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ON THE NATURAL VIBRATIONS OF TAPERED BEAMS WITH ATTACHED INERTIA ELEMENTS

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1. Introduction

Bapat and Bapat [1] developed an efficient approach based on the transfer matrix method to determine the natural frequencies of a straight beam supported by translational and torsional springs and carrying additional concentrated masses. In 1992, Matsuda *et al*. [2] studied vibration of a tapered Timoshenko beam restrained at any intermediate points and carrying a heavy tip body. The solutions were obtained by transforming the ordinary differential equations into integral equations and integrating them numerically.

Recently, Hamdan and Latif [3] have studied, by means of exact and approximate treatment, the dynamic behaviour of a beam with a constant cross-section in the presence of applied masses with rotating inertia. The procedure has as its main objective the numerical comparison between the exact methodology, expressed by the direct method, and the approximate procedures (Galerkin, Rayleigh–Ritz and FEM).

In reality, for certain structures used in the field of engineering, it is often necessary to analyze the dynamic behaviour of supporting structures in the presence of applied masses and with geometric characteristics which can be schematized by means of tapered beams. The simplest model of such a model is represented by a shelf with a supple joint both in translation and in rotation and with a linearly variable cross-section using the parameters $\alpha = h_2/h_1$ and $\beta = b_2/b_1$; see Figure 1.

The problem is dealt with by using two distinct methodologies of calculation: the first, of an exact type based on the direct method in which the solution is expressed by using the well-known Bessel functions; and another approximate technique by employing the Rayleigh–Ritz method [4, 5]. Some numerical examples, using calculation diagrams already analyzed by other authors for particular cases, confirm the validity of the procedures used.

2. Exact analysis

For a tapered beam, the classical Bernoulli–Euler beam theory is applicable; the effects of rotatory inertia and of transverse shear deformation are neglected. These assumptions are reasonably valid provided that the wavelength of the flexural motion is large compared with the section depth of the beam.

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The equation of motion may be written in the form

$$[EI(x)w_{1,xx}]_{,xx} - \rho\omega^2 A(x)w_1 = 0, \qquad 0 \leqslant x \leqslant cL,$$

$$[EI(y)w_{2,yy}]_{,yy} - \rho\omega^2 A(y)w_2 = 0, \qquad 0 \leqslant y \leqslant (1-c)L, \tag{1}$$

where w_1 and w_2 are the transverse displacements of the beam axis; $(_{,x}) = \mathrm{d}/\mathrm{d}x$ and $(_{,y}) = \mathrm{d}/\mathrm{d}y$.

After introducing the dimensionless variables

$$\xi = [1 + \{(\alpha - 1)/L\}x], \qquad \eta = [1 + (\alpha - 1)(c + y/L)], \tag{2}$$

the cross-sectional area and the moment of inertia are given by

$$A(\xi) = A_1\xi^n, \qquad I(\xi) = I_1\xi^{n+2}, \qquad 0 \leqslant \xi \leqslant (1 + (\alpha - 1)c), \tag{3}$$

$$A(\eta) = A_1\eta^n, \qquad I(\eta) = I_1\eta^{n+2}, \qquad (1 + (\alpha - 1)c) \leqslant \eta \leqslant \alpha, \tag{4}$$

where A_1 and I_1 are the area and inertia at $x = 0$, respectively, and n describes the taper of the cross-section; for $n = 1 \rightarrow (\alpha = h_2/h_1, \beta = 1)$, whereas, for $n = 2 \rightarrow (\alpha = \beta)$.

Substituting equations (2)–(4) into equations (1) yields the two equations

$$\xi^2 w_{1,\xi\xi\xi\xi} + 2(n+2)\xi w_{1,\xi\xi\xi} + 6n w_{1,\xi\xi} - q_a^4 w_1 = 0, \qquad 0 \leqslant \xi \leqslant 1 + (\alpha - 1)c, \tag{5}$$

$$\eta^2 w_{2,\eta\eta\eta\eta} + 2(n+2)\eta w_{2,\eta\eta\eta} + 6n w_{2,\eta\eta} - q_a^4 w_2 = 0, \qquad 1 + (\alpha - 1)c \leqslant \eta \leqslant \alpha, \tag{6}$$

where

$$q_a = p/(\alpha - 1), \qquad p^4 = \rho\omega^2 A_1 L^4/EI_1. \tag{7}$$

The solutions of equations (5) and (6) are [12]

