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LETTERS TO THE EDITOR

THE FREE VIBRATION OF THIN RECTANGULAR PLANFORM SHALLOW SHELLS WITH SLITS

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1. introduction

The problem of the vibration of thin, shallow shells has received considerable attention since such components are often to be found in engineering applications where dynamic excitation exists. Much of this work has been discussed in the excellent review articles by Leissa [1] and Qatu [2]. The effects of various characteristics of the shell, such as degree and nature of curvature and type of support conditions, have been examined by numerous researchers. However, a topic which has so far been almost neglected is the effect of the presence of a slit on a vibration of a shallow shell. This topic is of some practical importance as a slit can be considered as a first approximation to an open crack, which can indeed occur in practice. It may be noted that although a slit may be a relatively poor representation of a crack for the prediction of quantities such as stresses in the vicinity of the crack tip, it is a reasonable approximation for studies of more global characteristics such as natural frequencies and mode shapes of vibration. Some work that has been reported on the vibration of shallow shells with slits is that by Young and Dickinson [3, 4], in which natural frequencies are given for annular and circular spherical shells with radial slits, and that by Crossland, Young and Dickinson [5], which is a brief (two-page abstract) and preliminary report on the approach described here. The work described in references [3] and [4] was part of a more general study on the vibration of shallow shells with various different planforms [4, 6], and was approached by using a Ritz solution with one to four general sectorial elements being joined together through the use of artificial springs of very high stiffness. To the author's knowledge, no other researchers have treated the slit shell problem.

In the present work, uniform thickness, unstressed, shallow shells of rectangular planform, with slits parallel to one edge, are considered. Each shell is subdivided into several "free" rectangular shell elements which are joined together and to the boundaries by means of very stiff, artificial, line springs, thereby enforcing the continuity and boundary conditions. Slits are created by allowing the appropriate springs to have a stiffness of zero. The resulting slit can be termed "open" and does not interfere with the motion of the shell material in any of the three Cartesian directions. Natural frequency parameters are presented for a variety of problems and the effect of such variables as boundary conditions, curvature conditions, slit length and slit location are illustrated.

2. analytical approach

Consider the typical, rectangular planform, isotropic, thin, shallow shell shown in Figure 1, the corners of which lie in an x–y plane at $(x, y) = (0, 0)$, $(a, 0)$, $(0, b)$ and (a, b). The principal radii of curvature R_x and R_y are assumed to be constant and their axes coincide with the x- and y-axes; hence $1/R_{xy} = 0$. A slit of length c, approximating an open

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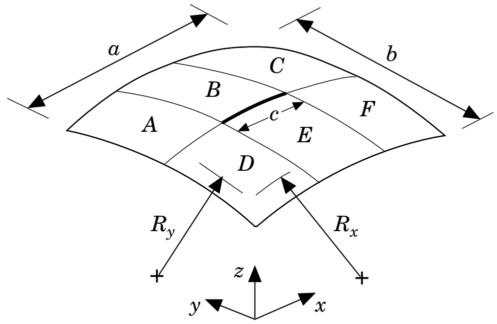

Figure 1. A shallow shell with an interior slit.

crack, lies parallel to the x-*axis*. It is further assumed that the middle surface of the shell is unstressed in the static equilibrium position. Since the shell is shallow, the displacements and energy expressions for both the entire shell and each element may be written in terms of the Cartesian co-ordinates associated with the projection of the shell on to the x–y plane. In this case, the shell may be idealized as composed of six rectangular planform, shallow shell elements, as indicated; for different locations of the slit, it may be desirable to use more or fewer elements. In using the artificial spring approach [7], each element is initially treated as completely free and is subsequently connected to adjacent elements and/or to the boundaries by means of line springs of very high stiffness. It is necessary to use up to four line springs for each connection, these being the three translational springs of stiffness K_u, K_v and K_w, and one rotational spring of stiffness R. Where a slit or a free boundary exists, the stiffness of any connecting spring is simply set to zero, effectively giving no connection. Similarly, if only partial support exists at a boundary, such as for a shear diaphragm support, then the stiffness of the appropriate spring(s) is (are) set to zero. In the event that elastic supports or inter-element connections are required, then the spring stiffnesses are adjusted to the appropriate finite values, modelling the actual connection.

