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# APPLICATION OF FRACTIONAL OPERATORS TO THE ANALYSIS OF DAMPED VIBRATIONS OF VISCOELASTIC SINGLE-MASS SYSTEMS

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Free damped vibrations of two hereditarily elastic oscillators, the hereditary properties of which are described by the Boltzmann–Volterra relationships with the weakly singular Rzhanitsyn kernel taken as the creep kernel (the first model) or as the relaxation kernel (the second model), are considered. These integral relationships are equivalent to the differential relationships involving infinite sums of various order time derivatives of excitation (force) or response (displacement) depending on whether the Rzhanitsyn kernel is used for the creep kernel or relaxation kernel. The problem is solved by the Laplace transform method. When passing from image to pre-image one is led to find the roots of an algebraic (characteristic) equation with fractional exponents. A method for solving such equations is proposed which allows one to investigate the roots' behaviour in a wide range of single-mass system parameters. It is shown that, if the Rzhanitsyn kernel is the creep kernel in the Volterra equations, then the characteristic equation does not possess real roots, but has two complex conjugate roots; i.e., the test single-mass system subjected to the impulse excitation does not pass into an aperiodic regime. On the contrary, the oscillator model with the Rzhanitsyn relaxation kernel may be both in vibrating motion and in the aperiodic regime, depending on the intervals over which the relaxation times for the given model vary, as well as on the order of fractional power and the ratio of the relaxed modulus (rubbery modulus) to the non-relaxed modulus (glassy modulus). However, contrary to the standard linear solid model with ordinary time derivatives for which the dimensions of the domain of aperiodicity as well as its existence are governed only by the magnitude of the ratio of the relaxed modulus to the non-relaxed modulus, for the model with the Rzhanitsyn relaxation kernel all of the above-listed factors essentially depend in addition on an order of a fractional operator parameter. The main characteristics of the vibratory and aperiodic motions of the single-mass system as functions of the relaxation time or creep time, which are equivalent to the temperature dependences, are constructed and analyzed for both models.



## 1. INTRODUCTION

Free damped vibrations of a hereditarily elastic oscillator, the hereditary properties of which are described by a rheological model containing fractional operators, were treated first by Rozovsky and Sinaisky [1]. The authors used for the fractional operators fractional derivatives to replace the ordinary derivatives in the standard linear solid model. The solution for such a generalized standard linear solid model has been constructed by the Laplace transformation method. During the Laplace transform inversion, rationalization of the characteristic equation with fractional powers has been made through substitution for a complex parameter of conversion, whereupon the Laplace transform has been decomposed into common fractions. As the result of inversion, the solution has been

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represented in terms of a linear combination of Rabotnov's fractional exponential functions [2] which are dependent upon the time and the roots of the rationalized characteristic equation. The solution constructed in such a manner is poorly amenable to analytical treatment, since for its realization tabulation of fractional exponential functions of a complex variable, which has not yet been carried out, is required; although the tables for fractional exponential functions of a real variable have long been in existence [Rabotnov *et al*., 3].

Another way of looking at the analysis of free damped vibrations of an oscillator described by the fractional calculus standard linear solid model, which is devoid of the enumerated limitations, was proposed by Rossikhin [4] and extended by Zelenev *et al*. [5, 6] and Meshkov *et al*. [7]. The Laplace transformation method was also applied in these papers, but the non-rationalized characteristic equation, i.e., the equation with fractional powers, was used for calculating the roots. It has been shown that the characteristic equation with fractional powers has no real roots but possesses two complex conjugate roots located in the half-plane of the complex plane. Upon determining the roots of the characteristic equation, for the calculation of which a highly efficient method was developed by Rossikhin [4], the solution has been constructured on the first sheet of a Riemann surface (a complex plane with a cut-out negative real semi-axis) with the use of the theory of residues.

The renewed interest in visoelastic models and their application to dynamic problems [8–10] centres around the elaboration of new damping systems in engineering and technology based on a continuum of damping elements distributed uninterruptedly throughout the relaxation or creep times instead of the discrete system of damping elements [11]. Among such damping systems are various kinds of coating, backing, substrate, sheeting and jacketing which are designed for damping of harmful vibrations.

However, despite a growing body of publications, the authors are not familiar with papers wherein hereditarily elastic models with more intricate fractional operators are applied for mathematical modelling of damped vibrations, instead of traditional operators of fractional differentiation and integration.

In the present paper, an attempt is made to eliminate this gap through the use of hereditarily elastic models with such fractional operators which combine the properties of the fractional derivative standard linear solid model and the features of the standard linear solid model with oridinary derivatives.

## 2. MODELS OF HEREDITARILY ELASTIC MEDIA WITH RZHANITSYN KERNEL

Let the hereditary features of an oscillator be described by the Boltzmann–Volterra relationships

$$\varepsilon(t) = J_\infty \left[ \sigma(t) + \nu_\sigma \int_0^t K_\sigma(t-t')\sigma(t')\,\mathrm{d}t' \right], \qquad \nu_\sigma = \frac{J_0 - J_\infty}{J_\infty}, \tag{1a}$$

$$\sigma(t) = E_\infty \left[ \varepsilon(t) - \nu_\varepsilon \int_0^t K_\varepsilon(t-t')\varepsilon(t')\,\mathrm{d}t' \right], \qquad \nu_\varepsilon = \frac{E_\infty - E_0}{E_\infty}, \tag{1b}$$

where the weakly singular Rzhanitsyn kernel is [12]

$$K_i(t) = \frac{t^{\gamma-1}}{\tau_i^\gamma \Gamma(\gamma)} \exp\left(-\frac{t}{\tau_i}\right), \qquad i = \varepsilon, \sigma. \tag{2}$$

Here $\sigma$ is the stress, $\varepsilon$ is the strain, $\tau_\sigma$ and $\tau_\varepsilon$ are the creep time and relaxation time, respectively, $J_0$ and $J_\infty$ are the relaxed and non-relaxed magnitudes of the elastic compliance, respectively, $E_0 = J_0^{-1}$ and $E_\infty = J_\infty^{-1}$ are the relaxed (rubbery) and non-relaxed (glassy) magnitudes of the elastic modulus, respectively, $(\tau_\varepsilon/\tau_\sigma)^\gamma = E_0/E_\infty = J_\infty/J_0$, $\Gamma(\gamma)$ is the Gamma function, and $\gamma$ is the order of the fractional operator (parameter of divisibility), $0 < \gamma \leqslant 1$.

Rzhanitsyn himself used the function (2) for the after-effect kernel $K_\sigma(t)$ to describe the deformation rates of a steel specimen [12], but the function (2) was applied for the relaxation kernel $K_\varepsilon(t)$ by Koltynov [13].

The resolvent kernel $K_\varepsilon(t)$ arising from the kernel (2) at $i = \sigma$ has been obtained by Wolfson [14] in the form

$$K_\varepsilon(t) = \nu_\varepsilon^{-1} \exp\left(-\frac{t}{\tau_\sigma}\right)\frac{t^{\gamma-1}}{\tau_\sigma^\gamma} \sum_{n=0}^{\infty} \frac{(-\nu_\sigma)^n (t/\tau_\sigma)^{\gamma n}}{\Gamma[\gamma(n+1)]} = \nu_\varepsilon^{-1} \exp(-t/\tau_\sigma) \ni_\gamma(-\nu_\sigma, t/\tau_\sigma), \tag{3}$$

where

$$\ni_\gamma(-\nu_i, t/\tau_i) = t^{\gamma-1}\tau_i^{-\gamma} \sum_{n=0}^{\infty} \frac{(-\nu_i)^n (t/\tau_i)^{\gamma n}}{\Gamma[\gamma(n+1)]}$$

is the Rabotnov fractional exponential function [2] which becomes an ordinary exponential at $\gamma = 1$.

It is known that the relaxation and creep kernels may be written in the integral form in terms of the distribution functions of the relaxation times $B_\varepsilon(\tau, \tau_\varepsilon)$ and creep times $B_\sigma(\tau, \tau_\sigma)$ by using the formulas [15]

$$K_i(t) = \int_0^\infty \tau^{-1} B_i(\tau, \tau_i)\, \mathrm{e}^{-t/\tau}\, \mathrm{d}\tau. \tag{4}$$

Thus, the distribution function of creep times compatible with the kernel (2) at $i = \sigma$ has the form [16]

$$B_\sigma(\tau, \tau_\sigma) = \frac{\sin \pi\gamma}{\pi} \frac{\tau^{\gamma-1}}{(\tau_\sigma - \tau)^\gamma} \mathrm{H}(\tau_\sigma - \tau), \tag{5}$$

but the distribution function of relaxation times appropriate to the kernel (3) is written as [17]

$$B_\varepsilon(\tau, \tau_\varepsilon) = \frac{\sin \pi\gamma}{\pi\tau\nu_\varepsilon} \frac{\mathrm{H}(\tau_\sigma - \tau)}{2\cos \pi\gamma + \nu_\sigma T^{-\gamma} + \nu_\sigma^{-1} T^\gamma}, \tag{6}$$

where $\mathrm{H}(\tau_\sigma - \tau)$ is the unit Heaviside function, and $T = \tau_\sigma \tau^{-1} - 1$.

Plots for the functions $B_\sigma(\tau, \tau_\sigma)$ and $B_\varepsilon(\tau, \tau_\varepsilon)$ at different magnitudes of $\gamma$ are presented in reference [16] for the case $\nu_\sigma = \nu_\varepsilon = \tau_\sigma = \tau_\varepsilon = 1$. Reference to these plots shows that the distribution functions have weak singularity at $\tau = 0$.