$$w_1(\xi) = \xi^{-0\cdot5n}\{C_1\mathrm{J}_n(2q_a\xi^{0\cdot5}) + C_2\mathrm{Y}_n(2q_a\xi^{0\cdot5}) + C_3\mathrm{I}_n(2q_a\xi^{0\cdot5}) + C_4\mathrm{K}_n(2q_a\xi^{0\cdot5})\},$$

$$w_2(\eta) = \eta^{-0\cdot5n}\{C_5\mathrm{J}_n(2q_a\eta^{0\cdot5}) + C_6\mathrm{Y}_n(2q_a\eta^{0\cdot5}) + C_7\mathrm{I}_n(2q_a\eta^{0\cdot5}) + C_8 K_n(2q_a\eta^{0\cdot5})\}, \tag{8}$$

where J and Y are Bessel functions of the first and second kinds, and I and K are modified functions of the first and second kinds. The integration constants C_i ($i = 1, \ldots, 8$) are determined from the boundary conditions of the support at the ends of the bar and continuity conditions at $x = cL$.

For $x = 0$, $\rightarrow \xi = 1$ [7, 8],

$$w_{1,\xi\xi} + (k_e^2 + d^2)[\mu_1/(\alpha - 1)]w_{1,\xi} - \mu_1[d/(\alpha - 1)^2]w_1 = 0, \tag{9}$$

$$w_{2,\eta\eta\eta} + (n+2)w_{2,\eta\eta} + \mu_1[d/(\alpha - 1)^2]w_{2,\eta} - [\mu_1/(\alpha - 1)^3]w_1 = 0, \tag{10}$$

where

$$m_t = \rho A_1 L \frac{2\alpha\beta + \alpha + \beta + 2}{6} = \rho A_1 LZ, \qquad \mu = \frac{M_e}{m_t}, \qquad k_e = L^{-1}\sqrt{\frac{J_e}{M_e}},$$

$$d = \frac{\bar{d}}{L}, \qquad \mu_1 = \mu Z p^4.$$

The continuity conditions at $x = cL$, $y = 0 \rightarrow \xi = \eta = 1 + (\alpha - 1)c$, are

$$w_{1,\xi\xi\xi} + (n+2)\xi^{-1}w_{1,\xi\xi} - v_1[\xi^{n+2}/(\alpha - 1)^3]w_1 - (n+2)\eta^{-1}w_{2,\eta\eta} - w_{2,\eta\eta\eta} = 0, \tag{11}$$

$$w_{1,\xi\xi} - k^2 v_1[p^2\xi^{n+2}/(\alpha - 1)]w_{1,\xi} - w_{2,\eta\eta} = 0, \qquad w_1 = w_2, \qquad w_{1,\xi} = w_{2,\eta}, \tag{12–14}$$

with

$$v_1 = vp^2[\alpha^n + (n-1)\alpha + 1]/(n+1), \qquad v = M/m_t, \qquad k = L^{-1}\sqrt{J/M}\,,$$

and, for the right end, $y = (1-c)L \to (\eta = \alpha)$,

$$w_2 - C_T(n+2)[(\alpha-1)^3/\alpha]w_{2,\eta\eta} + C_T(\alpha-1)^3 w_{2,\eta\eta\eta} = 0, \tag{15}$$

$$w_{2,\eta} + C_R(\alpha-1)w_{2,\eta\eta} = 0, \tag{16}$$

where

$$C_R = EI_2/k_R L, \qquad C_T = EI_2/k_T L^3. \tag{17}$$

Substituting equations (8) into equations (9)–(12) and (14)–(17), the corresponding characteristic equations, the roots (p_i) of which represent the free frequencies, can be obtained from

$$\det \mathbf{A} = \mathbf{0}, \tag{18}$$

where $\mathbf{A}$ is an 8×8 matrix; all of the elements are listed in the Appendix. The frequencies are calculated in dimensionless terms and are obtained by operating on equation (18) with use of the False Position Method and a symbolic calculation program to manage the Bessel function [13].

3. THE RAYLEIGH–RITZ METHOD

The Rayleigh–Ritz method can be used for this problem, as follows. It is assumed that $w(x)$ can be expressed as a series combination of beam functions which satisfy the specified boundary conditions at $x = 0$ and $x = L$: i.e.,

$$w(x) \simeq \sum_1^N q_i \phi_i(x), \tag{19}$$

where the q's are constants to be determined.