The displacements of the middle surface are defined as u, v and w in the x, y and z directions, respectively, and, for free, small amplitude vibration, the motion may be described by $u = U(x, y) \sin \omega t$, $v = V(x, y) \sin \omega t$ and $w = W(x, y) \sin \omega t$. The quantities U, V and W for each element are chosen to satisfy the geometrical free boundary conditions along the element edges and, in this work, simple polynomials are used. These are written

$$U(x, y) = \sum \sum A_{ij} x^i y^j, \qquad V(x, y) = \sum \sum B_{ij} x^i y^j \qquad \text{and} \qquad W(x, y) = \sum \sum C_{ij} x^i y^j,$$

where $i, j = 0, 1, 2, \ldots$, and the origin for the elemental x and y is at the corner of the element closest to the origin of the global co-ordinates.

The maximum energies, with respect to time, may now be written for each element using the standard expressions [8]. Further strain energy terms result from the inclusion of the artificial springs. For example, for the connection between two elements A and B, joined along an $x =$ constant edge, the additional strain energy terms may be written

$$U_{UAB} = \tfrac{1}{2} K_{UAB} \int [U_A(a_A, y) - U_B(0, y)]^2 \, \mathrm{d}y, \tag{1}$$

$$U_{VAB} = \tfrac{1}{2} K_{VAB} \int [V_A(a_A, y) - V_B(0, y)]^2 \, \mathrm{d}y, \tag{2}$$

$$U_{WAB} = \tfrac{1}{2} K_{WAB} \int [W_A(a_A, y) - W_B(0, y)]^2 \, \mathrm{d}y, \tag{3}$$

$$U_{RAB} = \tfrac{1}{2} R_{AB} \int \left[\left. \frac{\mathrm{d}W_A}{\mathrm{d}x} \right|_{x = a_A} - \left. \frac{\mathrm{d}W_B}{\mathrm{d}x} \right|_{x = 0} \right]^2 \mathrm{d}y, \tag{4}$$

where the integrals are carried out over $y = 0$ to $y = b_A$ and the side lengths of element A are a_A and b_A. The boundary conditions may also be accomodated by using similar expressions simply by setting the appropriate displacement or normal slope values to zero, in the event that rigid support is offered in a particular direction, or by setting the appropriate stiffness value to zero, if the edge is free to move in a particular direction. The energy contributions from all elements and artificial springs are then summed over the whole system and the Lagrangian formed and optimized with respect to the displacement coefficients A_{mn} to yield an eigenvalue equation of the standard form.

It should be noted that different numbers of terms could be used in the displacement functions for each element in both the x and y directions and for each of U, V and W but, in this work, all series were taken from 0 to n. As a result, a six-element shell structure has $18(n + 1)^2$ degrees of freedom in the entire system.

3. NUMERICAL RESULTS

Throughout this section, the following non-dimensional stiffness parameters are used: for translational line springs, Ka^3/D, and for rotational line springs, Ra/D, where D is the flexural rigidity of the shell, given by $Eh^3/(12(1 - v^2))$, in which E is Young's modulus of elasticity, h is the shell thickness and v is the Poisson ratio. Furthermore, all spring stiffness parameters for a particular shell are given the same numerical value κ and the frequency parameter reported is $\Omega = \omega a^2(\rho h/D)^{1/2}$, where ρ is the material density. In addition, the slit is always assumed to lie parallel to the x-axis.

