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ISODYNAMICAL TRACKS AND POTENTIALS

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The correspondence is found between a track in a vertical plane along which a bead is constrained to slide freely under the influence of gravity, and a one-dimensional potential, such that the motion due to the potential is exactly the same as (i.e., is isodynamical to) the motion of the bead, projected onto the horizontal axis. For any given track shape, the shape of the isodynamical one-dimensional potential function is explicitly and uniquely specified, and in general depends on the amplitude of oscillation. Various examples for quadratic and quartic functions are solved and displayed. In particular, the potential isodynamical to sufficiently large oscillations on a double valley shaped track has a triple well form. Other isodynamical situations dealt with include the case in which the non-linear potential and track functions are the same, and the case of motion along the arc length of the track path itself rather than its projection. The special cases of V-shaped and W-shaped tracks/potentials are solved in an Appendix.

1. INTRODUCTION

In discussing the motion of one-dimensional non-linear oscillators due to a potential, recourse is often made to an analogy with a ball moving under the influence of gravity along a frictionless track of the same shape as the potential function. The constrained two-dimensional ball motion is projected onto the horizontal axis, and some features of the original phase plane are deduced (see, e.g., reference [1, section 9.5]).

Some care needs to be exercised in making this analogy. For the original one-dimensional motion, the acceleration is proportional to the force as gradient of the potential, and hence to $\tan\psi$, the slope of the potential function. For the constrained two-dimensional motion along a track, the horizontal acceleration is instead proportional to the normal reaction force times $\sin\psi$. Actually, resolving forces along the X and Y directions (with gravity in the negative Y direction) and eliminating the normal reaction yields, in general, $\ddot{X}$ proportional to $\tan\psi[1 + \ddot{Y}/g]$. Thus the analogy really only holds precisely if the vertical acceleration $\ddot{Y}$ is constant, which is rather restrictive. Local equilibrium points and their stability nature, as well as motion turning points, may be deduced (see, e.g., reference [2, p. 18]), but the detailed shapes of the phase plane orbits may in general be only indicated qualitatively. (Since the ball should not jump off the track, and the angular kinetic energy due to rolling is ignored, the analogous system would really be better described by a bead on a frictionless wire.)

Recently, Gottwald *et al*. [3] analyzed the motion of a "cart" on a curved hill-and-two-symmetric-valleys "track" under gravity, and showed that the oscillatory motion *approximately* mimics the one-dimensional motion of a particle moving under the influence of a similarly shaped double-well potential, the force equation of which is the undamped Duffing's equation. Thus a mechanical system was used as an analogue of a non-linear differential equation. They carried out extensive experimental

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measurements on both free and forced oscillations of a rather elaborate cart-and-track system.

In this paper, the following question is dealt with: What is the relationship between the shape of a vertically planar curved track upon which a bead moves freely and smoothly under the influence of gravity, and the form of a one-dimensional potential producing motion along a straight line, such that the one-dimensional potential motion *exactly* mimics, i.e., reproduces, the motion resulting from the given track (projected onto a horizontal line; or along the track itself)? Such systems will be termed *isodynamical*. The converse problem, of finding the track shape which leads to motion isodynamical to a given potential, involves the numerical solution of non-linear differential equations, and is dealt with in a separate paper.

The interest here is in the actual potential, as motivated by the early part of reference [3]. The effects of damping and forcing, which formed the bulk of reference [3], are beyond the scope of this paper. Subsequent to reference [3], Shaw and Haddow [4] tackled the converse problem mentioned above, with emphasis on the arc length co-ordinate, whereas the present paper deals primarily with the "direct" problem of specifying the potential in terms of the given track shape, with emphasis on motion in terms of the horizontal co-ordinate. The type of problem considered, as well as the two approaches, are therefore quite different.

The basic equations for the two systems are presented in section 2. In section 3, an explicit formula is derived for the potential, which is isodynamical to a given track path shape for the latter's X-projected motion. It depends explicitly on the amplitude of the oscillations. For a parabolic track, the corresponding potential possesses points of inflexion. For large oscillations on a symmetric hill-and-two-valleys track ("double-valley" shape), the corresponding potential has, except for a small range of amplitudes, a triple-well form. (It should perhaps be stressed that the numerical simulations of Gottwald *et al*. [3] were for their full non-linear equation, not their Duffing-like approximation.)