The set of polynomials $\{\phi_1, \phi_2, \ldots, \phi_N\}$ is orthogonal on [0, 1] with respect to a weight function $r(x) = h(x) * b(x)$. The polynomials are constructed by employing the Gram–Schmidt process [4, 5, 10].

With the approximation (19), the expressions for the maximum kinetic energy and the maximum potential energy of the beam are known to be

$$T_{max} = \frac{\omega^2}{2}\int_0^L \rho A_x \left(\sum_1^N q_i\phi_i\right)^2 \mathrm{d}x + \frac{\omega^2}{2} M_e\left(\sum_1^N q_i\phi_i(0)\right)^2 - \frac{\omega^2}{2}(M_e\bar{d}^2 + J_e)\left(\sum_1^N q_i\phi_i(0)\right)^2_{,x}$$

$$+ \omega^2 M_e \bar{d}\left(\sum_1^N q_i\phi_i(0)\right)_{,x}\left(\sum_1^N q_i\phi_i(0)\right) + \frac{\omega^2}{2} M\left(\sum_1^N q_i\phi_i(cL)\right)^2$$

$$+ \frac{\omega^2}{2} J\left(\sum_1^N q_i\phi_i(cL)\right)^2_{,x}, \tag{20}$$

$$U_{max} = \tfrac{1}{2}\int_0^L EI_x\left[\sum_1^N q_i\phi_i(x)\right]^2_{,xx} \mathrm{d}x + \frac{k_R}{2}\left[\sum_1^N q_i\phi_i(L)\right]^2_{,x} + \frac{k_T}{2}\left[\sum_1^N q_i\phi_i(L)\right]^2. \tag{21}$$

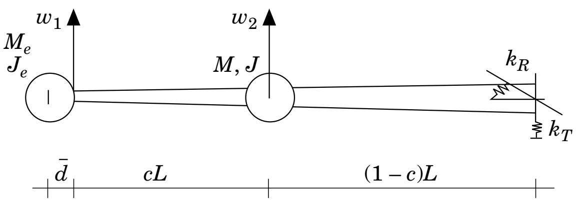

Figure 1. A definition sketch of the beam system.

In terms of the non-dimensional parameters, the functional governing the problem, with $\zeta = x/L$, can be written as

$$\Pi = \frac{EI_1}{2L^3}\left\{\begin{array}{l} \displaystyle\int_0^1 H(\zeta)\left(\sum_1^N q_i\phi_i\right)^2_{,\epsilon\epsilon}\mathrm{d}\zeta + \frac{H(1)}{C_R}\left(\sum_1^N q_i\phi_i(1)\right)^2_{,\zeta} \\ -p^4\left[\begin{array}{l}\displaystyle\int_0^1 G(\zeta)\left(\sum_1^N q_i\phi_i\right)^2\mathrm{d}\zeta + \mu Z(k_e^2+d^2)\left(\sum_1^N q_i\phi_i(0)\right)^2_{,\zeta} + \mu Z\left(\sum_1^N q_i\phi_i(0)\right)^2 \\ \displaystyle -\mu Zd\left(\sum_1^N q_i\phi_i(0)\right)_{,\zeta}\left(\sum_1^N q_i\phi_i(0)\right) + vZ\left(\sum_1^N q_i\phi_i(c)\right)^2 + vZk^2\left(\sum_1^N q_i\phi_i(c)\right)^2_{,\epsilon}\end{array}\right]\end{array}\right\}, \tag{22}$$

where

$$A_\zeta = A_1[(\alpha-1)\zeta+1][(\beta-1)\zeta+1] = A_1G(\zeta),$$

$$I_\zeta = I_1[(\alpha-1)\zeta+1]^3[(\beta-1)\zeta+1] = I_1H(\zeta). \tag{23}$$

The minimization conditions $\partial\Pi/\partial q_i = 0$ yield an eigenvalue problem of the type

$$(\mathbf{K} - p^4\mathbf{m})\mathbf{q} = \mathbf{0}, \tag{24}$$

TABLE 1

Comparison of the frequency parameters between the exact values and the R–R method for $\alpha = 1$; *R–R** (*present*); G[3] (*Galerkin*); *R—R*[3].