It may be shown (see the Appendix) that when $\gamma \to 1$ the distribution function (6) tends to the Dirac delta function

$$\lim_{\gamma \to 1} B_\varepsilon(\tau, \tau_\varepsilon) = \frac{(\nu_\sigma T)^{1/2}}{\tau} \delta(T - \nu_\sigma); \tag{7}$$

i.e., it is the delta-like sequence. This fact could be expected since the rheological model under consideration at $\gamma = 1$ goes over into the ordinary standard linear solid model, the relaxation and creep time distribution of which are described by the Dirac δ-function.

One can show that the Volterra relationships (1) with the kernels (2) are equivalent to rheological models with a certain fractional operator combining the properties of ordinary and fractional derivatives and integrals. Applying the Laplace transformation to equation (1a), wherein the kernel $K_\sigma$ has the form (2) at $i = \sigma$, yields

$$\bar{\varepsilon} = J_\infty[1 + v_\sigma(1 + p\tau_\sigma)^{-\gamma}]\bar{\sigma}, \tag{8}$$

where an overbar denotes the Laplace transform of the corresponding function.

Considering that

$$(1 + p\tau_\sigma)^{-\gamma} = 1 - \gamma p\tau_\sigma + \frac{\gamma(\gamma+1)}{2!}(p\tau_\sigma)^2 - \frac{\gamma(\gamma+1)(\gamma+2)}{3!}(p\tau_\sigma)^3 + \cdots \tag{9}$$

and substituting expansion (9) into the formula (8), after transition from image to pre-image in the resulting relation, one is led to the expression

$$\varepsilon = J_\infty[1 + v_\sigma(1 + \tau_\sigma\, \mathrm{d}/\mathrm{d}t)^{-\gamma}]\sigma, \tag{10}$$

which contains the operator $(1 + \tau_\sigma\, \mathrm{d}/\mathrm{d}t)^{-\gamma}$ having, as is shown in later sections, the features described above.

## 3. PROBLEM FORMULATION AND METHOD OF SOLUTION

Consider free damped vibrations of a single-mass system subjected to impulse excitation at the initial instant of time. Due to the hereditary elastic Boltzmann–Volterra relationships (1), the equation of motion can be written in the two equivalent forms in terms of the relaxation $K_\varepsilon(t)$ and the creep kernel $K_\sigma(t)$, respectively:

$$\ddot{x} + \omega_\infty^2\left[x - v_\varepsilon\int_0^t K_\varepsilon(t-t')x(t')\,\mathrm{d}t'\right] = F\delta(t), \tag{11}$$

$$\ddot{x} + \omega_\infty^2 x + v_\sigma\int_0^t K_\sigma(t-t')\ddot{x}(t')\,\mathrm{d}t' = F[\delta(t) + v_\sigma K_\sigma(t)]. \tag{12}$$

Here $x$ is the co-ordinate, $F$ is the amplitude of force impulse per unit mass, $\omega_\infty$ is the frequency of elastic vibrations corresponding to the non-relaxed magnitude of the elastic modulus, and overdots denote time derivatives.

Applying the Laplace transformation to equations (11) and (12) yields

$$\bar{x}(p) = \frac{F}{p^2 + \omega_\infty^2[1 - v_\varepsilon\bar{K}_\varepsilon(p)]} = \frac{F[1 + v_\sigma\bar{K}_\sigma(p)]}{\omega_\infty^2 + p^2[1 + v_\sigma\bar{K}_\sigma(p)]}. \tag{13}$$

The solution in the space of inverse transforms is determined according to the inversion formula

$$x(t) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\bar{x}(p)\,\mathrm{e}^{pt}\,\mathrm{d}p. \tag{14}$$

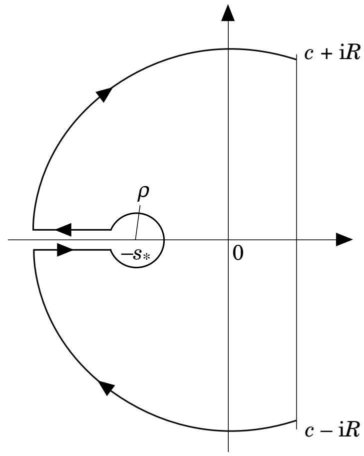


Figure 1. The contour used to calculate the complex inversion integral in the Laplace method.

To calculate the integral (14), it is necessary to find all singular points of the complex function $\bar{x}(p)$. The weakly singular kernels discussed above have branch points $p = -s_*$, $s_* \geqslant 0$ and $p = -\infty$ and ordinary poles at the same magnitudes of $p$ which cause the demoninator in formula (13) to vanish: i.e., they are the roots of the equations

$$p^2 + \omega_\infty^2[1 - v_\varepsilon \bar{K}_\varepsilon(p)] = 0, \qquad \omega_\infty^2 + p^2[1 + v_\sigma \bar{K}_\sigma(p)] = 0. \tag{15a, b}$$

For multi-valued functions possessing a branch point, the inverse transform theorem is applicable only for the first sheet of the Riemann surface; i.e., when $-\pi < \arg p < \pi$. Thus the closed contour should be chosen in the form presented in Figure 1. Due to the Jordan lemma, the curvilinear integrals taken along the arcs $c_R$ tend to zero at $R \to \infty$. For weakly

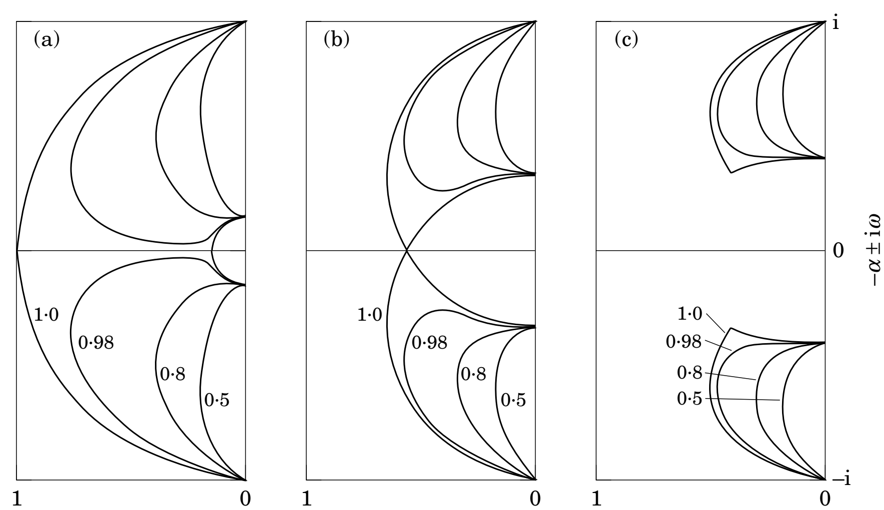


Figure 2. The behaviour of the complex conjugate roots $p_{1,2} = -\alpha \pm i\omega$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn after-effect kernel: (a) $\xi = 1/50$; (b) $\xi = 1/9$; (c) $\xi = 1/6$.

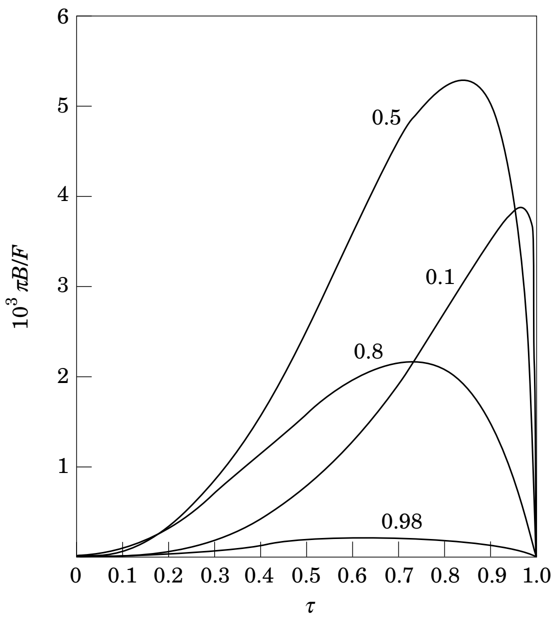


Figure 3. The distribution function $B$ versus $\ln \tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn after-effect kernel at $\xi = 1/50$.

singular kernels, the integral taken along $c_\rho$ also tends to zero when $\rho \to 0$. By using the main theorem of the theory of residues, the solution of equations (11) and (12) thus may be written as

$$x(t) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty [\bar{x}(s\,\mathrm{e}^{-\mathrm{i}\pi}) - \bar{x}(s\,\mathrm{e}^{\mathrm{i}\pi})]\mathrm{H}(s - s_*)\,\mathrm{e}^{-st}\,\mathrm{d}s + \sum_k \mathrm{res}[\bar{x}(p_k)\,\mathrm{e}^{p_k t}], \tag{16}$$

where the summation is taken over all isolated singular points (poles).