In order to engender confidence in the analysis and to establish a satisfactory value of the parameter κ, studies examining the convergence of the frequency parameters were conducted and comparisons made with results available from elsewhere. These results are not tabulated here, but a brief description of their interpretation follows. The first two systems considered were shells with no slits, these being the cantilevered cylindrical shell treated by Lim, Liew and Ong [9], who used a single domain, two-dimensional polynomial of fifteenth degree, Ritz solution, and the spherically curved, shear diaphragm supported shell treated by Leissa and Kadi [10], who presented an exact solution. It was found that there was no change in the fifth significant figure of the frequency parameters for the first few modes of either system when using $\kappa = 10^{11}$ or $\kappa = 10^{13}$. The agreement with the previously published results was excellent. Two problems involving slits were then treated: a fully clamped, rectangular flat plate with a centrally located slit, previously investigated experimentally by Maruyama and Ichinomiya [11] and theoretically by Yuan and Dickinson, using the analysis given in reference [7]; and a fully free, square, spherical shell with a slit that started at one edge and extended along a centreline. The theoretical results used for both comparisons were obtained using similar analyses to that described here—although each was formulated differently—and, as would be expected, the agreement achieved was excellent. It was observed that the rate of convergence with increased number of terms in the displacement series decreased in the presence of a slit, as may be seen in the results that follow.

For the slit shell, the number of parameters and degree to which each may be varied is enormous and renders a comprehensive parametric study impractical. As a consequence, only a brief parametric study is given here. A somewhat more comprehensive parametric study, including the depiction of mode shapes for some cases, may be found in the work by Crossland [12]. In all cases discussed, the following parameters were used for each shell: $\kappa = 10^{13}$, $a/b = 1$ and $a/h = 200$. Two descriptive quantities are introduced to aid in the discussion and the presentation of the results. The "convergence index", abbreviated to C.I. in the tables, is defined as the percentage difference between the $n = 5$ and $n = 7$ term solutions (based on the $n = 7$ solution) for a particular case. While this does not indicate that the solution has converged to within that percentage of the true solution, it does give some indication of the degree to which convergence has occurred. The "percentage reduction" (P.R. in the tables) is the percentage reduction which is observed between the natural frequency for the slit shell and that of the equivalent shell with no slit, based upon the natural frequency of the shell with no slit. In all but one of the tables presented, the mode type is described by means of S and A, the first indicating symmetry or antisymmetry about the central x-axis and the second about the y-axis. For these cases, numerical results for the first two modes of each type are reported.

For all cases in which the slit is wholly in the interior of the shell, the six-element idealization shown in Figure 1 was used. For shells with a slit which abuts an $x =$ constant edge but does not lie along a $y =$ constant edge, the element configuration shown in Figure 2(a) was used. For shells with a slit along a $y =$ constant edge, the configuration shown in either Figure 2(b) or Figure 2(c) was used.

To investigate the effect of slit length, a fully clamped, spherically curved shell ($a/R_x = a/R_y = \frac{1}{2}$), with a centrally located slit, was chosen for study. Four different slit length ratios, $c/a = 0$, $c/a = \frac{1}{4}$, $c/a = \frac{1}{2}$ and $c/a = \frac{3}{4}$, were considered. A brief convergence study is given in Table 1 and the convergence indices and percentage reductions in Table 2. (In all subsequent cases, the convergence study, although conducted, is not given: only the convergence indices are reported, together with the frequency parameters computed using $n = 7$.) It may be seen that the reductions in frequency increase as the slit lengthens, as would be expected with the decrease in stiffness resulting from the presence of a slit. The convergence rate decreases in the presence of a slit and tends to deteriorate as the slit lengthens. Also included in Table 1 are values calculated for the first four modes by using the commercial finite element program ABAQUS [13] for the case of $c/a = \frac{1}{2}$. These are designated FE 16 × 16 and FE 32 × 32 and were calculated using 16 × 16 and 32 × 32 eight-noded, quadrilateral shell elements, giving 3525 and 15 365 degrees of freedom,

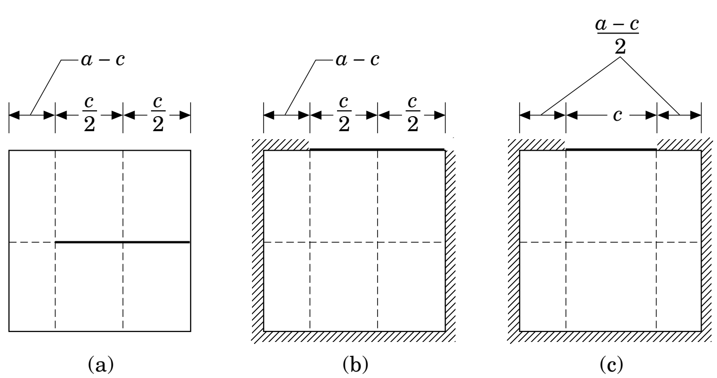

Figure 2. The element configuration for slit shells.