In section 4 the possibility is explored that the isodynamical (non-linear) track and potential have the *same* shape. This is solvable, but results in a repulsive potential/decreasing track function which produces unidirectional motion and is therefore not oscillatory.

In section 5, the motion *along* the track path arc length (rather than its X-projection) is required to be isodynamical to the potential motion. Two examples are solved: the potential corresponding to a parabolic track; and also, as just one example of the converse problem, the track corresponding to a quadratic (harmonic oscillator) potential.

The special case of piecewise *linear* track and potential shapes which are isodynamical is dealt with in an Appendix. The motion for a W-shape is found exactly, and may serve as an analytically solvable model of a simplified double well.

2. THE EQUATIONS OF MOTION

2.1. ONE-DIMENSIONAL POTENTIAL MOTION

The equation of motion for a particle of mass m moving along the x-axis due to a potential $v(x)$ is

$$\ddot{x} = -(1/m)\,\mathrm{d}v/\mathrm{d}x, \tag{2.1}$$

where an overdot denotes differentiation with respect to time. The first (energy) integral may in general be written in the form

$$\dot{x}^2 + (2/m)v(x) = \text{constant} \tag{2.2a}$$

or, more specifically,

$$\dot{x}^2 = (2/m)[v(a) - v(x)], \tag{2.2b}$$

where $\dot{x} = 0$ at $x = a > 0$, it being assumed that such a local maximum amplitude of x (end-point) exists.

2.2. PLANAR MOTION UNDER GRAVITY

Consider a particle of mass M sliding freely along, but constrained to, a curved path in a vertical plane (e.g., a bead on a wire) the single valued shape equation of which is of the form $Y = Y(X)$, under the influence of the constant gravitational force, g, acting vertically downwards. Upper case co-ordinates are used throughout for this two-dimensional motion. The potential energy is just MgY. (This gravitational potential energy should in no way be confused with the one-dimensional potential $v(x)$ of section 2.1, which is the subject of investigation in the remainder of this paper.)

2.2.1. *Projected motion onto the X-axis*

By resolving forces and eliminating the normal reaction, the equation of motion for the X co-ordinate (i.e., the projection onto the X-axis) is found to be

$$\ddot{X} = -(g + Y''\dot{X}^2)Y'/[1 + (Y')^2], \tag{2.3}$$

where a prime denotes differentiation with respect to X of the *given* path equation $Y(X)$. (A recent incorrect derivation ([5], p. 700) omits consideration of the normal reaction and hence misses the important second term in the numerator.) A first integral (or, alternatively, use of energy considerations) yields, in general,

$$[1 + (Y'(X))^2](\dot{X})^2 + 2gY(X) = \text{constant} \tag{2.4a}$$

or, more specifically,

$$(\dot{X})^2 = 2g[Y(A) - Y(X)]/[1 + (Y'(X))^2], \tag{2.4b}$$

where $\dot{X} = 0$ at $X = A > 0$, it being again assumed here that such a local maximum amplitude exists.

2.2.2. *Motion along the curved path*

By resolving forces, the equation of motion along the curved path with arc length S is given by (cf., reference [4])

$$\ddot{S} = -g\,\mathrm{d}Y/\mathrm{d}S, \tag{2.5}$$

where

$$(\mathrm{d}S)^2 = (\mathrm{d}X)^2 + (\mathrm{d}Y)^2; \tag{2.6}$$

so there is a variety of useful expressions:

$$\mathrm{d}S = [1 + (Y'(X))^2]^{1/2}\,\mathrm{d}X \tag{2.7a}$$

$$= [1 + (\mathrm{d}X/\mathrm{d}Y)^2]^{1/2}\,\mathrm{d}Y \tag{2.7b}$$

$$= \mathrm{d}X/[1 - (\mathrm{d}Y/\mathrm{d}S)^2]^{1/2} \tag{2.7c}$$

$$= \mathrm{d}Y/[1 - (\mathrm{d}X/\mathrm{d}S)^2]^{1/2}. \tag{2.7d}$$

A first integral (or consideration of energy) now yields

$$\dot{S}^2 = 2g[Y(A) - Y(X)], \tag{2.8}$$

where $\dot{S} = 0$ when $X = A$.