## 4. HEREDITARY ELASTIC OSCILLATOR MODEL WITH THE RZHANITSYN AFTER-EFFECT KERNEL

Applying the Laplace transformation to the formula (2) at $i = \sigma$ yields

$$\bar{K}_\sigma(p) = (1 + p\tau_\sigma)^{-\gamma}. \tag{17}$$

Considering the expression (17) in relation (13), one finds that

$$\bar{x}(p) = \frac{F[(1 + p\tau_\sigma)^\gamma + v_\sigma]}{(p^2 + \omega_\infty^2)(1 + p\tau_\sigma)^\gamma + p^2 v_\sigma}. \tag{18}$$

To obtain the poles of the function (18), one finds the roots of the characteristic equation

$$(p^2 + \omega_\infty^2)(1 + p\tau_\sigma)^\gamma + p^2 v_\sigma = 0. \tag{19}$$

It can be demonstrated that equation (19) has no real negative roots. Setting $p = -y$, $y > 0$ in equation (19) yields

$$y^2 = -\omega_\infty^2 (1 - y\tau_\sigma)^\gamma / [(1 - y\tau_\sigma)^\gamma + v_\sigma],$$

which is inconsistent with the input assumption.

For finding the complex roots of equation (19), one substitutes $p = r\,\mathrm{e}^{\mathrm{i}\psi}$. Then, separating the real and imaginary parts, one obtains the system of two equations

$$\omega_\infty^2 r^{-2} \cos 2\psi + v_\sigma R^{-\gamma} \cos \gamma\Phi + 1 = 0, \tag{20a}$$

$$\omega_\infty^2 r^{-2} \sin 2\psi + v_\sigma R^{-\gamma} \sin \gamma\Phi = 0, \tag{20b}$$

where $R^2 = 1 + 2\tau_\sigma r \cos\psi + \tau_\sigma^2 r^2$ and $\mathrm{tg}\,\Phi = \tau_\sigma r \sin\psi(1 + \tau_\sigma r \cos\psi)^{-1}$.

From equation (20b), it will be obvious that the system is rootless at every $0 < |\psi| < \pi/2$. To calculate the roots of equations (20) at $\pi/2 < |\psi| < \pi$, one multiplies equation (20a) by $\sin 2\psi$ and equation (20b) by $\cos 2\psi$, and substracts the second equation from the first one. Introducing the new variable $x = \tau_\sigma r$ and putting $\omega_\infty = 1$ yields

$$\sin 2\psi + v_\sigma(1 + 2x\cos\psi + x^2)^{-\gamma/2} \sin\left[2\psi - \gamma\,\mathrm{arctg}\left(\frac{x\sin\psi}{1 + x\cos\psi}\right)\right] = 0. \tag{21}$$

From equation (21), at every fixed angle $\pi/2 < |\psi| < \pi$, and given $v_\sigma$ and $\gamma$, one can determine the value $x$. Then substituting that value $x$ into equation (20b), first one finds that

$$r = (-R^\gamma \sin 2\psi / v_\sigma \sin\gamma\phi)^{1/2},$$

and thereafter one can calculate the value $\tau_\sigma = x r^{-1}$.

The behaviour of the roots in the complex plane as functions of the parameter $\tau_\sigma$ for the three magnitudes $v_\sigma = 49$, 8 and 5 ($\xi = 1/50$, 1/9 and 1/6) at $\omega_\infty = 1$ is presented in Figures 2(a), (b) and (c), respectively, wherein the magnitudes of $\gamma$ are indicated numerically. It is seen that the curves for the two complex conjugate roots $p_{1,2} = -\alpha + \mathrm{i}\omega$ at $\gamma \neq 1$ leave the points $\pm\mathrm{i}$ and converge in the points $\pm\mathrm{i}\xi^{1/2}$; in so doing they do not meet the real negative semi-axes and remain inside the curves for the roots of the characteristic equation with $\gamma = 1$ (the ordinary standard linear solid model). In other words, at $\gamma \neq 1$ the behaviour of the roots of the characteristic equation for the fractional calculus model is governed by the magnitudes of the value of $\xi$, which may be considered as the deficiency of the elastic modulus.

The asymptotics for the characteristic equation (19) may be written as

$$p^{2+\gamma} + p^2 v_\sigma \tau_\sigma^{-\gamma} + \omega_\infty^2 p^\gamma = 0, \qquad \tau_\sigma \gg 1, \tag{22a}$$

$$p^3\gamma + p^2\tau_\sigma^{-1}(1 + v_\sigma) + \omega_\infty^2 p\gamma + \omega_0^2 \tau_\sigma^{-1} = 0, \qquad \tau_\sigma \ll 1, \tag{22b}$$

where $\omega_0$ is the frequency of elastic vibrations corresponding to the relaxed magnitude of the elastic modulus, from which it follows that the root behaviour is determined by the fractional calculus standard linear solid model and the ordinary standard linear solid model for large $\tau_\sigma$ and small $\tau_\sigma$, respectively.

Knowing the behaviour of roots of the characteristic equation and considering that the branch points are $s_* = \tau_\sigma^{-1}$ and $-\infty$, one can write the solution (16) in the form

$$x(t) = A_0(t) + A\exp(-\alpha t)\sin(\omega t - \varphi), \tag{23}$$

where

$$A = 2F[(h^2 + q^2)^{-1}(v_\sigma^2 + 2R^\gamma v_\sigma \cos\gamma\Phi + R^{2\gamma})]^{1/2}, \tag{24}$$

$$\operatorname{tg}\varphi = -[v_\sigma \sin\chi + R^\gamma \sin(\gamma\psi + \chi)]/[v_\sigma \cos\chi + R^\gamma \cos(\gamma\psi + \chi)], \qquad \operatorname{tg}\chi = h/q,$$
$$h = 2rR^\gamma \cos(\psi + \Phi) + r^2 R^{\gamma-1}\gamma\tau_\sigma \cos[2\psi + (\gamma-1)\Phi] + \tau_\sigma\omega_\infty^2\gamma R^{\gamma-1}\cos(\gamma-1)\Phi + 2rv_\sigma\cos\psi,$$
$$q = 2rR^\gamma \sin(\psi + \Phi) + r^2 R^{\gamma-1}\gamma\tau_\sigma \sin[2\psi + (\gamma-1)\Phi] + \tau_\sigma\omega_\infty^2\gamma R^{\gamma-1}\sin(\gamma-1)\Phi + 2rv_\sigma\sin\psi. \tag{25}$$

The function $A_0(t)$ decribing the drift of the equilibrium position of the single-mass system may be represented in the form (4): i.e.,

$$A_0(t) = \int_0^\infty \tau^{-1} B(\tau, \tau_\sigma)\, e^{-t/\tau}\, d\tau. \tag{26}$$

Here

$$B(\tau, \tau_\sigma) = \frac{\sin\pi\gamma}{\pi}\frac{F\omega_\infty^2(1 + \omega_\infty^2\tau^2)^{-1}\tau^3 H(\tau_\sigma - \tau)}{[D_\sigma(\tau)]^{-1} + D_\sigma(\tau) + 2\cos\pi\gamma}, \tag{27}$$
$$D_\sigma(\tau) = \tau^\gamma(1 + \omega_\infty^2\tau^2)^{-1}(\tau_\sigma - \tau)^{-\gamma} v_\sigma,$$

gives the distribution of the creep parameters (retardation parameters) of the dynamic system. The character of this function versus $\ln\tau$ is shown in Figure 3 for $v_\sigma = 49$ and

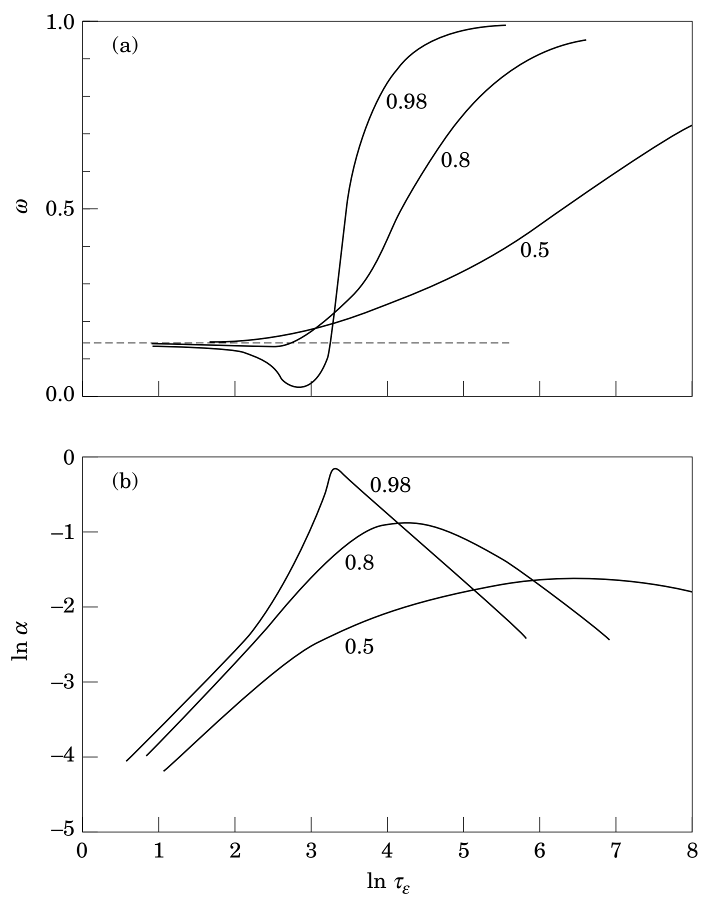


Figure 4. The values $\alpha$ and $\omega$ as function of $\ln\tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn after-effect kernel at $\xi = 1/50$.