TABLE 1

Frequency parameters, Ω, for shells with slits of various lengths

c/a		SS1	SA1	AS1	AA1	SS2	SA2	AS2	AA2
0	$n = 3$	368·39	363·54	370·24	382·67	478·98	474·73	411·32	500·70
	$n = 5$	349·93	355·65	355·94	369·53	367·80	390·31	389·83	421·34
	$n = 7$	349·79	355·58	355·58	369·18	367·13	388·88	388·88	419·20
$\frac{1}{4}$	$n = 3$	295·7	356·9	370·2	361·9	474·2	497·8	413·5	544·4
	$n = 5$	250·9	343·1	350·3	335·1	360·7	387·3	384·5	421·5
	$n = 7$	245·7	340·9	348·3	331·3	359·5	386·0	381·9	418·9
$\frac{1}{2}$	$n = 3$	231·3	304·7	361·0	321·4	470·7	448·8	413·7	494·7
	$n = 5$	179·2	224·7	265·5	229·5	356·3	384·8	368·3	382·4
	$n = 7$	176·9	217·5	251·5	221·4	348·6	381·3	366·4	371·5
	FE 16×16	175·1	223·1	249·8	219·3	–	–	–	–
	FE 32×32	173·9	216·1	247·1	215·1	–	–	–	–
$\frac{3}{4}$	$n = 3$	269	272	356	324	530	628	510	654
	$n = 5$	187	167	239	174	347	384	366	371
	$n = 7$	167	155	187	162	298	322	327	334

respectively. The agreement with the present solution using $m = 7$ in the displacement series, for which the number of degrees of freedom for the whole shell is only 1152, is reasonably close.

The $c/a = \frac{1}{4}$ case has a fairly small average reduction in frequency, with by far the largest change occurring for the SS1 mode. For the $c/a = \frac{3}{4}$ case, much larger reductions in frequency occur, but it must be noted that all the modes have fairly large convergence indices. Remembering that the Ritz solution converges from above, it may be concluded that the true values of the frequencies could be considerably lower than those reported. As would be expected, the behaviour of the $c/a = \frac{1}{2}$ case falls between these two cases both with respect to percentage reductions in frequency and convergence indices. Based upon these results, in order to create a balance between a reasonable convergence rate and significant change in frequency, a slit length of $c/a = \frac{1}{2}$ was chosen for use in all of the subsequent cases.

In order to examine the effect of different boundary conditions, a spherically curved shell $(a/R_x = a/R_y = \frac{1}{2})$ was again considered, both with no slit and with a centrally located slit. This problem had already been treated for the fully clamped condition and, here, two additional sets of boundary conditions were considered: all four edges shear diaphragm

TABLE 2

C.I. and P.R. for shells with slits of various lengths

c/a		SS1	SA1	AS1	AA1	SS2	SA2	AS2	AA1	Average
0	C.I.	0·040	0·020	0·101	0·095	0·182	0·368	0·244	0·510	0·195
$\frac{1}{4}$	C.I.	2·12	0·645	0·574	1·15	0·334	0·337	0·681	0·621	0·807
	P.R.	29·8	4·13	2·05	10·3	2·08	0·741	1·79	0·072	6·37
$\frac{1}{2}$	C.I.	1·30	3·31	5·57	3·66	2·20	0·918	0·519	2·93	2·55
	P.R.	49·4	38·8	29·3	40·0	5·05	1·96	5·78	11·4	22·7
$\frac{3}{4}$	C.I.	12·0	7·74	27·8	7·41	16·4	19·3	11·9	11·1	14·2
	P.R.	52·3	56·4	47·4	56·1	18·8	17·2	15·9	20·3	35·6

C.I. convergence index;
P.R. percentage reduction in frequency.