3. POTENTIAL CORRESPONDING TO A GIVEN PATH

If the track path function $Y(X)$ is given, and it is desired to find the corresponding potential $v(x)$ the motion $x(t)$ of which exactly mimics, i.e., is isodynamical to, the projected motion $X(t)$ of the particle under the influence of gravity, then equations (2.2b) and (2.4b) must have exactly the same form. Thus

$$v(x) - v(A) = mg[Y(x) - Y(A)]/[1 + (Y'(x))^2], \tag{3.1}$$

where the prime now denotes differentiation with respect to the one-dimensional co-ordinate x. (For the projections to agree, now $a = A$.) It is henceforth convenient to use the "normalized" potential

$$\hat{v}(x) = v(x)/(mg). \tag{3.2}$$

Without loss of generality, axes may be chosen such that

$$Y(0) = 0 = v(0), \tag{3.3}$$

so that

$$\hat{v}(A) = Y(A)/[1 + (Y'(0))^2], \tag{3.4}$$

and thus the desired isodynamical correspondence expression is, explicitly,

$$\hat{v}(x) = \frac{\{[1 + (Y'(0))^2]Y(x) + Y(A)[(Y'(x))^2 - (Y'(0))^2]\}}{[1 + (Y'(0))^2][1 + (Y'(x))^2]}. \tag{3.5}$$

It is important to note that, for given non-linear path functions Y, this potential depends explicitly on the amplitude A.

In the common case of tracks producing oscillations with

$$Y'(0) = 0, \tag{3.6}$$

the corresponding potential is given explicitly by

$$\hat{v}(x) = [Y(x) + Y(A)(Y'(x))^2]/[1 + (Y'(x))^2]. \tag{3.7}$$

Then, in this case, there are the end-point identities for points x_A (including $x = A$) such that $Y(x_A) = Y(A)$,

$$\hat{v}(x_A) = Y(A), \qquad \hat{v}'(x_A) = Y'(x_A)/[1 + (Y'(x_A))^2], \tag{3.8a, b}$$

and the zero-slope identities for points x_z (including $x = 0$) such that $Y'(x_z) = 0$,

$$\hat{v}(x_z) = Y(x_z), \qquad \hat{v}'(x_z) = 0. \tag{3.9a, b}$$

3.1. EXAMPLES

3.1.1. *Parabolic path*

For the parabolic track shape

$$Y(X) = \tfrac{1}{2}KX^2 \qquad (K > 0), \tag{3.10}$$

the potential which is isodynamical to the projected X-motion on the interval $[-A, A]$ is given through equation (3.7) by

$$\hat{v}(x) = \tfrac{1}{2}K(1 + K^2A^2)x^2/[1 + K^2x^2], \tag{3.11}$$

which has a minimum at $x = 0$ and horizontal asymptote, as $x \to \pm\infty$, $\hat{v}(x) \to [1/(2K)](1 + K^2A^2) = \hat{v}(A) + 1/(2K)$. (Note that the motion, however, takes place between $\pm A$.)

In Figure 1 is shown the shape of the potential isodynamical to a parabolic track shape, for the case $K = 1$, $A = 1$. The potential has points of inflexion which the track does not possess. It is clear that equations (3.8a) and (3.9a, b) are satisfied. It is emphasized that this particular figure only covers the case with amplitude $A = 1$; i.e., $\dot{X} = 0 = \dot{x}$ at $X = 1 = x$.

The detailed X-motion oscillations with track (3.10) and the period thereof may be completely described [6] in terms of incomplete and complete (respectively) elliptic integrals of the second kind [7], and therefore so is the x-motion due to the more complicated potential (3.11), with exactly the same time dependence for given amplitude A, by the isodynamics. Explicitly (see reference [6, pp. 107–108]), for $X(t) = x(t)$,

$$t = [(1 + K^2A^2)/(gK)]^{1/2} E[\arccos(X/A), k], \qquad X \geqslant A \geqslant 0, \tag{3.12}$$

where

$$k = A[A^2 + (1/K^2)]^{-1/2}, \tag{3.13}$$

and

$$E[\theta, k] = \int_0^\theta [1 - k^2 \sin^2\phi]^{1/2}\, \mathrm{d}\phi \tag{3.14}$$

is the incomplete elliptic integral of the second kind [7, p. 908]. The period is $4t|_{X=0}$, where $E[\pi/2, k] = E(k)$ is the complete elliptic integral of the second kind.