c	k_e	μ	k	v	1	2	3	4	5	
0·4	1	5	1	5	0·577528	1·010010	1·614418	2·988467	7·922812	Exact
					0·577772	1·015738	1·661745	3·033598	8·094794	G [3]
					0·577895	1·018616	1·689432	3·064918	8·216193	R–R*
					0·579576	1·040670	1·665689	3·322992	8·104343	R–R [3]
	1	0·1	1	0·1	1·413511	2·492382	3·521664	6·195630	8·503077	Exact
					1·413829	2·494927	3·611185	6·395637	8·636190	G [3]
					1·413994	2·496215	3·662253	6·531052	8·745954	R–R*
					1·415931	2·556673	3·613588	6·903891	8·653406	R–R [3]
0·5	$(1/5)^{0\cdot5}$	5	$(1/5)^{0\cdot5}$	5	0·752515	1·383353	2·137078	2·708818	9·480262	Exact
					0·752600	1·385825	2·219663	2·727709	9·574332	G [3]
					0·752636	1·386844	2·259297	2·740545	9·609262	R–R*
					0·753286	1·420494	2·220540	2·893973	9·696748	R–R [3]

TABLE 2

As Table 1 *but for* $\alpha = \beta = 1{\cdot}5$, $d = 0$

$\alpha = \beta = 1{\cdot}5$, $d = 0$					1		2		3		4		5	
k_e	μ	k	v	c	R–R	Exact	R–R	Exact	R–R	Exact	R–R	Exact	R–R	Exact
0	1	0	1	0·25	1·434281	1·434276	3·699618	3·698415	6·980029	6·972650	11·233998	11·221403	14·730853	14·591590
				0·50	1·510086	1·510084	3·529150	3·527558	7·708643	7·702148	10·604666	10·555656	14·659575	14·356159
				0·75	1·531616	1·531616	4·294797	4·293975	6·636747	6·627278	10·234898	10·213547	14·539022	14·355622
1	1	0	0·5	0·25	1·032491	1·032491	1·919194	1·919185	4·999696	4·998286	7·562921	7·550923	11·583156	11·554261
				0·50	1·036175	1·036175	1·988969	1·988965	4·475050	4·473880	8·802252	8·800871	11·318705	11·247578
				0·75	1·036839	1·036839	2·019795	2·019795	5·088835	5·088045	7·779940	7·771827	11·387267	11·362250
1	1	0·5	0·5	0·25	1·022082	1·022055	1·866983	1·865512	3·403777	3·231818	6·834476	6·164930	7·819994	7·631800
				0·50	1·032755	1·032749	1·901937	1·900305	4·233868	3·917682	4·540404	4·495800	10·965108	10·373952
				0·75	1·036204	1·036203	1·980488	1·979920	3·777396	3·661910	7·367104	6·693051	7·952626	7·820918
1	1	1	1	0·25	0·951796	0·950739	1·613478	1·595284	2·450036	2·330174	6·246183	5·741479	7·694178	7·427130
				0·50	1·006223	1·005827	1·561958	1·546253	3·161435	2·914690	4·068787	4·064626	10·793113	10·290430
				0·75	1·031388	1·031339	1·731349	1·716177	2·723710	2·624439	6·668965	6·072039	7·398589	7·367998
1	5	1	5	0·25	0·638403	0·637679	1·085976	1·072436	1·689935	1·604106	4·244971	3·966270	7·607324	7·255010
				0·50	0·675362	0·675093	1·054125	1·042436	2·158985	1·995186	2·859398	2·852968	10·505777	10·090052
				0·75	0·692159	0·692127	1·179222	1·166959	1·850189	1·785647	4·751063	4·446769	7·140726	6·945028