TABLE 3

Ω, C.I. *and* P.R. *for shells with various boundary conditions*

B.C.	Slit	SS1	SA1	AS1	AA1	SS2	SA2	AS2	AA2	Average
SD.	No	327·5 (0)	332·7 (0)	332·7 (0)	338·9 (0)	344·1 (0·003)	353·9 (0·031)	353·9 (0)	384·5 (0·026)	– (0·008)
SD.	Yes	124·3 (2·50)	211·7 (3·37)	250·3 (5·60)	220·4 (3·59)	316·6 (2·72)	342·6 (0·353)	334·1 (0·081)	341·5 (1·44)	– (2·46)
	P.R.	62·0	36·4	24·8	35·0	7·99	3·19	5·59	11·2	23·2
FF.	No	20·376 (0·034)	37·788 (0·042)	37·780 (0·056)	13·651 (0·007)	52·028 (0·092)	92·931 (0·185)	92·921 (0·266)	72·736 (0·122)	– (0·101)
FF.	Yes	20·126 (0·214)	37·738 (0·058)	37·749 (0·058)	13·648 (0·007)	51·993 (0·096)	92·863 (0·219)	92·886 (0·250)	72·713 (0·117)	– (0·127)
	P.R.	1·23	0·106	0·082	0·022	0·067	0·073	0·038	0·032	0·206

SD., shear diaphragm; FF., fully free;
( ), convergence index; P.R., percentage reduction in frequency.

supported and all four fully free. The frequency parameters for these two cases are given in Table 3, together wtih the convergence indices and the percentage reductions. Inspection of Table 3 and the appropriate section of Table 2 (middle two rows) shows that, for the shear diaphragm supported and clamped shells, the percentage reduction in frequency in both cases is of similar order, with no clear pattern emerging other than that the lower mode of each symmetry class is the more affected. The convergence indices are also of similar order for the two cases and are significantly higher in the presence of a slit than without. For the fully free case, the percentage reduction in frequency is two or more orders of magnitude less than for the supported shells.

The effects of magnitude and type of curvature were now examined. Here, a clamped shell was selected for study, again with the centrally located slit, with three types of curvature: spherical ($R_x/R_y = 1$), hyperbolic paraboloid ($R_x/R_y = -1$) and cylindrical ($R_x/R_y = 0$ or $R_y/R_x = 0$). Each of the spherical and hyperbolic curvatures were given magnitudes of $|a/R_x| = |a/R_y| = 0{\cdot}5$. The curvatures for the two cylindrical cases were selected as $a/R_x = 0{\cdot}5$, $a/R_y = 0$ and $a/R_y = 0{\cdot}5$, $a/R_x = 0$ and correspond respectively to situations in which the slit runs parallel and perpendicular to the axis along which the curvature exists. These cases are identified by "parallel" and "perpendicular" in Table 4, where the results are presented. To investigate the effect of magnitude of curvature for a particular shell, three additional cases were selected for study, these being shells with spherical curvatures with ratios $a/R_x = a/R_y = 0{\cdot}3$, 0·1 and 0, the last corresponding to a flat plate. The numerical results for the spherically curved shell with curvature ratio 0·5 are omitted since they are given in Tables 1 and 2. As before, the rate of convergence deteriorates in the presence of a slit and, on average, is not significantly affected by the magnitude or nature of the curvature, although for particular modes, significant differences may be seen.

The effect of slit location was studied by considering a spherical ($a/R_x = a/R_y = 0{\cdot}5$), fully clamped shell with the slit placed in different locations within the shell. The slit lies along the line $y = \beta$, between the points $x = \alpha$ and $x = \alpha + a/2$, and its location may then be defined by the two parameters α/a and β/b. (Note that here x and y are the co-ordinates