3.1.2. *Symmetric hill-and-two-valleys path*

For the track shape

$$Y(X) = -K_2X^2 + K_4X^4 \qquad (K_i > 0), \tag{3.15}$$

which looks like a double well, the corresponding isodynamical potential (to projected X-motion) is explicitly, from equation (3.7),

$$\hat{v}(x) = x^2 \frac{[(K_4x^2 - K_2) + 4A^2(K_4A^2 - K_2)(2K_4x^2 - K_2)^2]}{[1 + 4x^2(2K_4x^2 - K_2)^2]}. \tag{3.16}$$

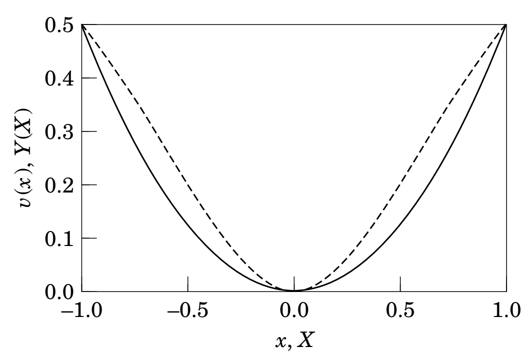

Figure 1. The potential function $v(x)$ (normalized with respect to (mg); - - - -) isodynamical to a parabolic track shape $Y(X)$. (——) See equations (3.11) and (3.10), with $K = 1$ and $A = 1$.

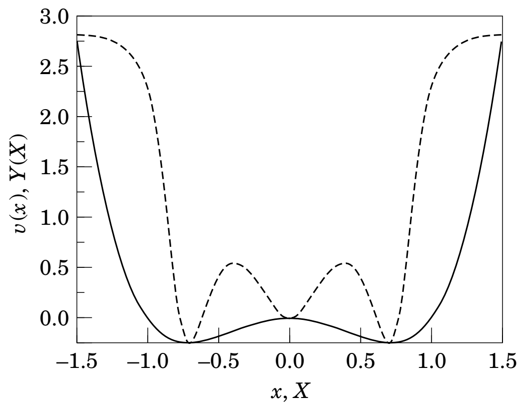

Figure 2. The potential function $v(x)$ (normalized with respect to (mg); - - - -) isodynamical to a symmetric hill-and-two-valleys track shape $Y(X)$ (——). See equations (3.16) and (3.15), with $K_2 = K_4 = 1$ and $A = 3/2$.

This is the actual potential that the track-and-cart system of Gottwald et al. [3] *was modelling*. It is quite different in functional form from equation (3.15), and it depends on amplitude A.

If $Y(A) > 0$ is chosen for full motion across the hill and both valleys, then for this (symmetric) case there are "large orbit" oscillations and the motion takes place between $+A$ and $-A$. In Figure 2 is shown a hill-and-two-valleys shaped track and the shape of the corresponding isodynamical potential, for the case $K_2 = K_4 = 1$ and $A = 3/2$. The potential looks quite different from the track: it has two humps and three wells. Nevertheless, it may be noted how equation (3.8a) and the zero-slope identities (3.9a, b) are satisfied. The (scaled) potential has minima at the turning points of the track curve. Although this may seem inconsistent, it should be remembered that for the track it is the X-projected motion that is at issue, so that, even though the kinetic energy decreases when the track height increases, as the curve becomes more nearly parallel to the X-axis $|\dot{X}|$ can increase.

There is actually a very small range of amplitudes A for which the potential $\hat{v}(x)$ does have a double-well type shape; i.e., $\hat{v}''(0)$ is negative. To analyze this, the *general* result (3.7) is differentiated twice to yield (cf., equation (3.9))

$$\hat{v}''(x_z) = Y''(x_z)[1 + 2Y''(x_z)(Y(A) - Y(x_z))]. \tag{3.17}$$

Since $Y(A) \geqslant Y(x)$,

$$Y''(x_z) > 0 \quad \Rightarrow \quad \hat{v}''(x_z) > 0. \tag{3.18}$$

However, from equation (3.17),

$$Y''(x_z) < 0 \quad \Rightarrow \quad \hat{v}''(x_z) < 0 \qquad \text{if and only if} \quad Y(A) - Y(x_z) < -1/(2Y''(x_z)). \tag{3.19}$$

For the track (3.15) with $K_2 = K_4 = 1$, putting $x_z = 0$ so that $Y(0) = 0$ and $Y''(0) = -2$, then $\hat{v}''(0) < 0$ if $Y(A) < 1/4$; i.e., if $A^2 < (1 + \sqrt{2})/2$, which implies that $1 \leqslant A < 1{\cdot}098684$. For instance, for $A = 1{\cdot}05$, a plot of equation (3.16) with $K_2 = K_4 = 1$ shows $\hat{v}(x)$ to be clearly of a double-well form with $\hat{v}(x) \geqslant Y(x)$, satisfying equations (3.8a) and (3.9a, b) but with a point of inflexion in each outermost "arm".