TABLE 3

As Table 1 but for $\alpha = \beta = 1{\cdot}5$, $d = 0{\cdot}5$

$\alpha = \beta = 1{\cdot}5$, $d = 0{\cdot}5$					1		2		3		4		5	
k_e	μ	k	v	c	R–R	Exact	R–R	Exact	R–R	Exact	R–R	Exact	R–R	Exact
0	1	0	1	0·25	1·102446	1·102446	2·546940	2·546679	6·425156	6·424175	9·393009	9·376359	12·009206	11·933431
				0·50	1·116931	1·116931	2·864886	2·864408	5·677412	5·672076	9·797352	9·792720	12·195035	12·038143
				0·75	1·119969	1·119969	3·365277	3·365189	5·588675	5·583811	8·510574	8·495759	12·490234	12·431437
1	1	0	0·5	0·25	0·926231	0·926231	2·069971	2·069954	5·115926	5·114649	7·600544	7·588255	11·591353	11·561968
				0·50	0·928493	0·928493	2·166397	2·166388	4·543524	4·541396	8·833998	8·832728	11·334960	11·262925
				0·75	0·928912	0·928911	2·212935	2·212935	5·132603	5·131769	7·809190	7·801025	11·410726	11·385465
1	1	0·5	0·5	0·25	0·919981	0·919970	2·003709	2·001126	3·400575	3·229735	7·056127	6·352086	7·821087	7·646247
				0·50	0·926382	0·926379	2·043424	2·040400	4·337941	3·989838	4·563012	4·546850	11·003025	10·412486
				0·75	0·928508	0·928507	2·151935	2·150702	3·833893	3·715232	7·402959	6·716683	7·960546	7·843321
1	1	1	1	0·25	0·876532	0·876045	1·657818	1·629200	2·471116	2·364727	6·518009	5·983691	7·698586	7·432873
				0·50	0·910544	0·910390	1·606955	1·585697	3·301447	3·051091	4·152836	4·149163	10·830672	10·331122
				0·75	0·925530	0·925511	1·800006	1·777924	2·883103	2·785631	6·675749	6·085078	7·438853	7·404168
1	5	1	5	0·25	0·587407	0·587076	1·116457	1·095421	1·712437	1·636574	4·424359	4·130614	7·608847	7·255048
				0·50	0·610386	0·610282	1·085882	1·070062	2·278767	2·110453	2·904808	2·901233	10·514070	10·099783
				0·75	0·620408	0·620395	1·227089	1·209274	1·977935	1·916442	4·751092	4·447414	7·151919	6·956110

with

$$k_{ij} = \int_0^1 H(\zeta)\phi_{i,\zeta\zeta}\phi_{j,\zeta\zeta}\,\mathrm{d}\zeta + \frac{H(1)}{C_R}\phi_{1,\zeta}(1)\phi_{j,\epsilon}(1) + \frac{H(1)}{C_T}\phi_i(1)\phi_j(1),$$

$$m_{ij} = \int_0^1 G(\zeta)\phi_i\phi_j\,\mathrm{d}\zeta + \mu Z(d^2 + k_e^2)\phi_{i,\zeta}(0)\phi_{j,\zeta}(0) + \mu Z\phi_i(0)\phi_j(0)$$

$$- \mu Zd[(\phi_i(0)\phi_{j,\zeta}(0) + \phi_j(0)\phi_{i,\zeta}(0)] + \mu Z\phi_i(c)\phi_j(c) + \mu Zk^2\phi_{i,\zeta}(c)\phi_{j,\zeta}(c). \qquad (25)$$

The stiffness matrix **K** and mass matrix **m** are positive definite, and hence all the eigenvalues p_i are real and positive. The accuracy of the Rayleigh–Ritz method result depends on the number N of the assumed mode shape functions ϕ_i.

4. NUMERICAL RESULTS

The first five free vibration frequencies for the structural model in Figure 1 have been calculated, by using both the exact approach (henceforth Exact) and the Rayleigh–Ritz method (henceforth R–R).

In Table 1 the non-dimensional frequencies are compared with the results given by Hamdan and Latif [3] for a beam with a constant cross-section. The exact results obtained by using the procedure outlined in section 2 coincide with the results given in reference [3], whereas our approximate Rayleigh–Ritz results can be considered as a better result, with respect to the analogous results given in reference [3]. This is probably due to our choice of the polynomial functions, while the Hamdan and Latif results are comparable with those obtained by using a Galerkin procedure [3].

In Tables 2 and 3 the first five non-dimensional frequencies p_i are given, for a beam with taper ratio $\alpha = \beta = 1{\cdot}5$, and for different values of the non-dimensional parameters of the masses inertial properties. As can be noted the first free vibration frequencies are practically exact even with only seven approximating functions, whereas the higher frequencies show some significant discrepancies, up to a maximum of 9%.

