\infty$. As the wave is incident on the periodic section of the beam, it will be exposed to a process of successive reflections, and consequently, it is partly reflected from and partly transmitted through the periodic structure. Perturbation techniques are next employed to analyze the reflection process.

3. STRAIGHTFORWARD ASYMPTOTIC EXPANSION

An approximate solution for V is sought in the form

$$V(x)=V_0(x)+\varepsilon V_1(x)+\cdots \tag{4}$$

by substituting equation (4) into equation (3) and equating each of the coefficients of ε^0 and ε^1 to zero, to obtain

$$O(1):\qquad \mathrm{d}^4V_0/\mathrm{d}x^4-k^4V_0=0, \tag{5}$$

$$\begin{aligned}O(\varepsilon):\qquad \mathrm{d}^4V_1/\mathrm{d}x^4-k^4V_1=&-[\delta\cos(k_ex+\theta)-\cos(k_ex)](6k_e\,\mathrm{d}^3V_0/\mathrm{d}x^3)\\&+[\delta\sin(k_ex+\theta)-\sin(k_ex)](3k_e^2\,\mathrm{d}^2V_0/\mathrm{d}x^2-2k^4V_0).\end{aligned} \tag{6}$$

The general solution of equation (5) can be written as

$$V_0=A\,\mathrm{e}^{\mathrm{i}kx}+B\,\mathrm{e}^{-\mathrm{i}kx}+C\,\mathrm{e}^{kx}+D\,\mathrm{e}^{-kx}, \tag{7}$$

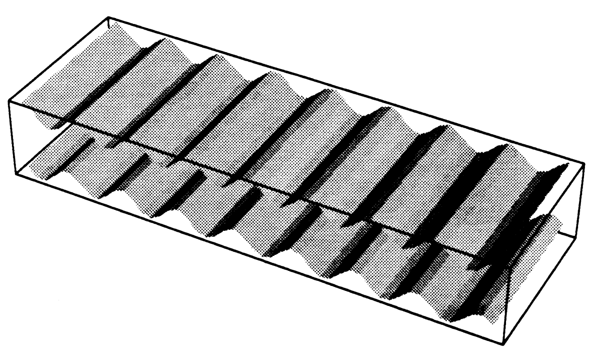

Figure 1. Beam geometry.

where A,B,C, and D are arbitrary constants. By considering only propagating waves, V_0 can be substituted in equation (6) and the trigonometric functions expressed in polar form. This yields terms including $e^{i(k_e \mp k)x}$ at the right-hand side. Such terms produce secular terms in the particular solution of V_1 when

$$\pm 2k \approx k_e, \tag{8}$$

which leads to the break down of the asymptotic expansion. To remedy this, the method of multiple scales is employed in the next section [18].

4. METHOD OF MULTIPLE SCALES

To find a uniform expansion (free of secular terms) near resonance, a first-order perturbation expansion is sought for V in powers of ε in the form

$$V(x) = V_0(X_0, X_1) + \varepsilon V_1(X_0, X_1) + \dots, \tag{9}$$

where $X_0 = x$ is a short length scale of the order of the wavelength in the beam and $X_1 = \varepsilon x$ is a long length scale which characterizes the spatial amplitude and phase modulations due to the geometric periodicity. The derivatives with respect to x are expanded in terms of ε as

$$\frac{d}{dx} = \frac{\partial}{\partial X_0} + \varepsilon \frac{\partial}{\partial X_1} + \cdots, \quad \frac{d^2}{dx^2} = \frac{\partial^2}{\partial X_0^2} + 2\varepsilon \frac{\partial^2}{\partial X_0\, \partial X_1} + \cdots,$$

$$\frac{d^3}{dx^3} = \frac{\partial^3}{\partial X_0^3} + 3\varepsilon \frac{\partial^3}{\partial X_0^2\, \partial X_1} + \cdots, \quad \text{and} \quad \frac{d^4}{dx^4} = \frac{\partial^4}{\partial X_0^4} + 4\varepsilon \frac{\partial^4}{\partial X_0^3\, \partial X_1} + \cdots. \tag{10}$$