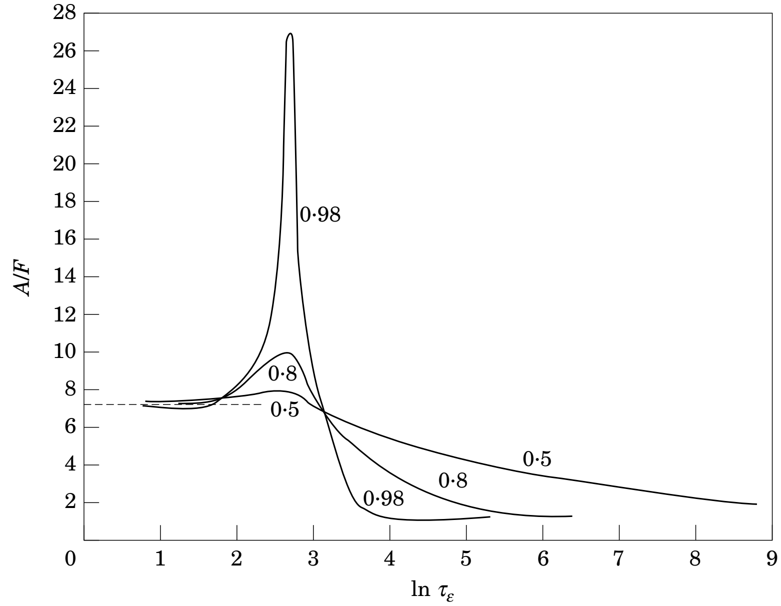


Figure 5. The amplitude $A$ versus $\ln \tau$ for a single-mass based on the fractional calculus model with the Rzhanitsyn after-effect kernel at $\xi = 1/50$.

$\tau_\sigma = 1$, from which it is seen that the fractional operator parameter $\gamma$ is some structural parameter that characterizes "fuzzifying" of the creep spectrum. The function $\beta$ at $m \neq 0$, similar to the function $B$ at $m = 0$, is not the δ-like sequence at $\gamma \to 1$.

In the quasi-static case, the distribution function for the relaxation and creep parameters of the dynamic system transforms into the distribution function of the creep times (5) for the rheological model with the Rzhanitsyn after-effect kernel.

Relationships (23)–(27) define the damped vibrations around the drifting equilibrium position with the natural frequency $\omega$ and the damping factor $\alpha$. An aperiodic regime is lacking for $\gamma \neq 1$.

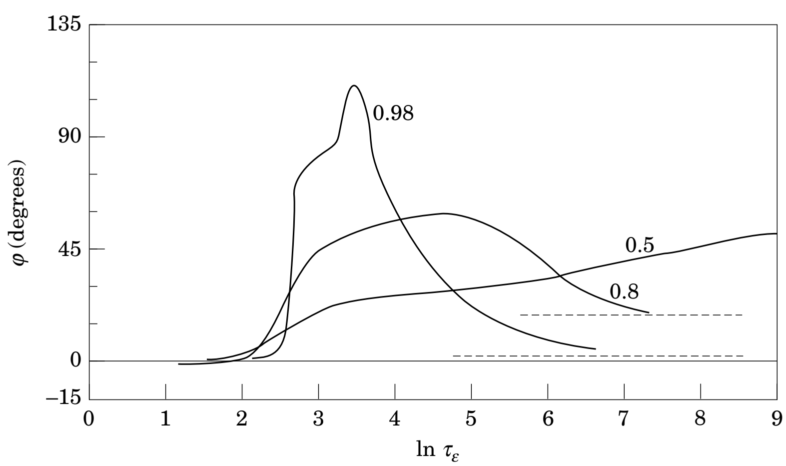


Figure 6. The phase shift $\varphi$ versus $\ln \tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn after-effect kernel at $\xi = 1/50$.

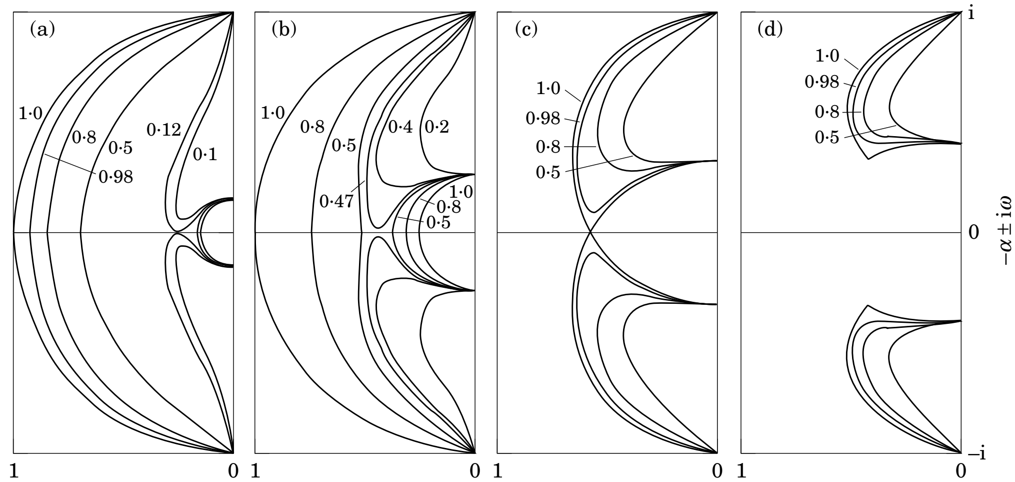


Figure 7. The behaviour of the complex conjugate roots $p_{1,2} = -\alpha \pm i\omega$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn relaxation kernel: (a) $\xi = 1/50$; (b) $\xi = 1/9$; (c) $\xi = 1/6$; (d) $\xi = 1/6$.

One can trace the temperature dependence of the damping factor $\alpha$ and frequency $\omega$. Since for the majority of relaxation processes $\tau_i = \tau_{i0} \exp(U/RT)$, where $i = \varepsilon$ or $\sigma$, $U$ is the activation energy, $R$ is the characteristic gas constant, and $T$ is the absolute temperature, then the $\ln \tau_\epsilon$ or $\ln \tau_\sigma$ dependence of the physical values is equivalent to the temperature dependence.

The characteristic behaviour of the values $\alpha$ and $\omega$ as functions of $\ln \tau_\sigma$ is presented in Figure 4 for $\xi = 1/50$. Figures near the curves point to the magnitudes of the parameter $\gamma$. Reference to Figure 4 shows that at every magnitude $\gamma \neq 1$ as $\tau$ varies from 0 to $\infty$, the damping factor passes a maximum and the vibration frequency increases monotonically only when $\gamma < 0{\cdot}85$. With $\gamma > 0{\cdot}85$ as $\tau$ increases, the vibration frequency first decreases from $\xi^{1/2}$ to some value and then increases up to 1 with further increase in $\tau$.

The $\ln \tau$ dependences of the functions $A$ and $\varphi$ are presented in Figures 5 and 6, respectively. From Figure 5 it follows that the function $A$ varies between $\xi^{-1/2}$ to 1, either decreasing monotonically ($\gamma < 0{\cdot}8$) or, after passing a maximum, decreasing monotonically, ($0{\cdot}8 \leqslant \gamma < 1$). From Figure 6 it is seen that $\mathrm{tg}\, \varphi \to -\sin(\gamma\pi/2)/[\nu_\sigma + \cos(\gamma\pi/2)]$ when $\tau_\sigma \to 0$, but the phase shift $\varphi \to \pi(1-\gamma)/2$ when $\tau_\sigma \to \infty$.

## 5. HEREDITARY ELASTIC OSCILLATOR MODEL WITH THE RZHANITSKYN RELAXATION KERNEL

Consider now the Rzhanitsyn kernel (2) at $i = \varepsilon$ as the relaxation kernel in the Boltzmann–Volterra relationships (1). Then formula (13) can be written in the form

$$\bar{x}(p) = F(1 + p\tau_\varepsilon)^\gamma / [(p^2 + \omega_\infty^2)(1 + p\tau_\varepsilon)^\gamma - \omega_\infty^2 \nu_\varepsilon]. \tag{28}$$

To define the poles of the function $\bar{x}(p)$, the roots of the characteristic equation

$$(p^2 + \omega_\infty^2)(1 + p\tau_\varepsilon)^\gamma - \omega_\infty^2 \nu_\varepsilon = 0 \tag{29}$$

should be found. It may be shown that equation (29) possesses real roots. Putting $p = -y$, $y > 0$ in equation (29), one finds the relation

$$y^2 = \omega_\infty^2 [v_\varepsilon - (1 - y\tau_\varepsilon)^\gamma ](1 - y\tau_\varepsilon)^{-\gamma}, \tag{30}$$

which is fulfilled under the condition $\tau_\varepsilon^{-1}(1 - v_\varepsilon^{1/\gamma}) \leqslant y < \tau_\varepsilon^{-1}$.