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REFERENCES

1. C. N. Bapat and C. Bapat 1987 *Journal of Sound and Vibration* **112**, 177–182. Natural frequencies of a beam with non-classical boundary conditions and concentrated masses.
2. H. Matsuda, C. Morita and T. Sakiyama 1992 *Journal of Sound and Vibration* **158**, 331–339. A method for vibration analysis of a tapered Timoshenko beam with constraint at any points and carrying a heavy tip body.
3. M. N. Hamdan and L. Abdel Latif 1994 *Journal of Sound and Vibration* **169**, 527–545. On the numerical convergence of discretization methods for the free vibrations of beams with attached inertia elements.
4. R. Bhat 1985 *Journal of Sound and Vibration* **102**, 493–499. Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh–Ritz method.
5. R. O. Grossi and R. Bhat 1991 *Journal of Sound and Vibration* **147**, 174–178. A note on vibrating tapered beams.
6. H. H. Mabie and C. B. Rogers 1974 *Journal of the Acoustical Society of America* **55**, 986–991. Transverse vibrations of double-tapered beams with end support and with end mass.

7. R. O. GROSSI, A. ARANDA and R. BHAT 1993 *Journal of Sound and Vibration* **160**, 175–178. Vibration of tapered beams with one end spring hinged and the other end with tip mass.
8. M. J. MAURIZI, P. BELLES and M. ROSALES 1990 *Journal of Sound and Vibration* **138**, 170–172. A note on free vibrations of a constrained cantilever beam with a tip mass of finite length.
9. N. M. AUCIELLO 1996 *Journal of Sound and Vibration* **194,** 25–34. Transverse vibrations of a linearly tapered cantilever beam with tip mass of finite length.
10. R. L. BURDEN and J. D. FAIRES *Numerical Analysis*. Boston, MA: PWS–KENT P.C.; fourth edition.
11. T. W. LEE 1976 *Journal of Applied Mechanics*, 366–367. Transverse vibrations of a tapered beam carrying a concentrated mass.
12. G. B. WATSON 1966 *A Treatise on the Theory of Bessel Functions*. Cambridge: Cambridge University Press.
13. S. WOLFRAM 1993 *Mathematica*, *Version* 2.2. Champaign, IL: Wolfram Research Inc.

APPENDIX

$$a_{11} = p\mathrm{J}_{n+2}(a) - k_e^2\mu_1\mathrm{J}_{n+1}(a), \qquad a_{12} = pY_{n+2}(a) - k_e^2\mu_1\mathrm{Y}_{n+1}(a)$$

$$a_{13} = p\mathrm{I}_{n+2}(a) + k_e^2\mu_1\mathrm{I}_{n+1}(a), \qquad a_{14} = p\mathrm{K}_{n+2}(a) - k_e^2\mu_1\mathrm{K}_{n+1}(a)$$

$$a_{21} = -p^3\mathrm{J}_{n+3}(a) + (n+2)(\alpha-1)p^2\mathrm{J}_{n+2}(a) - \mu_1\mathrm{J}_n(a),$$

$$a_{22} = -p^3\mathrm{Y}_{n+3}(a) + (n+2)(\alpha-1)p^2\mathrm{Y}_{n+2}(a) - \mu_1\mathrm{Y}_n(a),$$

$$a_{23} = p^3\mathrm{I}_{n+3}(a) + (n+2)(\alpha-1)p^2\mathrm{I}_{n+2}(a) - \mu_1\mathrm{I}_n(a),$$

$$a_{24} = -p^3\mathrm{K}_{n+3}(a) + (n+2)(\alpha-1)p^2\mathrm{K}_{n+2}(a) - \mu_1\mathrm{K}_n(a),$$

$$a_{1i} = 0, \qquad a_{2i} = 0, \qquad i = 5, \ldots, 8,$$

$$a_{31} = -pb^{n+0\cdot5}\mathrm{J}_{n+3}(bb) + Q_1\mathrm{J}_{n+2}(bb) + v_1\mathrm{J}_n(bb),$$

$$a_{32} = -pb^{n+0\cdot5}\mathrm{Y}_{n+3}(bb) + Q_1\mathrm{Y}_{n+2}(bb) + v_1\mathrm{Y}_n(bb)$$

$$a_{33} = pb^{n+0\cdot5}\mathrm{I}_{n+3}(bb) + Q_1\mathrm{I}_{n+2}(bb) + v_1\mathrm{I}_n(bb),$$

$$a_{34} = -pb^{n+0\cdot5}\mathrm{K}_{n+3}(bb) + Q_1\mathrm{K}_{n+2}(bb) + v_1\mathrm{K}_n(bb),$$