Substituting equations (9) and (10) into equation (3), and equating the coefficients of ε^0 and ε^1 on both sides, one has

$$O(1): \qquad \partial^4 V_0/\partial X_0^4 - k^4 V_0 = 0 \tag{11}$$

$$O(\varepsilon): \qquad \partial^4 V_1/\partial X_0^4 - k^4 V_1 = -4\partial^4 V_0/\partial X_0^3\, \partial X^1 - [\delta \cos(k_e X_0 + \theta) - \cos(k_e X_0)] \times (6k_e\, \partial^3 V_0/\partial X_0^3) + [\delta \sin(k_e X_0 + \theta) - \sin(k_e X_0)](3k_e^2\, \partial^2 V_0/\partial X_0^2 - 2k^4 V_0). \tag{12}$$

The general solution of equation (11) is given in the form

$$V_0 = A^+(X_1)\, e^{ikX_0} + A^-(X_1)\, e^{-ikX_0} + B^+(X_1)\, e^{kX_0} + B^-(X_1)\, e^{-kX_0}, \tag{13}$$

where A^+ (A^-) are the amplitudes of the incident (reflected) wave, and B^+ (B^-) are the amplitudes of the near field incident (reflected) wave. Attention will be given only to the propagating waves. The wave amplitudes are unknown functions of the long scale X_1 at this level of approximation.

In order to describe quantitatively the nearness of the wavenumber k to resonance, a detuning parameter σ of $O(1)$ is introduced such that

$$2k = k_e + \varepsilon\sigma. \tag{14}$$

Equation (13) is substituted into equation (12), the resonance condition (14) is imposed, and trigonometric functions are expressed in polar form. Then, the terms that produce secular terms in V_1 are eliminated to ensure a uniform expansion. This is accomplished by setting each of the coefficients of $e^{\mp ikX_0}$ to zero. The result is

$$dA^+/dX_1 = C_c A^-\, e^{-i\sigma X_1}, \qquad dA^-/dX_1 = C_c^* A^+\, e^{i\sigma X_1}, \tag{15, 16}$$

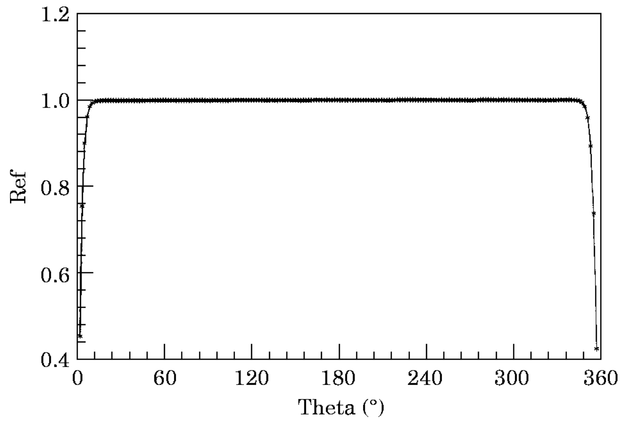

Figure 2. Resonant reflection coefficient versus the phase angle θ when $\alpha = 1$.

where the superscript * indicates a complex conjugate, and

$$C_c = (1/8k)(6kk_e - 3k_e^2 - 4k^2)(\delta\, e^{i\theta} - 1). \quad (17)$$

Equations (15) and (16) are the coupled-wave equations. They describe the resonant wave interaction in the periodic section of the beam. From the coupled-wave equations, one realizes that the physical meaning of the resonance condition given by equation (8) is that a wave travelling in the $+x$-direction generates necessarily its counterpart wave travelling in the $-x$-direction, resulting in a stopband interaction. This corresponds to the well known Bragg condition in the field of solid-state physics.

Before this analysis is closed, it must be mentioned that the other possibility of building a beam with a periodic cross-sectional area is to have the width of the beam vary periodically. It is found that a beam with a periodic width has weak reflection properties and it does not show any resonance phenomenon. The analysis of this case is included in Appendix A.