Setting $p = r\,\mathrm{e}^{\mathrm{i}\psi}$ in equation (29) and separating real and imaginary parts yields

$$r^2 \cos 2\psi - \omega_\infty^2 v_\varepsilon R^{-\gamma} \cos \gamma\Phi + \omega_\infty^2 = 0, \qquad r^2 \sin 2\psi - \omega_\infty^2 v_\varepsilon R^{-\gamma} \sin \gamma\Phi = 0, \quad \text{(31a, b)}$$

where $R^2 = 1 + 2\tau_\varepsilon r \cos\psi + \tau_\varepsilon^2 r^2$, and $\operatorname{tg}\Phi = \tau_\varepsilon r \sin\psi(1 + \tau_\varepsilon r \cos\psi)^{-1}$. From equation (31b), it is seen that the system is rootless at any $|\psi| < \pi/2$. To calculate the roots of the

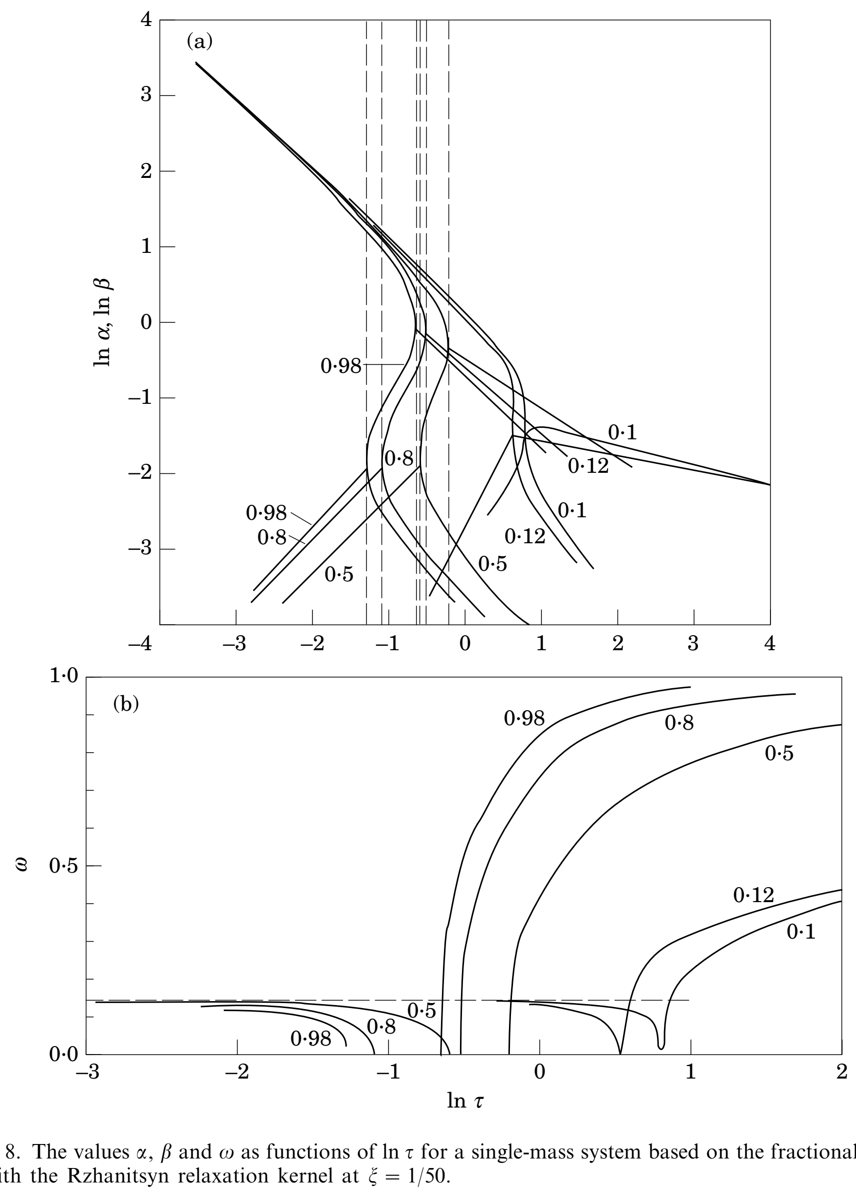


Figure 8. The values $\alpha$, $\beta$ and $\omega$ as functions of $\ln\tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn relaxation kernel at $\xi = 1/50$.

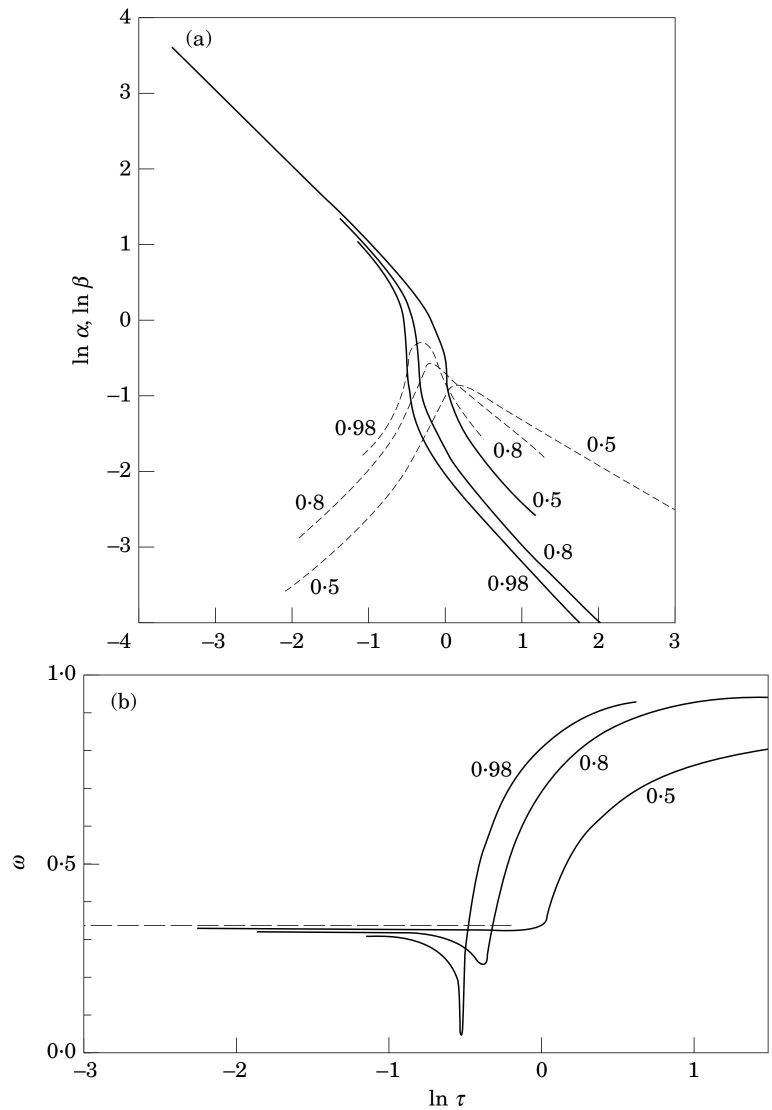


Figure 9. The values $\alpha$, $\beta$ and $\omega$ as functions of $\ln \tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn relaxation kernel at $\xi = 1/9$. ——, $\beta$; - - - - -, $\alpha$.

system (31) at $\pi/2 < |\psi| < \pi$, one multiplies equations (31a) and (31b) by $\sin 2\psi$ and $\cos 2\psi$, respectively, and substracts the second equation from the first one. Introducing the new variable $x = \tau_\varepsilon r$ and putting $\omega_\infty = 1$, as a result one obtains

$$\sin 2\psi - \nu_\varepsilon (1 + 2x \cos \psi + x^2)^{-\gamma/2} \sin \left[ 2\psi + \gamma \operatorname{arctg} \left( \frac{x \sin \psi}{1 + x \cos \psi} \right) \right] = 0. \tag{32}$$

The value $x$ is determined from equation (32) at every fixed angle $\pi/2 < |\psi| < \pi$ and given $\nu_\varepsilon$ and $\gamma$. Then, substituting the found value $x$ into equation (31b), first one finds that

$$r = (-R^\gamma \sin 2\psi / \nu_\varepsilon \sin \gamma \Phi)^{-1/2},$$

and thereafter one can calculate the value $\tau_\varepsilon = x r^{-1}$.

The behaviour of the roots in the complex plane as a function of the parameter $\tau_\varepsilon$ for four magnitudes $v_\varepsilon = 0{\cdot}98$, 14/15, 8/9 and 5/6 ($\xi = 1/50$, 1/15, 1/9 and 1/6) at $\omega_\infty = 1$ is presented in Figures 7(a), (b), (c) and (d), respectively, wherein the magnitudes of $\gamma$ are indicated numerically. Reference to Figure 7 shows that as $\tau_\varepsilon$ changes from 0 to $\infty$, the curves for the two complex conjugate roots $p_{1,2} = -\alpha \pm \mathrm{i}\omega$ at any $0 < \gamma < 1$ issue out of the points $\pm \mathrm{i}\xi^{1/2}$ and converge in the points $\pm\mathrm{i}$. At $v_\varepsilon = 0{\cdot}98$ and $0{\cdot}12 \leqslant \gamma < 1$ (Figure 7(a)) or $v_\varepsilon = 14/15$ and $0{\cdot}47 \leqslant \gamma < 1$ (Figure 7(b)) a domain of aperiodicity is observed, which narrows with decrease in $\gamma$ from 1 to 0·12 or from 1 to 0·47 and degenerates into a point at $\gamma = 0{\cdot}12$ or $\gamma = 0{\cdot}47$. This domain disappears completely at $0 \leqslant \gamma < 0{\cdot}12$ or $0 < \gamma < 0{\cdot}47$, respectively. At $v_\varepsilon = 8/9$ (Figure 7(c)) and 5/6 (Figure 7(d)) the domain of aperiodicity is entirely absent, although a domain of aperiodicity at $v_\varepsilon = 8/9$ and $\gamma = 1$ exists in the form of a point.