TABLE 4

Ω, C.I. *and* P.R. *for shells of various curvatures*

Curvature	Slit	SS1	SA1	AS1	AA1	SS2	SA2	AS2	AA2	Average
HYP	No	248·0 (0·177)	243·9 (0·131)	243·9 (0·213)	244·4 (0·205)	329·8 (0·506)	325·9 (0·927)	325·9 (0·175)	340·9 (0·648)	– (0·373)
HYP	Yes	217·4 (0·570)	218·6 (1·01)	232·7 (0·709)	229·4 (1·10)	265·2 (0·773)	284·2 (1·40)	283·5 (1·73)	290·5 (2·28)	– (1·20)
	P.R.	12·3	10·4	4·59	6·14	19·6	12·8	13·0	14·8	11·7
CYL	No	141·3 (0·113)	146·0 (0·075)	208·7 (0·120)	218·8 (0·101)	280·6 (0·549)	228·2 (0·351)	307·1 (0·218)	288·1 (0·281)	– (0·226)
CYL	Parallel	135·6 (0·693)	142·1 (0·528)	187·0 (2·65)	172·8 (2·30)	269·7 (0·415)	226·7 (1·38)	286·6 (1·80)	275·1 (1·59)	– (1·42)
	P.R.	4·03	2·67	10·4	21·0	3·88	0·675	6·68	4·51	6·73
CYL	Perpendicular	90·89 (2·50)	103·3 (2·97)	176·2 (2·12)	190·8 (1·10)	166·2 (2·68)	167·3 (1·54)	256·3 (1·94)	262·7 (3·06)	– (2·24)
	P.R.	35·7	29·2	15·6	12·8	40·8	26·7	16·5	8·82	23·3
SPH {0·3}	No	227·3 (0·044)	225·4 (0·009)	225·4 (0·049)	241·4 (0·041)	244·8 (0·217)	269·7 (0·367)	269·7 (0·044)	317·7 (0·444)	– (0·152)
SPH {0.3}	Yes	133·9 (0·717)	169·3 (1·88)	175·9 (2·21)	177·8 (1·86)	231·9 (1·07)	265·2 (0·739)	237·8 (0·467)	278·8 (2·30)	– (1·41)
	P.R.	41·1	24·9	22·0	26·3	5·27	1·67	11·8	12·2	18·2
SPH {0·1}	No	96·75 (0·005)	102·8 (0)	102·8 (0·039)	130·4 (0·015)	148·6 (0·330)	179·7 (0·334)	179·7 (0·039)	251·7 (0·350)	– (0·139)
SPH {0.1}	Yes	70·93 (0·876)	93·50 (0·713)	86·74 (1·44)	118·6 (0·514)	133·7 (1·96)	173·0 (1·18)	134·9 (1·45)	217·5 (2·07)	– (1·28)
	P.R.	26·7	9·05	15·6	9·05	10·0	3·73	24·9	13·6	14·1
PLA	No	35·99 (0·006)	73·39 (0·011)	73·39 (0·048)	108·2 (0·009)	131·6 (0·220)	165·0 (0·339)	165·0 (0·048)	242·18 (0·322)	– (0·125)
PLA	Yes	32·73 (1·39)	72·28 (0·379)	58·99 (2·02)	105·1 (0·324)	111·5 (2·47)	158·0 (1·35)	118·8 (1·99)	209·3 (2·00)	– (1·49)
	P.R.	9·06	1·51	19·6	2·87	15·3	4·24	28·0	13·6	11·8

SPH spherical; HYP hyperbolic paraboloid; CYL cylindrical; PLA plate; ( ) convergence index; P.R. percentage reduction in frequency.

of the whole shell.) Most of the systems studied in this section were not geometrically symmetrical; hence the SA/AS nomenclature is no longer appropriate and, in Table 5, where the numerical results for the first six modes of vibration are given, they are simply numbered in order of ascending frequency. Inspection of Table 5 shows that, for most cases, the closer the slit is to the geometric centre of the shell ($\alpha/a = 1/4$, $\beta/b = 1/2$), the