For $0 < A < 1$, the motion takes place just within the right-hand "valley", and is asymmetric. Plots with, say, $A = 0{\cdot}8$, show that, in contrast to Figure 2, $Y(X)$ and $\hat{v}(x)$ are quite close together (but still distinct) in the physical region.

It is again stressed that each different A is a different case, with in general differently shaped $\hat{v}(x)$ (see equation (3.5)), and with $\dot{x} = 0$ at $x = A$. The motion must therefore be regarded as starting from rest at $x = A$.

3.1.3. *Other quartic paths*

For the concave-up track shapes $Y = K_4X^4$ and $Y = K_4X^4 + K_2X^2$, the corresponding isodynamical potentials given explicitly by equation (3.7), e.g., with $A = 1$ and $K_4 = 1$, $K_2 = 1$, may be described in much the same way as Figure 1 for the parabolic track: the potential has a point of inflexion at a positive x, and at a negative x by symmetry.

4. REFLEXIVELY ISODYNAMICAL NON-LINEAR FUNCTION

In this section, the following question is dealt with: What shape do the isodynamical potential and track take if they are the same function; i.e.,

$$v(x) \propto Y(x)? \tag{4.1}$$

The "trivial" case of (piecewise) linearity is dealt with and then solved in detail in the Appendix. From now on, it will be assumed that $Y'(X) \neq$ constant.

It turns out that, with origin chosen so that $v(0) = 0$ and hence $Y(0) = 0$, it is necessary to have $a = 0$ and hence $A = 0$, so equation (2.2b) is replaced by

$$\dot{x}^2 = -(2/m)v(x). \tag{4.2}$$

In this case, equation (2.4b) is not a suitable form for the first integral of equation (2.3), since it does not allow for the possibility $|Y'(A)| = \infty$. Rather, the general first integral form (2.4a) is used here:

$$\dot{X}^2[1 + (Y'(X))^2] = -2gY(X) + \text{constant}. \tag{4.3}$$

Thus, for equations (4.2) and (4.3) to agree, evidently first of all

$$(\dot{X})^2(Y')^2 = \text{constant} \tag{4.4}$$

must be satisfied. Then, equation (4.4) with equations (4.1) and (4.2) give

$$(v')^2 \propto -1/v. \tag{4.5}$$

The functions obtained are

$$v(x) = -kx^{(2/3)}, \qquad Y(X) = -KX^{(2/3)}, \tag{4.6a}$$

$$K = k/(mg), \tag{4.6b}$$

with explicit solution for the motion

$$x(t) = (2/3)^{(3/2)}(2k/m)^{(3/4)}t^{(3/2)}, \qquad X(t) = (2/3)^{(3/2)}(2gK)^{(3/4)}t^{(3/2)}. \tag{4.7a, b}$$

Then equation (4.3) reads $\dot{X}^2 = -2gY$, with $(\dot{X})^2(Y')^2 = (8/9)gK^3 =$ constant in equations (4.3) and (4.4).

This is not oscillatory motion. The solutions (4.6) and (4.7) correspond to a decreasing concave-up curve starting at the origin $X = 0$ (where $Y'(0) = -\infty$); see Figure 3 for $K = 1$. The potential $v(x)$ is repulsive, and the corresponding one-dimensional motion accelerates off from rest at the origin along the $+x$ direction. The X-projection of a bead, starting at the origin with appropriate downwards vertical speed and moving, due to gravity, down the track $Y(X)$ *of exactly the same shape*, has exactly the same time-dependence along the positive X-axis at does the potential motion along the x-axis. In this problem, the motion is unbounded. Thus, as alluded to in the Introduction, there is no genuine non-linear *oscillatory* potential the motion of which can accurately be described by consideration of a track of exactly the same shape.