To calculate the real roots of equation (29), one can introduce the new variable $x = y\tau_\varepsilon$ in (30). Then for every fixed $(1 - v_\varepsilon^{1/\gamma}) < x < 1$, from equation (30) one obtains $y$, and thereafter knowing $x$ and $y$ one finds $\tau_\varepsilon = x/y$.

In other words, the structural parameter $\gamma$ along with the parameter $\xi$ influences not only the dimensions of the periodicity domain, but the existence of this domain as well. This circumstance profitably distinguishes the model under consideration from the previous model, since it allows one to use this model for describing dissipative processes of high intensity.

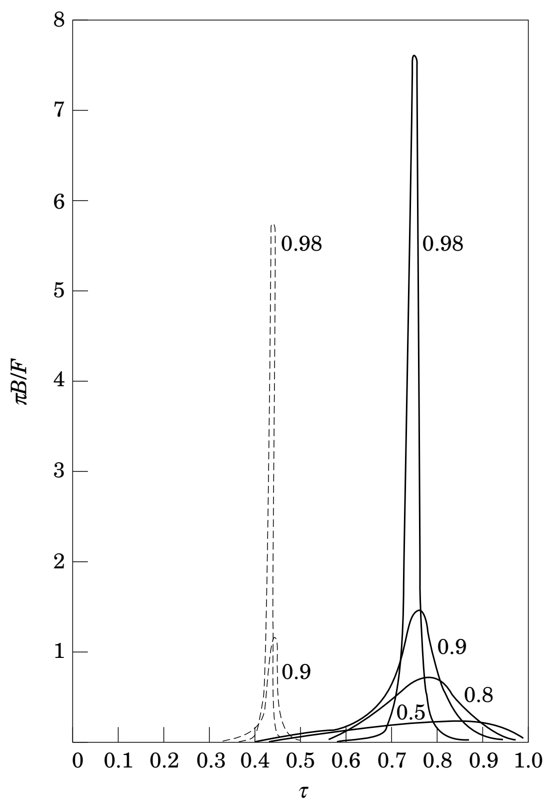


Figure 10. The distribution function $B$ versus $\ln\tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn relaxation kernel at $\xi = 1/50$. ——, $\tau_\varepsilon = 1$; - - - - -, $\tau_\varepsilon = 0.5$.

TABLE 1

*The influence of the fractional operator parameter $\gamma$ and the parameter $v_\varepsilon$ on the behaviour of the single-mass system with the Rzhanitsyn relaxation kernel* (12) *when $i = \varepsilon$*

| $v_\varepsilon$ | $\gamma$ | $\tau_\varepsilon$ | The type of solution for $x(t)$ |
|---|---|---|---|
| 0·98 | $0 < \gamma < 0{\cdot}12$ | $0 \leqslant \tau_\varepsilon < \infty$ | (33) |
| | | $0 < \tau_\varepsilon < 1{\cdot}955$ | (33) |
| | 0·12 | $1{\cdot}955 < \tau_\varepsilon < \infty$ | |
| | | $\tau_\varepsilon = 1{\cdot}955$ | (36) |
| | | $0 < \tau_\varepsilon < 0{\cdot}543$<br>$0{\cdot}819 < \tau_\varepsilon < \infty$ | (33) |
| | 0·5 | $\tau_\varepsilon = 0{\cdot}543$<br>$\tau_\varepsilon = 0{\cdot}819$ | (60) |
| | | $0{\cdot}543 < \tau_\varepsilon < 0{\cdot}891$ | (34) |
| | | $0 \leqslant \tau_\varepsilon < 0{\cdot}355$<br>$0{\cdot}599 < \tau_\varepsilon < \infty$ | (33) |
| | 0·8 | $\tau_\varepsilon = 0{\cdot}355$<br>$\tau_\varepsilon = 0{\cdot}599$ | (35) |
| | | $0{\cdot}355 < \tau_\varepsilon < 0.599$ | (34) |
| | | $0 \leqslant \tau_\varepsilon < 0{\cdot}283$<br>$0{\cdot}52 < \tau_\varepsilon < \infty$ | (33) |
| | 0·98 | $\tau_\varepsilon = 0{\cdot}283$<br>$\tau_\varepsilon = 0{\cdot}52$ | (35) |
| | | $0{\cdot}283 < \tau_\varepsilon < 0{\cdot}52$ | (34) |
| 8/9 or 5/6 | $0 < \gamma < 1$ | $0 \leqslant \tau_\varepsilon < \infty$ | (33) |
| 14/15 | $0 < \gamma < 0{\cdot}47$ | $0 \leqslant \tau_\varepsilon < \infty$ | (33) |
| | 0·47 | $0 \leqslant \tau_\varepsilon < 0{\cdot}962$<br>$0{\cdot}962 < \tau_\varepsilon < \infty$ | (33) |
| | | $\tau_\varepsilon = 0{\cdot}962$ | (36) |
| | | $0 \leqslant \tau_\varepsilon < 0{\cdot}881$<br>$0{\cdot}885 < \tau_\varepsilon < \infty$ | (33) |
| | 0·5 | $\tau_\varepsilon = 0{\cdot}881$<br>$\tau_\varepsilon = 0{\cdot}885$ | (35) |
| | | $0{\cdot}881 < \tau_\varepsilon < 0{\cdot}885$ | (34) |
| | | $0 \leqslant \tau_\varepsilon < 0{\cdot}587$<br>$0{\cdot}663 < \tau_\varepsilon < \infty$ | (33) |
| | 0·8 | $\tau_\varepsilon = 0{\cdot}587$<br>$\tau_\varepsilon = 0{\cdot}633$ | (35) |
| | | $0{\cdot}587 < \tau_\varepsilon < 0{\cdot}633$ | (34) |

The $\ln \tau_\varepsilon$ dependence of the values $\ln \alpha$ and $\ln \beta$ ($\beta$ are the real roots of the characteristic equations) are given in Figures 8 and 9 for $v_\varepsilon = 0{\cdot}98$ and $v_\varepsilon = 8/9$, respectively. It is evident that at $v_\varepsilon = 0{\cdot}98$ and $0{\cdot}12 \leqslant \gamma < 1$ (Figure 8(a)) the curves $\ln \alpha$, increasing monotonically, approach the boundaries of the aperiodicity domain from the left and from the right, and at the boundary points those curves continuously transform into the curve of real roots, such that the curves $\ln \alpha$ together with the segments of the curves $\ln \beta$ between two boundary points generate continuous lines having a maximum on the right boundary of the domain of aperiodicity (this maximum decreases with decrease in the parameter $\gamma$). The real root decreases from $+\infty$ to $-\infty$ when $\ln \tau_\varepsilon$ changes from $-\infty$ to $+\infty$, and in so doing its monotonic decrease to the left and to the right of the aperiodicity domain boundaries alternate in a zigzag diminution, resulting in three values $\ln \beta$ at every magnitude $\ln \tau_\varepsilon$ inside the aperiodicity domain. When $0 < \gamma < 0{\cdot}12$, the curves $\ln \alpha$ pass a maximum, but the curves $\ln \beta$ decrease monotonically from $+\infty$ to $-\infty$ as $\ln \tau_\varepsilon$ changes from $-\infty$ to $+\infty$, which results in one real root. The values $\ln \alpha$ and $\ln \beta$ as functions of $\ln \tau_\varepsilon$ behave similarly at $v_\varepsilon = 8/9$ (Figure 9(a)) for every magnitude of the parameter $0 < \gamma \leqslant 1$.

The character of the value of $\omega$ as a function of $\ln \tau_\varepsilon$ is presented in Figures 8(b) and 9(b) for various magnitudes of the parameter $\gamma$. It is seen that as $\ln \tau_\varepsilon$ increases from $-\infty$ to $\infty$, almost all curves first decrease from $\xi^{1/2}$ to 0 or to some finite value, but then increase to 1 during further increase in $\ln \tau_\varepsilon$. Only for a small number of curves (they correspond to small $\gamma$) is a monotonic increase in $\omega$ from $\xi^{1/2}$ to 1 observed.

Knowing the behaviour of the characteristic equation roots and considering that the branch points are $s_* = -\tau_\varepsilon^{-1}$ and $-\infty$, one can write the solution (23) as follows: for the domains of vibration motions (one real and two complex conjugate roots)

$$x(t) = A_0(t) + A_1 \exp(-\alpha_1 t) + A_2 \exp(-\alpha_2 t) \sin(\omega t - \varphi), \tag{33a}$$

or (real root disappears)

$$x(t) = A_0(t) + A_2 \exp(-\alpha_2 t) \sin(\omega t - \varphi); \tag{33b}$$

for the domain of aperiodic motions (three real different roots)

$$x(t) = A_0(t) + \sum_{i=1}^{3} H_i \exp(-\beta_i t); \tag{34}$$

for the boundaries of the domain of aperiodic motions (one real root, for example, $\beta_1$ is the simple root, and the other real root $\beta = \beta_2 = \beta_3$ is one repeated root)

$$x(t) = A_0(t) + H_1 \exp(-\beta_1 t) + B_1 \exp(-\beta t) + B_2 t \exp(-\beta t), \tag{35}$$

or (in the case that the domain of the aperiodic motions degenerates into a point, the real root $\beta^* = \beta_1 = \beta_2 = \beta_3$ becomes a three-fold root)

$$x(t) = A_0(t) + B_1^* \exp(-\beta^* t) + B_2^* t \exp(-\beta^* t) + B_3^* t^2 \exp(-\beta^* t), \tag{36}$$

where $\alpha_1$, $\beta_i$ $(i = 1, 2, 3)$, $\beta$ and $\beta^*$ are the real roots of equation (29) which are located between $\tau_\varepsilon^{-1}(1 - v_\varepsilon^{1/\gamma})$ and the branch point $\tau_\varepsilon^{-1}$, and $\alpha_2 \pm i\omega$ are the complex conjugate roots of equation (29).