5. DISCUSSION AND EXAMPLES

To solve the coupled-wave equations (15) and (16), they are provided with relevant boundary conditions. Without any loss of generality, these conditions are given by

$$A^+ = 1, \quad \text{at } X_1 = 0; \qquad A^- = 0, \quad \text{at } x_1 = \ell. \quad (18)$$

Note that the first condition represents the excitation amplitude of the incident wave, and the second condition expresses the fact that the reflected wave vanishes at the end of the periodic section. System (15)–(18) forms a standard two-point boundary value problem that can be solved analytically for the missing boundary conditions.

To facilitate the algebra, system (15)–(18) is transformed into an autonomous one by introducing the transformations

$$A^- = a^-, \qquad A^+ = a^+\, e^{-i\sigma X_1}, \quad (19)$$

which leads to

$$da^+/dX_1 = C_c a^- + i\sigma a^+, \qquad da^-/dX_1 = C_c^* a^+, \quad (20, 21)$$

$$a^+(0) = 1, \qquad a^-(\ell) = 0. \quad (22)$$

The system of equation (20)–(22) is now solved for the missing boundary condition at $x = 0$, i.e.,

$$a^-(0) = C_c(e^{\lambda_1 \ell} - e^{\lambda_2 \ell})/[\lambda_2 e^{\lambda_1 \ell} - \lambda_1 e^{\lambda_2 \ell}], \tag{23}$$

where

$$\lambda_{1,2} = \tfrac{1}{2}[-i\sigma \mp (4C_c C_c^* - \sigma^2)^{1/2}]. \tag{24}$$

To evaluate the reflection process in the periodic section of the beam, the reflection coefficient is used as an indicator. It can be calculated from the relation $R = |a^-(0)/a^+(0)|$.

For the numerical example, a beam is assumed to be made of aluminum with $\rho = 2{\cdot}7 \times 10^3$ kg/m^3 and $E = 7{\cdot}1 \times 10^{10}$ N/m^2. The beam has an average thickness, $\hat{h}_0$ 1 cm and a width, $\hat{b} = 5$ cm. The geometric periodicity is built in such a way that the beam wavenumber is equal to twice the wavenumber of the flexural wave propagating at a frequency of 1250 Hz. Then, the periodicity of the beam is described as $b = 5[1 + \varepsilon\delta \sin(5{\cdot}15x + \theta) - \sin(5{\cdot}15x)]$ cm. Hence, the wavelength of the periodic beam is $\Lambda = 1{\cdot}22$ cm. The small parameter ε is taken to be 0·1, and the length of the periodic section is assumed to be $\ell = 10\Lambda$.