5. CORRESPONDENCE BETWEEN ARC PATH AND POTENTIAL

The *arc* length motion $S(t)$ of a particle moving along a path $Y(X)$ under gravity will be isodynamical to the potential motion $x(t)$ due to a given one-dimensional potential $v(x)$ if equations (2.2b) and (2.8) have the same form. Thus the path co-ordinate Y, when expressed in terms of its arc length S, which is here denoted by the function $Y_3(S)$, must take the same functional form as the potential v does of its argument:

$$Y(X(S)) \equiv Y_3(S) = (mg)^{-1}v(S) \equiv \hat{v}(S), \tag{5.1a}$$

and the end-points must be related by

$$S_A \equiv S|_{X=A} = a. \tag{5.1b}$$

(In equation (5.1a), S is taken as negative when the particle is to the "left" of the origin.)

If a path $Y(X)$ is given and if the relation between Y and the corresponding path arc length variable S, $Y_3(S)$, can be found via equation (2.7), then the isodynamical potential is given by

$$\hat{v}(x) = Y_3(x), \tag{5.2a}$$

with

$$a = S_A. \tag{5.2b}$$

On the other hand, if $S(Y)$ is known, then $v(x)$ is obtained implicitly via

$$x = S(\hat{v}), \tag{5.3}$$

and such a curve can be drawn.

For the converse problem, if the potential is given, and equation (5.1) can be solved explicitly for S as a function of Y, then the track path may be found in Cartesian co-ordinates from equation (2.7b) written in the form

$$\mathrm{d}X/\mathrm{d}Y = \pm[(\mathrm{d}S/\mathrm{d}Y)^2 - 1]^{1/2}. \tag{5.4}$$

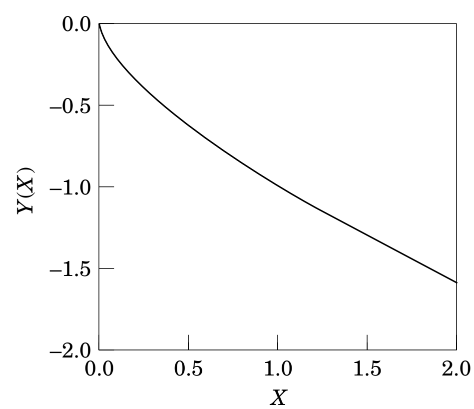

Figure 3. The track shape $Y(X)$ isodynamical to potential $v(x)$ (normalized with respect to mg) of the *same* shape. See equations (4.6) with $K = 1$.

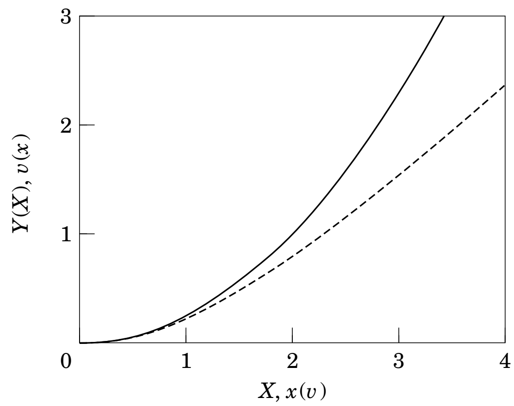

Figure 4. The potential function $v(x)$ (normalized with respect to (mg); - - - -) giving motion $x(t)$ isodynamical to motion $S(t)$ along the arc length of a parabolic track $Y(X)$ (——). See equations (5.7) and (5.5), with $K=1/2$. The curves are extended to negative $x = X$ by symmetry.

In these cases, $S(t)$ is measured by the particle's motion along the path. For instance, in the context of the "cart" of Gottwald *et al*. [3] moving along a "track", their potentiometer signal actually is a *direct* measure of the cart's position S along the track curve itself. This was the approach adopted by Shaw and Haddow [4].

5.1. EXAMPLES

5.1.1. *Parabolic path*

Suppose that the parabolic path

$$Y = \tfrac{1}{2}KX^2 \tag{5.5}$$

is given. (The dynamical solution for $X(t)$ has been described in section 3.1.1.) Then integration (see, e.g., reference [8, p. 43]) of equation (2.7a) or equation (2.7b) is found to give the corresponding arc-length S in term of Y as

$$S = \{Y[(2K)^{-1} + Y]\}^{1/2} + (2K)^{-1} \operatorname{arcsinh} \{[(2K)Y]^{1/2}\}. \tag{5.6}$$

By analogy with a cycloid function (see equation (5.11) below), this may be termed a "chycloid" function.