The amplitudes $A_i$, $H_i$, $B_i^*$ $(i = 1, 2, 3)$, $B_1$ and $B_2$, as well as $\operatorname{tg} \varphi$, are expressed in terms of the damping coefficients $\alpha_1$, $\alpha_2$, $\beta_i$, $\beta$, $\beta^*$ and the natural frequency $\omega$ as follows:

$$A_1 = \frac{F(1 - \alpha_1 \tau_\varepsilon)}{\alpha_1^2 \tau_\varepsilon (2 + \gamma) - 2\alpha_1 + \gamma \tau_\varepsilon \omega_\infty^2}, \qquad H_i = \frac{F(1 - \beta_i \tau_\varepsilon)}{\beta_i^2 \tau_\varepsilon (2 + \gamma) - 2\beta_i + \gamma \tau_\varepsilon \omega_\infty^2},$$

$$A_2 = 2FR^\gamma (h^2 + q^2)^{-1/2}, \qquad \operatorname{tg} \varphi = -\operatorname{tg}(\gamma\psi + \chi), \qquad \operatorname{tg} \chi = h/q,$$

$$h = 2rR^\gamma \cos(\psi + \gamma\Phi) + \gamma r^2 R^{\gamma-1} \tau_\varepsilon \cos[2\psi + (\gamma - 1)\Phi] + \gamma \omega_\infty^2 R^{\gamma-1} \tau_\varepsilon \cos(\gamma - 1)\Phi,$$

$$q = 2rR^\gamma \sin(\psi + \gamma\Phi) + \gamma r^2 R^{\gamma-1} \tau_\varepsilon \sin[2\psi + (\gamma - 1)\Phi] + \gamma \omega_\infty^2 R^{\gamma-1} \tau_\varepsilon \sin(\gamma - 1)\Phi,$$

$$B_1 = 2F\gamma(1 - \beta\tau_\varepsilon)\tau_\varepsilon l^{-1}, \qquad B_2 = 2F(1 - \beta\tau_\varepsilon)^2 l^{-1},$$

$$l = \beta^2 \tau_\varepsilon^2 (2 + \gamma)(1 + \gamma) - 4\beta\tau_\varepsilon(1 + \gamma) + \gamma(\gamma - 1)\tau_\varepsilon^2 \omega_\infty^2,$$

$$B_1^* = 3F\gamma(\gamma-1)(1-\beta\tau_\varepsilon)\tau_\varepsilon^2 l_1^{-1}, \qquad B_2^* = 6F\gamma(\gamma-\beta\tau_\varepsilon)^2\tau_\varepsilon l_1^{-1}, \qquad B_3^* = 3F(1-\beta\tau_\varepsilon)^3 l_1^{-1},$$

$$l_1 = \beta^2\tau_\varepsilon^3(2+\gamma)(1+\gamma)\gamma - 6\beta\tau_\varepsilon^2(1+\gamma)\gamma + 4\tau_\varepsilon(1+\gamma) + \gamma(\gamma-1)(\gamma-2)\tau_\varepsilon^2\omega_\infty^2.$$

The influences of the fractional operator parameter $\gamma$ and the parameter $v_\varepsilon$ upon the behaviour of the single-mass system are illustrated in Table 1.

The value $A_0(t)$ is determined by formula (26), wherein the distribution function $B(\tau, \tau_\varepsilon)$ of the relaxation and creep parameters of the dynamical system has the form

$$B(\tau, \tau_\varepsilon) = \frac{\sin\pi\gamma}{\pi} \frac{F(1+\omega_\infty^2\tau^2)^{-1}\tau^3\mathrm{H}(\tau_\varepsilon-\tau)}{[D_\varepsilon(\tau)]^{-1} + D_\varepsilon(\tau)\tau^4 + 2\tau^2\cos\pi\gamma}, \tag{37}$$

$$D_\varepsilon(\tau) = \tau^\gamma(1+\omega_\infty^2\tau^2)^{-1}(\tau_\varepsilon-\tau)^{-\gamma}\omega_\infty^2 v_\varepsilon.$$

In the quasi-static case the distribution function (37) transforms into the distribution function (6).

The $\tau$-dependence of the distribution function $B$ is presented in Figure 10 at $v_\varepsilon = 0{\cdot}98$ and $\tau_\varepsilon = 1$. Note that the function $B$ at $m \neq 0$, similar to the analogous function $B$ at $m = 0$ (see the Appendix), is the δ-like sequence at $\gamma \to 1$.

To evaluate the contribution of each term in the expressions (33) and (34), the $\ln\tau_\varepsilon$ dependences of the amplitudes $A_1$ and $A_2$ at $v_\varepsilon = 8/9$ and the amplitudes $H_1$, $H_2$, $H_3$, $A_1$

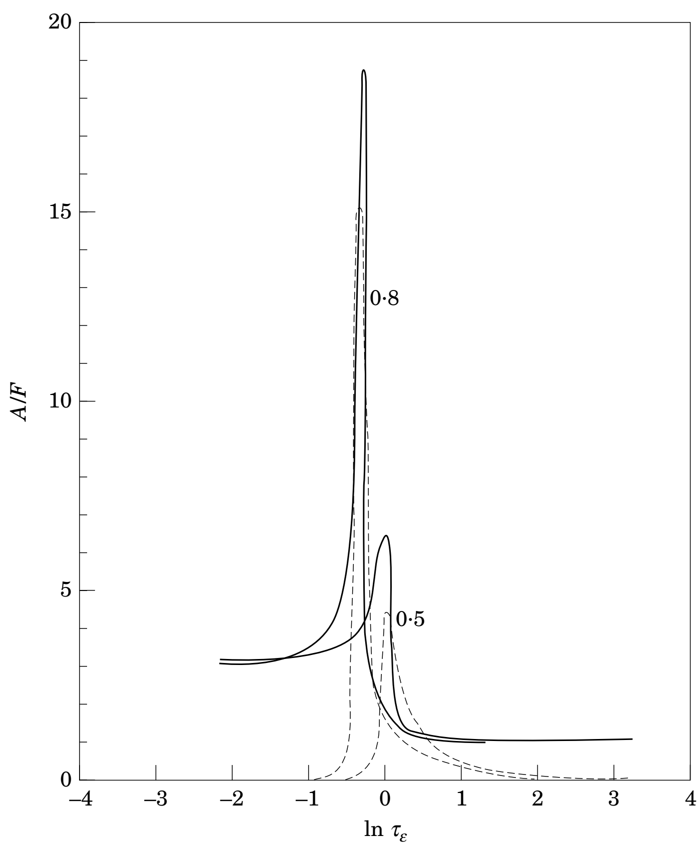


Figure 11. The amplitudes $A_1$ and $A_2$ versus ln $\tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn relaxation kernel at $\xi = 1/9$. - - -, $A_1$; ——, $A_2$.

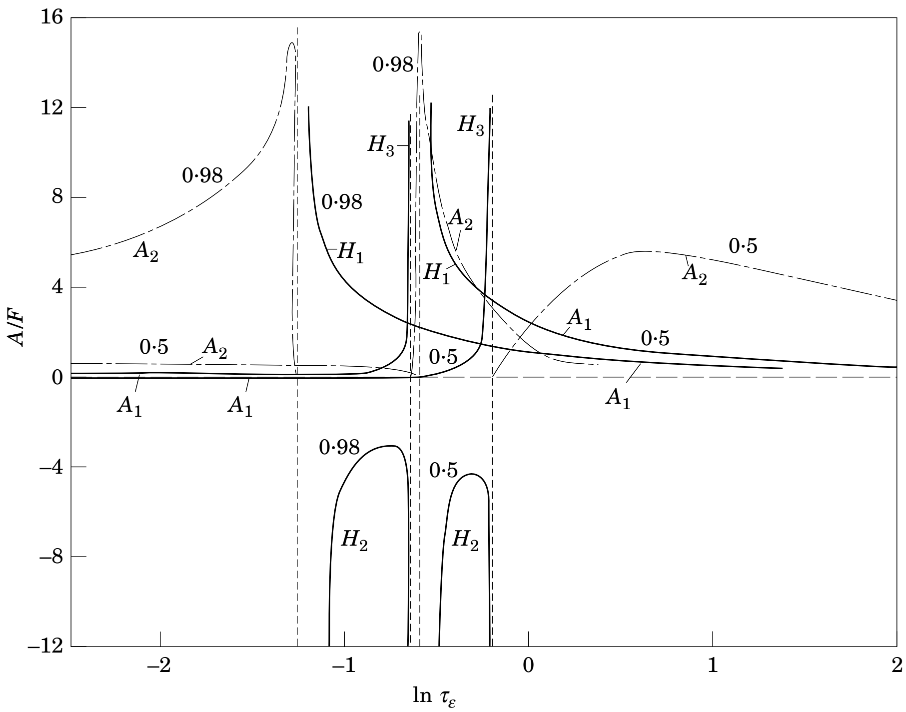


Figure 12. The amplitudes $A_1$, $A_2$, $H_i$ versus ln $\tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn relaxation kernel at $\xi = 1/50$.