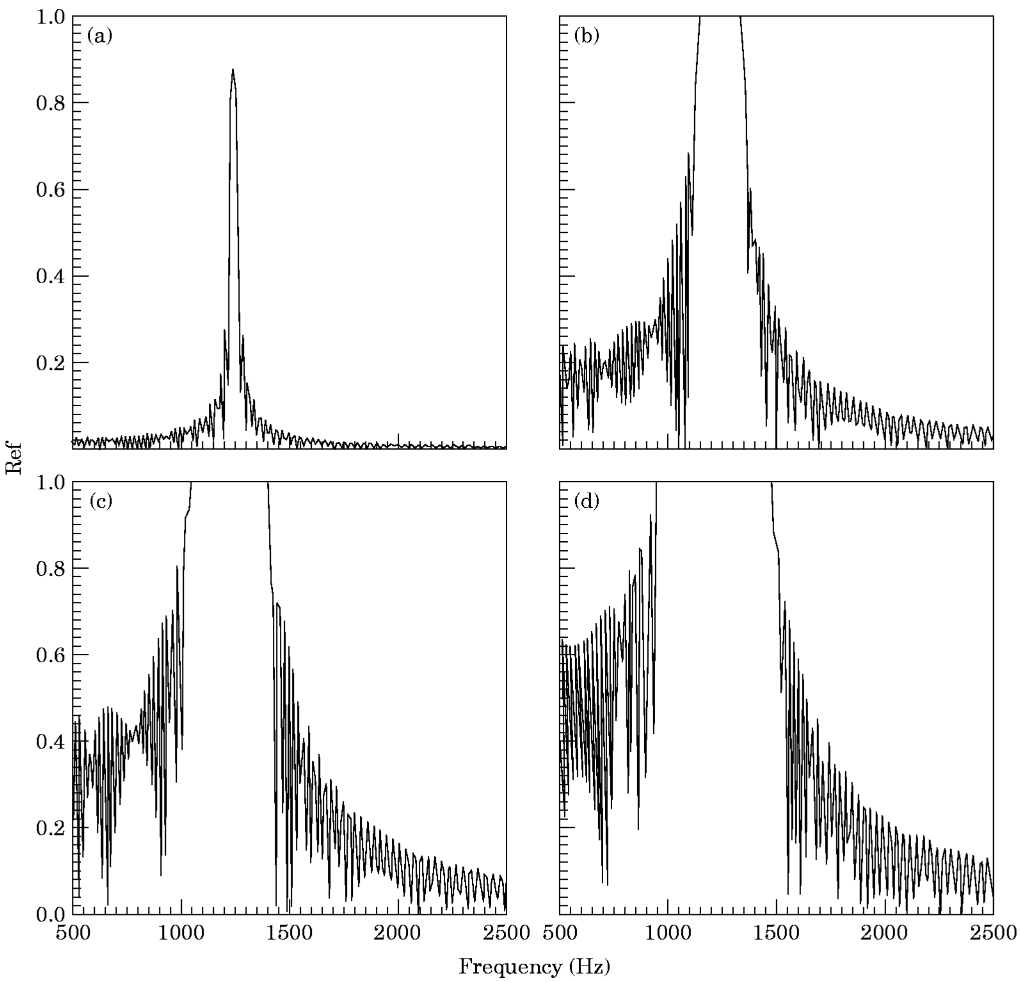

Figure 3. Reflection coefficient versus frequency when $\alpha = 1$ for (a) $\theta = 5°$, (b) $\theta = 45°$, (c) $\theta = 90°$, and (d) $\theta = 180°$.

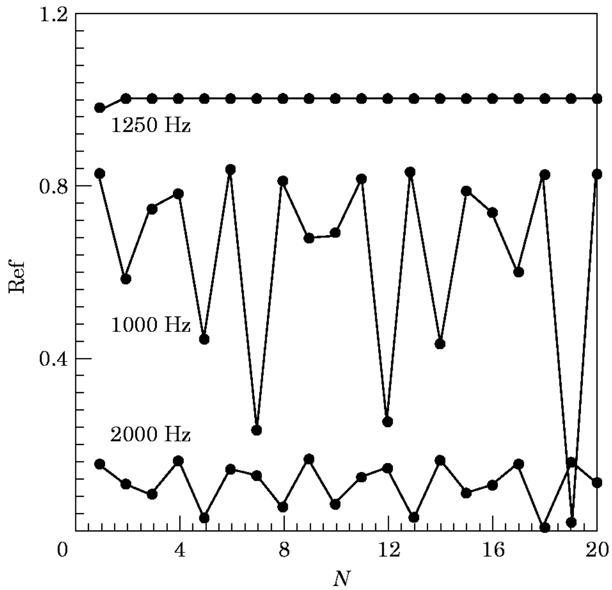

Figure 4. Reflection coefficient versus *N* (number of resonant wavelengths of the periodic section of the beam).

An important consideration when studying the reflection characteristics of the beam is to determine the roots of the coupling coefficient C_c. These roots correspond to the cases of decoupling, i.e., unmodulated propagation. This occurs when $\delta\, e^{i\theta} - 1 = 0$. For the case when $\delta = 1$, decoupling occurs at $\theta = 0$ or 2π, corresponding to the case of a wavy beam. The variation of the reflection coefficient at resonance with the phase angle θ for the case when $\delta = 1$ is shown in Figure 2. One notes that periodic beams act well as wave filters except when θ is equal 0 or 2π. If $\delta \neq 1$, the coupling coefficient, C_c, has no roots, which means that the wave will always suffer some reflection for every θ.