This gives S as a function of Y, with slope $\mathrm{d}S/\mathrm{d}Y = \infty$ at $Y = 0$ (where $S = 0$) by equation (2.7b). Thence Y as a function of S is obtained implicitly (i.e., by interchanging axes): a curve shape $Y_3(S)$, with slope zero (minimum) at the origin. Replacing S by the potential's independent variable x gives a curve $Y_3(x)$, extended to negative x by symmetry. Replacing Y_3 as ordinate by $v(x)/(mg) = \hat{v}(x)$ according to equation (5.2a) then gives the (scaled) shape of the desired isodynamical potential $v(x)$ implicitly: a symmetrized "chycloid", by equations (5.3) and (5.6).

As an example, with $K = \frac{1}{2}$,

$$x = \hat{v}^{1/2}(1 + \hat{v})^{1/2} + \operatorname{arcsinh}(\hat{v}^{1/2}). \tag{5.7}$$

This (scaled) potential $\hat{v}(x)$, depicted in Figure 4, produces the same motion $x(t)$ with equation (5.2b) as the arc motion $S(t)$ of a bead moving under the influence of gravity along a wire of parabolic shape $Y = \frac{1}{4}X^2$. This track shape is also shown in Figure 4; it agrees with $\hat{v}(x)$ for small x because then, by equation (5.7), $\hat{v}(x) \approx \frac{1}{4}x^2$.

5.1.2. *Quadratic potential (harmonic oscillator)*

Conversely, if the potential $v(x)$ representing a simple harmonic oscillator

$$v(x) = \tfrac{1}{2}kx^2 \tag{5.8}$$

is given, then, by equation (5.1a),

$$Y_3(S) = KS^2, \qquad K = k/(2mg). \tag{5.9a, b}$$

Since, then, $S = \pm(Y/K)^{1/2}$, the ordinary differential equation (5.4) to be solved for the path is

$$\mathrm{d}X/\mathrm{d}Y = \pm\{[(4K)^{-1} - Y]/Y\}^{1/2}. \tag{5.10}$$

Integration, together with $Y = 0$ when $X = 0$, gives

$$\pm X = \{Y[(4K)^{-1} - Y]\}^{1/2} + (4K)^{-1} \arcsin\{[(4K)Y]^{1/2}\}, \tag{5.11}$$

which is the equation to a cycloid, with $|X| \leqslant \pi/(8K)$, $0 \leqslant Y \leqslant 1/(4K)$, and thus, by equation (5.9a), $S \leqslant 1/(2K)$.

Thus, if $k/m = 2gK$ (cf., equation (5.9b)), then the x-motion of a harmonic oscillator (equation (2.1) with equation (5.8)) $m\ddot{x} + kx = 0$ and the arc-length S-motion of a bead sliding on a smooth wire under gravity (equation (2.5) with equation (5.9a)) $\ddot{S} + 2gKS = 0$ for the cycloidal path shape given in equation (5.11) are identical (isodynamical) and given by $x = a\cos\omega t$ and $S = a\cos\omega t$ respectively, in which the radian frequency $\omega = (k/m)^{1/2} = (2gK)^{1/2}$, independent of the amplitude $a \leqslant 1/(2K)$. This is the isodynamical approach to the situation of a more usual mechanics problem which starts with a cycloidal shape and proves that the resulting motion, under gravity, along the arc is simple harmonic (see, e.g., reference [6, p. 302]). This problem was dealt with in Example 3.1 of reference [4], where the track shape was plotted but apparently not identified explicitly as the cycloid as given by equation (5.11) above.

6. CONCLUSIONS

For a given track shape, the potential isodynamical to the projected horizontal motion is explicitly and uniquely specified. The formula for the potential is in general more complicated than its isodynamical track, and does depend on the amplitude. In particular, corresponding to a double-well shaped track (cf., reference [3]) (with motion under gravity, projected onto the horizontal axis), the isodynamical potential may have a triple-well form. The converse problem (given the potential, find the isodynamical track) is the subject of a forthcoming paper.

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APPENDIX: PIECEWISE LINEAR SHAPES

The case of constant (magnitude of) slope deserves separate attention, since then the whole motion may be solved exactly analytically. Furthermore, the corresponding track and potential shapes are geometrically similar, so phase-plane analogies are fully justified in these cases.

Given a straight track

$$Y = KX, \tag{A.1}$$

by equation (3.5) the isodynamical potential is again a straight line:

$$v(x) = kx, \tag{A.2}$$

with

$$k = mgK/(1 + K^2). \tag{A.3}$$

This is independent of the (common) amplitude A.