and $A_2$ at $v_\varepsilon = 0{\cdot}98$ are given in Figures 11 and 12. The value $\gamma$ is the fractional operator parameter, the values of which are indicated by figures near curves. From Figure 11 it is seen that $A_1$ and $A_2$ pass a maximum which shifts to greater $\tau_\varepsilon$ with a decrease in the parameter $\gamma$. Reference to Figure 12 shows that as $\ln \tau_\varepsilon$ varies from $-\infty$ to the magnitude which determines the left boundary of the periodicity domain, and from the magnitude which governs its right boundary to $+\infty$, the amplitude $A_2$ passes a maximum tending to zero at $\ln \tau_\varepsilon \to \pm\infty$ and vanishing on the boundaries of the aperiodicity domain. As this takes place, the amplitude $A_1$ approaches zero at $\ln \tau_\varepsilon \to \pm\infty$, but on the left and right boundaries of the aperiodicity region it goes uninterruptedly into the amplitudes $H_3$ and $H_1$, respectively. Within the aperiodicity domain, the amplitudes $H_3$ and $H_1$ asymptotically approach the right and left boundaries, respectively. The amplitude $H_2$ residing within the aperiodicity domain asymptotically approaches its right and left boundaries.

The ln $\tau_\varepsilon$ dependence of the phase shift $\varphi$ is shown in Figure 13 for $v_\varepsilon = 0{\cdot}98$. When $\tau_\varepsilon \to 0$, the value $\varphi \to (\pi - \gamma\pi/2)$, but when $\tau_\varepsilon \to \infty$, the function $\varphi \to 0$. From Figure 13 it is seen that when passing through the aperiodicity domain the phase shift varies over $\pi/2$ without regard to the value of the fractional operator parameter $\gamma$.

Thus, the weakly singular Rzhanitsyn function chosen as the relaxation kernel makes it possible to account for the influence of the fractional operator parameter, which governs "fuzzifying" of the relaxation spectrum, on the dynamic characteristics of the single-mass system not only within the domain of vibration, but within the domain of aperiodic motions as well.

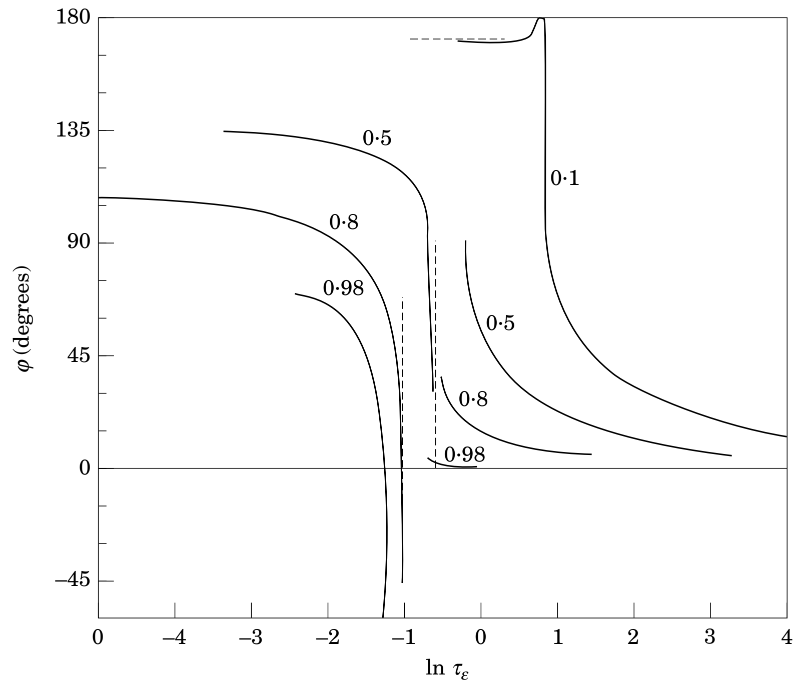


Figure 13. The phase shift $\varphi$ versus ln $\tau$ for a single-mass system based on the fractional calculus model with the Rzhanitsyn relaxation kernel.

## 6. CONCLUSIONS

Comparison studies of damped vibrations of the hereditarily elastic oscillators, the hereditary properties of which are described by the fractional calculus models with the weakly singular Rzhanitsyn kernel, allow one to make the following conclusions.

1. The oscillators with fractional operators constitute a continuum of ordinary viscoelastic elements distributed uninterruptedly over the relaxation or creep times in terms of some functions (distribution functions), as distinct from the oscillators with ordinary operators at $\gamma = 1$, which involve either one viscoelastic element or a finite number of ordinary viscoelastic elements distributed discretely over the relaxation or creep times. This enables the fractional calculus models to be used for analyzing and calculating damping systems which are founded on the harnessing of damping backing, coating and isolators of various dimensions and shapes.

2. The parameter $\gamma$ plays the role of some structural parameter, the modification of which implies "fuzzifying" of the relaxation or creep times spectrum and equalizing of the spectral density along the entire time axis.

3. It is known that, under a sufficiently large intensity of dissipative processes, real vibrating systems may experience an aperiodic regime. The peculiarity of the vibrational process of viscoelastic single-mass systems, which are modelled by fractional calculus models with the weakly singular Rzhanitsyn kernel as the creep kernel, resides in the impossibility of the transition from the vibrating motions to the aperiodic regime.

4. The more complicated rheological model (the model with the weakly singular Rzhanitsyn kernel as the relaxation kernel) is suggested for describing the viscoelastic properties of a single-mass system, which allows one to trace the influence of the fractional operator parameter $\gamma$ on the dynamic characterisitcs of the system, not only in the region of vibration, but in the domain of the aperiodic motions as well. Moreover, it has been

shown that the occurrence or vanishing of the region of the aperiodic motions for the model put forward is governed not only by the magnitudes of $v_\varepsilon$, but by the magnitudes of $\gamma$ as well.

5. An efficient method for solving the characteristic equations for the generalized fractional calculus viscoelastic models, i.e., algebraic equations with fractional powers, is suggested for use at any magnitudes of the fractional operator parameter $\gamma$.

6. In a manner similar to that for quasi-static cases, when investigating the vibrations of single-mass systems based on the generalized fractional calculus viscoelastic rheological models, in every specific case it has been possible to construct the distribution function for the relaxation or creep parameters of the dynamic system, which involves the system mass $m$ in addition to the rheological parameters. This function defines the drift of the equilibrium position for the vibrating system. When $m \to 0$ (quasi-static case), the dynamic distribution function transforms to the distribution function of the corresponding rheological model.

## ACKNOWLEDGMENTS

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## APPENDIX

When $\gamma \to 1$, the functions $\sin \pi\gamma$ and $\cos \pi\gamma$ may be approximated by the expression

$$\sin \pi\gamma \sim \pi(1-\gamma), \qquad \cos \pi\gamma \sim -1 + 1/2\pi^2(1-\gamma)^2, \tag{A1}$$

respectively. With due account taken of expressions (A1), the distribution function (6) may be approximated by the expression

$$B_\varepsilon(\tau, \tau_\varepsilon) \sim \frac{(T\nu_\sigma)^{1/2}\mathrm{H}(\tau_\sigma - \tau)}{\pi\tau} \frac{(T\nu_\sigma)^{1/2}\pi(1-\gamma)}{(T-\nu_\sigma)^2 + T\nu_\sigma\pi^2(1-\gamma)^2}, \tag{A2}$$

where $T = \tau_\sigma/\tau - 1$. Noting that

$$\frac{(T\nu_\sigma)^{1/2}\pi(1-\gamma)}{(T-\nu_\sigma)^2 + T\nu_\sigma\pi^2(1-\gamma)^2} = -\mathrm{Im}\,\frac{1}{T-\nu_\sigma + \mathrm{i}(T\nu_\sigma)^{1/2}\pi(1-\gamma)},$$

and considering the Sokhotsky formula [18]

$$\lim_{\gamma\to 1} \frac{1}{T-\nu_\sigma + \mathrm{i}(T\nu_\sigma)^{1/2}\pi(1-\gamma)} = \frac{1}{T-\nu_\sigma} - \mathrm{i}\pi\delta(T-\nu_\sigma) \tag{A3}$$

yields

$$\lim_{\gamma\to 1} B_\varepsilon(\tau, \tau_\varepsilon) = \mathrm{H}(\tau_\sigma - \tau)\,\frac{(T\nu_\sigma)^{1/2}}{\tau\nu_\varepsilon}\,\delta(T-\nu_\sigma) = \mathrm{H}(\tau_\sigma - \tau)\,\frac{(T\nu_\sigma)^{1/2}}{\tau_\sigma\nu_\varepsilon}\,\delta(\tau_\varepsilon - \tau): \tag{A4}$$

i.e., in the limiting case the distribution is described by the δ-function with the evident fulfillment of the normalizing condition, since at $T = \nu_\sigma$ or $\tau = \tau_\varepsilon$ one obtains that $\mathrm{H}(\tau_\sigma - \tau_\varepsilon) = 1$, and $\nu_\sigma\nu_\varepsilon^{-1}\tau_\varepsilon\tau_\sigma^{-1} = 1$.