The reflection coefficient spectra for periodic beams with $\delta = 1$ and $\theta = 5$, 45, 90, and 180° are shown in Figures 3a–d. Reflection at resonance is noticed to be complete in all beams except in that with a phase angle of 5°. The widths of the stopbands shown in the reflection coefficient spectra are respectively 100, 340, 430, and 620 Hz. This indicates that a periodic beam with best performance is the one having $\delta = 1$ and $\theta = 180^\circ$.

The influence of the length of the periodic section of the beam on its reflection characteristics is also examined. For a beam with $\delta = 1$ and $\theta = 90^\circ$, the reflection coefficient is monitored against the length of the periodic section for representative frequencies in Figure 4. The length of periodic section is measured by the number of periods. It is evident that the wave is efficiently reflected for all lengths at the resonance frequency 1250 Hz. This is an expected result because the beam periodicity is tuned to filter out the wave at that frequency. For frequencies which lie in the "rippled region" of the reflection coefficient spectrum, such as 1000 Hz and 2000 Hz, reflection varies critically when changing the length of the periodic section.

It is finally worth mentioning that any periodic function can be expressed in terms of a summation of sinusoidal periodicities using Fourier series. Thus, the above analysis could be considered as a building block to handle beams with a piece-wise, continuous type of periodicity.

6. CONCLUSION

The stopband reflection of flexural waves in a periodic beam, whose thickness or width are sinusoidally varying, has been analyzed. The analysis has been performed using asymptotic expansions, leading to the derivation of the coupled-wave equations. The performance of the beam, as flexural wave filters, was presented in terms of the reflection coefficient. It was found that a beam with periodic thickness reflects flexural waves more efficiently than a beam with periodic width. It is concluded that the interaction could be weakened or strengthened by carefully designing the phase difference. The choice depends on the type of application for which the beam is to be employed.

REFERENCES

1. M. A. HECKL 1964 *Journal of the Acoustical Society of America* **36**, 1335–1343. Investigations on vibrations of grillages and other simple beam structures.
2. E. E. UNGAR 1965 *Journal of the Acoustical Society of America* **39**, 887–894. Steady-state responses of one-dimensional periodic flexural systems.
3. Y. K. LIN and T. J. McDANIEL 1969 *Journal of Engineering for Industry* **17**, 1133–1141. Dynamics of beam-type periodic structures.
4. D. J. MEAD 1975 *Journal of Sound and Vibration* **11**, 181–197. Free wave propagation in periodically supported, infinite beams.
5. D. J. MEAD and K. K. PUJARA 1971 *Journal of Sound and Vibration* **14**, 525–541. Space-harmonic analysis of periodically supported beams: response to convicted random loading.
6. A. L. ABRAHAMSON 1973 *Journal of Sound and Vibration* **28**, 247–258. Flexural wave mechanics—an analytical approach to the vibration of periodic structures forced by convected pressure field.
7. R. M. ORRIS and M. PETYT 1974 *Journal of Sound and Vibration* **33**, 223–236. A finite element study of harmonic wave propagation in periodic structures.
8. D. J. MEAD and A. K. MALLIK 1976 *Journal of Sound and Vibration* **47**, 457–471. An approximate method of predicting the response of periodically supported beams subjected to random convected loading.
9. D. J. MEAD and S. MARKUS 1983 *Journal of Sound and Vibration* **90**, 1–24. Coupled flexural-longitudinal wave motion in periodic beam.
10. D. J. MEAD 1986 *Journal of Sound and Vibration* **104**, 9–27. A new method of analyzing wave propagation in periodic structures: application to periodic Timoshenko beams and stiffened plates.
11. A. M. ZHANG and W. H. ZHANG 1991 *Journal of Sound and Vibration* **151**, 1–7. The reduction of vibrational energy flow in a periodically supported beam.
12. D. J. MEAD, R. G. WHITE and X. M. ZHANG 1994 *Journal of Sound and Vibration* **169**, 5581–561. Power transmission in a periodically supported beam excited at a single point.
13. S. MUKHERJEE and S. PARTHAN 1993 *Journal of Sound and Vibration* **162**, 57–66. Free wave propagation in rotationally restrained infinite periodic beams.
14. E. TASSILLY 1987 *International Journal of Engineering Science* **25**, 85–94. Propagation of bending waves in a periodic beam.
15. S. Y. LEE, H. Y. KE and M. J. KAO 1990 *Journal of Applied Mechanics* **57**, 779–783. Flexural waves in a periodic beam.
16. A. H. NAYFEH and M. A. HAWWA 1990, *Proceedings of the* 16*th International Conference of Experimental Mechanics*, 397–401. Vibration and wave propagation characteristics of multisegmented elastic beams.
17. S. Y. LEE and H. Y. KE 1992 *Journal of Applied Mechanics* **59**, S189–S196. Flexural wave propagation in an elastic beam with periodic structure.
18. A. H. NAYFEH 1981 *Introduction to Perturbation Techniques*. New York: Wiley-Interscience.
19. L. BRILLOUIN 1953 *Wave Propagation in Periodic Structures*. New York: Dover Publications.
20. C. ELACHI 1976 *Proceedings of the IEEE* **64**, 1666–1698. Waves in active and passive periodic structures: a review.