Conversely, given $v = kx$, for some k, the relation (A.3) again holds, but now

$$K = K_{\pm} = [1 \pm (1 - 4\lambda^2)^{1/2}]/(2\lambda), \tag{A.4}$$

where

$$\lambda = k/(mg). \tag{A.5}$$

Thus there are two distinct track slopes yielding motion isodynamical to the potential (A.2), provided that $\lambda < \frac{1}{2}$; i.e., k in equation (A.2) satisfies $k < \frac{1}{2}mg$. The track $Y(X)$ is not unique: the two tracks (A.1) with (A.4) are isodynamical with each other.

In fact, with gradient $K = \tan\theta$, since $K_- = 1/K_+$, $\tan\theta_+ = \cot\theta_-$, so $\theta_+ = (\pi/2) - \theta_-$. This result may also be readily obtained from a simple mechanics problem for the time of descent for motion on an inclined line under gravity; it has been obtained here via the isodynamical potential procedure.

The solution $x(t)$ to equation (2.2b) with equation (A.2), with $x(0) = A$ (and $\dot{x}(0) = 0$) is

$$x(t) = A - \tfrac{1}{2}(k/m)t^2 \tag{A.6}$$

or, for equation (2.4b) with equation (A.1),

$$X(t) = A - \tfrac{1}{2}[gK/(1 + K^2)]t^2, \tag{A.7}$$

which are equivalent via equations (A.3) and (A.4). The time to origin/time of descent to $x = 0 = X$ is therefore given by

$$t_1 = (2mA/k)^{1/2} = [2(1 + K^2)A/(gK)]^{1/2}. \tag{A.8}$$

The period for a complete oscillation cycle in a V-shaped potential well with slope $\pm k$, or for a V-shaped track with (point) bead under gravity with slope $\pm K$ related by equation (A.3), is therefore given by

$$T = 4t_1, \tag{A.9}$$

where t_1 is given by equation (A.8). For the potential problem, with corresponding force proportional to sgn (x), this agrees with reference [9, p. 113] and with the more recent result in reference [10] (after use of the energy equation to re-express T in terms of the amplitude).

Other shapes with straight-line segments (piecewise linear) may be similarly analyzed. For the projected extremities of the isodynamical motions to agree, it is necessary that K^2 is the same throughout the track (and likewise for the potential), which consequently may have segments with slopes $\pm K$ only, for fixed $K > 0$. (This ensures that $\dot{X}$ is continuous.)

For example, a W-shaped potential well

$$v(x) = -kx, \qquad 0 \leqslant x \leqslant D, \tag{A.10a}$$

$$= k(x - 2D), \qquad D \leqslant x \leqslant A, \tag{A.10b}$$

where $A > 2D$, with symmetric extension for $-A \leqslant x \leqslant 0$, is solved by

$$x_1 = -\tfrac{1}{2}\mu t^2 + A, \qquad A \geqslant x \geqslant D, \tag{A.11a}$$

$$0 \leqslant t \leqslant t_1 = [2(A - D)/\mu]^{1/2}; \tag{A.11b}$$

$$x_2 = \tfrac{1}{2}\mu t^2 - 2[2\mu(A - D)]^{1/2} t + (3A - 2D), \qquad D \geqslant x \geqslant 0, \tag{A.12a}$$

$$t_1 \leqslant t \leqslant t_2 = [2(2(A - D))^{1/2} - (2(A - 2D))^{1/2}]/\mu^{1/2}; \tag{A.12b}$$

and by extension for $t > t_2$. A similar solution holds for the W-shaped track with

$$Y(X) = -KX, \qquad 0 \leqslant X \leqslant D, \tag{A.13a}$$

$$= K(X - 2D), \qquad D \leqslant X \leqslant A \tag{A.13b}$$

(extended symmetrically to $-A \leqslant X \leqslant 0$). In equation (A.11) and (A.12), for equation (A.10) or equation (A.13) respectively,

$$\mu = k/m = gK/(1 + K^2). \tag{A.14}$$

The period for one complete oscillation cycle of the motion for the W-shape is given by

$$T = 4t_2, \tag{A.15}$$

with t_2 as in equation (A.12b) with equation (A.14). (As $D \to 0$, the V-shape result given in equations (A.8) and (A.9) is recovered.)

This W-shaped potential/track could be of interest as an explicitly solvable simplified model of a double-well type potential/track: the correspondence is exact in this case.