TABLE 5

Ω, C.I. *and* P.R. *for shells with slits in various locations*

			Mode number						
α/a	β/b		1	2	3	4	5	6	Average
$\frac{1}{4}$	$\frac{1}{2}$	Ω	176·9	217·5	221·4	251·5	348·6	366·4	–
			(1·27)	(3·32)	(3·65)	(5·55)	(2·19)	(0·535)	(2·75)
		P.R.	49·4	38·8	37·7	31·5	5·58	5·78	28·1
	$\frac{2}{3}$	Ω	184·0	216·5	231·8	253·9	349·7	359·1	–
			(1·43)	(3·85)	(3·25)	(5·39)	(1·56)	(2·55)	(3·01)
		P.R.	47·4	39·1	34·8	30·8	5·28	7·66	27·5
	$\frac{5}{6}$	Ω	214·2	221·2	307·5	322·0	347·3	356·6	–
			(3·56)	(4·76)	(1·97)	(2·01)	(1·20)	(2·07)	(2·60)
		P.R.	38·8	37·8	13·5	12·3	5·93	8·30	19·4
	1	Ω	256·7	261·2	346·9	352·3	354·5	365·7	–
			(5·17)	(6·16)	(0·092)	(0·133)	(0·302)	(0·339)	(2·03)
		P.R.	26·6	26·5	2·44	4·04	3·98	5·96	11·6
$\frac{1}{3}$	$\frac{1}{2}$	Ω	181·6	220·3	223·2	253·2	348·6	361·8	–
			(1·47)	(3·23)	(3·66)	(5·48)	(2·21)	(1·39)	(2·91)
		P.R.	48·1	38·0	37·2	31·0	5·57	6·96	27·8
	$\frac{2}{3}$	Ω	188·3	219·3	234·1	255·5	350·4	357·4	–
			(1·63)	(3·74)	(3·23)	(5·31)	(1·51)	(2·05)	(2·91)
		P.R.	46·2	38·3	34·2	30·4	5·09	8·10	27·0
	$\frac{5}{6}$	Ω	217·0	223·7	308·2	323·9	347·9	356·7	–
			(3·75)	(4·53)	(1·93)	(2·13)	(1·14)	(1·93)	(2·57)
		P.R.	38·0	37·1	13·3	11·8	5·76	8·28	19·0
	1	Ω	257·6	261·9	347·4	352·2	354·7	365·7	–
			(5·21)	(6·14)	(0·101)	(0·125)	(0·313)	(0·295)	(2·03)
		P.R.	26·4	26·3	2·30	4·07	3·92	5·96	11·5
$\frac{1}{2}$	$\frac{1}{2}$	Ω	208·8	244·5	248·3	277·3	355·8	357·4	–
			(0·546)	(1·40)	(1·31)	(0·923)	(0·191)	(0·406)	(0·796)
		P.R.	40·3	31·2	30·2	24·5	3·62	8·10	23·0
	$\frac{2}{3}$	Ω	214·8	246·4	254·9	279·0	355·2	356·9	–
			(0·633)	(1·87)	(0·953)	(0·911)	(0·710)	(0·429)	(0·918)
		P.R.	38·6	30·7	28·3	24·0	3·79	8·22	22·3
	$\frac{5}{6}$	Ω	240·7	253·1	318·2	339·0	351·6	356·6	–
			(1·30)	(2·42)	(0·305)	(0·802)	(1·06)	(0·788)	(1·11)
		P.R.	31·2	28·8	10·5	7·66	4·76	8·30	15·2
	1	Ω	266·4	285·8	349·5	352·9	355·9	365·9	–
			(4·80)	(5·46)	(0·100)	(0·091)	(0·278)	(0·118)	(1·81)
		P.R.	23·8	19·6	1·71	3·88	3·60	5·91	9·75

( ), Convergence index;
P.R., percent reduction in frequency.

higher is the percentage reduction in frequency caused by the slit. (It should be recognized that this is for a fully clamped shell and, probably, a similar result would be seen for other shells supported on all edges. However, if a free edge were to exist, it is expected that the most significant reductions in frequency would occur when the slit reaches a free edge with which it is mutually perpendicular, whereupon flapping motion could occur.)

Clearly, this has been a preliminary study and there is considerable scope for further investigation of the vibration of shells with slits or cracks, including the important problem of the effects of a pre-stressed middle surface.

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