APPENDIX A: BEAM WITH A PERIODIC WIDTH

An elastic beam which has a sinusoidally varying width is considered. The width is described by $b(\hat{x}) = \hat{b}_0\{1 + \varepsilon[\delta \sin(k_e\hat{x} + \theta) - \sin(k_e\hat{x})]\}$, where $\hat{b}_0$ is the average width of the beam, k_e is the wavenumber of the edges, ε is a small dimensionless parameter and equal is to the ratio of the amplitude of the sinusiodal variation of an edge to $\hat{b}_0$, δ is a parameter allowing for different amplitudes of the periodic edges, and θ is a phase angle.

Using the average width of the beam $\hat{b}_0$ as the characteristic length, defining $k = (\rho\hat{A}_0/E\hat{I}_0)^{1/4}\hat{\omega}^{1/2}\hat{b}_0$, and expanding $[I(x)]^{-1}$ in terms of a power series, the following governing equation of flexural motion is obtained:

$$\mathrm{d}^4V/\mathrm{d}x^4 + 2\varepsilon k_e[\delta\cos(k_ex+\theta) - \cos(k_ex)]\,\mathrm{d}^3V/\mathrm{d}x^3 - \varepsilon k_e^2[\delta\sin(k_ex+\theta) - \sin(k_ex)]\,\mathrm{d}^2V/\mathrm{d}x^2 - k^4V + H.O.T. = 0. \quad (11)$$

An approximate solution for V is sought in terms of a straightforward asymptotic expansion in the form

$$V(x) = V_0(x) + \varepsilon V_1(x) + \cdots. \quad (2)$$

Then, one obtains the following two problems:

$$O(1): \qquad \mathrm{d}^4V_0/\mathrm{d}x^4 - k^4V_0 = 0, \quad (3)$$

which has the solution (7) of section 3, and

$$O(\varepsilon): \qquad \frac{\mathrm{d}^4V_1}{\mathrm{d}x^4} - k^4V_1 = -2k_e[\delta\cos(k_ex+\theta) - \cos(k_ex)]\,\mathrm{d}^3V_0/\mathrm{d}x^3 + k_e^2[\delta\sin(k_e+\theta) - \sin(k_ex)]\frac{\mathrm{d}^2V_0}{\mathrm{d}x^2}. \quad (4)$$

Substituting for V_0 from equation (7) into the right-hand side of the $O(\varepsilon)$ problem leads to the following particular solution of V_1:

$$V_{1p} = \frac{\mathrm{i}k^2(\delta\,\mathrm{e}^{\mathrm{i}\theta} - 1)}{2[(k_e-k)^2 + k^2]}A\,\mathrm{e}^{\mathrm{i}(k_e-k)x} - \frac{\mathrm{i}k^2(\delta\,\mathrm{e}^{-\mathrm{i}\theta} - 1)}{2[(k_e-k)^2 + k^2]}B\,\mathrm{e}^{-\mathrm{i}(k_e-k)x}, \quad (5)$$

which is a uniform solution even when $2k \approx k_e$. Hence, resonance does not occur in this case, and the flexural wave is weakly reflected.