$$a_{35} = pb^{n+0\cdot5}\mathrm{J}_{n+3}(bb) - Q_1\mathrm{J}_{n+2}(bb), \qquad a_{36} = pb^{n+0\cdot5}\mathrm{Y}_{n+3}(bb) - Q_1\mathrm{Y}_{n+2}(bb),$$

$$a_{37} = -pb^{n+0\cdot5}\mathrm{I}_{n+3}(bb) - Q_1\mathrm{I}_{n+2}(bb), \qquad a_{38} = pb^{n+0\cdot5}\mathrm{K}_{n+3}(bb) - Q_1\mathrm{K}_{n+2}(bb),$$

$$a_{41} = \mathrm{J}_{n+2}(bb) + Q_2\mathrm{J}_{n+1}(bb), \qquad a_{42} = \mathrm{Y}_{n+2}(bb) + Q_2\mathrm{Y}_{n+1}(bb),$$

$$a_{43} = \mathrm{I}_{n+2}(bb) - Q_2\mathrm{I}_{n+1}(bb), \qquad a_{44} = \mathrm{K}_{n+2}(bb) + Q_2\mathrm{K}_{n+1}(bb),$$

$$a_{45} = -\mathrm{J}_{n+2}(bb), \qquad a_{46} = -\mathrm{Y}_{n+2}(bb), \qquad a_{47} = -\mathrm{I}_{n+2}(bb), \qquad a_{48} = -\mathrm{K}_{n+2}(bb),$$

$$a_{51} = \mathrm{J}_n(bb) = -a_{55}, \qquad a_{52} = \mathrm{Y}_n(bb) = -a_{56},$$

$$a_{53} = \mathrm{I}_n(bb) = -a_{57}, \qquad a_{54} = \mathrm{K}_n(bb) = -a_{58},$$

$$a_{61} = \mathrm{J}_{n+1}(bb) = -a_{65}, \qquad a_{62} = \mathrm{Y}_{n+1}(bb) = -a_{66},$$

$$a_{63} = -\mathrm{I}_{n+1}(bb) = -a_{67}, \qquad a_{64} = \mathrm{K}_{n+1}(bb) = -a_{68},$$

$$a_{75} = \alpha^2\mathrm{J}_n(aa) - (n+2)(\alpha-1)^3C_Tp_a^2\mathrm{J}_{n+2}(aa) + p_a^3\alpha^{0\cdot5}\mathrm{J}_{n+3}(aa),$$

$$a_{76} = \alpha^2\mathrm{Y}_n(aa) - (n+2)(\alpha-1)^3C_Tp_a^2\mathrm{Y}_{n+2}(aa) + p_a^3\alpha^{0\cdot5}\mathrm{Y}_{n+3}(aa),$$

$$a_{77} = \alpha^2\mathrm{I}_n(aa) - (n+2)(\alpha-1)^3C_Tp_a^2\mathrm{I}_{n+2}(aa) - p_a^3\alpha^{0\cdot5}\mathrm{I}_{n+3}(aa),$$

$$a_{75} = \alpha^2\mathrm{K}_n(aa) - (n+2)(\alpha-1)^3C_Tp_a^2\mathrm{K}_{n+2}(aa) + p_a^3\alpha^{0\cdot5}\mathrm{K}_{n+3}(aa),$$

$$a_{85} = -\alpha^{0\cdot5}\mathrm{J}_{n+1}(aa) + C_R p \mathrm{J}_{n+2}(aa), \qquad a_{86} = -\alpha^{0\cdot5}\mathrm{Y}_{n+1}(aa) + C_R p \mathrm{Y}_{n+2}(aa),$$

$$a_{87} = \alpha^{0\cdot5}\mathrm{I}_{n+1}(aa) + C_R p \mathrm{I}_{n+2}(aa), \qquad a_{88} = -\alpha^{0\cdot5}\mathrm{K}_{n+1}(aa) + C_R p \mathrm{K}_{n+2}(aa),$$

$$a_{7i} = a_{8i} = 0, \qquad i = 1, \ldots, 4.$$

Here it is assumed that

$$a = 2p_a, \qquad aa = 2p_a\alpha^{0\cdot5}, \qquad b = 1 + (\alpha - 1)c, \qquad bb = 2p_a b^{0\cdot5}$$

$$Q_1 = (n+2)b^n(\alpha - 1), \qquad Q_2 = pk^2 v_1 b^{-n-1\cdot5}.$$