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TRANSVERSE VIBRATIONS OF RECTANGULAR PLATES OF NON-UNIFORM THICKNESS WITH A FREE, CONCENTRIC CIRCULAR HOLE

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1. APPROXIMATE ANALYTICAL SOLUTION

Considering the structural systems depicted in Figure 1, it is convenient to define the dimensionless parameters $\lambda = a/b$, $\delta_0 = 2r_0/a$, $\delta_1 = 2r_1/a$ and $\eta = h_1/h_0$. In the case of Figure 1(a), the thickness variation is defined by

$$h(\bar{x}, \bar{y}) = \begin{cases} h_0 & (\bar{x}, \bar{y}) \in \bar{P}_0, \\ h_0 g(\bar{x}, \bar{y}), \quad g(\bar{x}, \bar{y}) = \eta, & (\bar{x}, \bar{y}) \in \bar{P}_1. \end{cases} \tag{1}$$

On the other hand, the plate thickness of the mechanical system, shown in Figure 1(b), is defined by

$$h(x, y) = \begin{cases} h_0 & (x, y) \in \bar{P}_0, \\ h_0 g(\bar{x}, \bar{y}), & (\bar{x}, \bar{y}) \in \bar{P}_1, \end{cases} \tag{2}$$

where

$$g(\bar{x}, \bar{y}) = \frac{1-\eta}{r_0(1-\delta_r)} \sqrt{\left(\bar{x} - \frac{a}{2}\right)^2 + \left(\bar{y} - \frac{b}{2}\right)^2} + \frac{\eta - \delta_r}{1-\delta_r}, \qquad \delta_r = \frac{\delta_1}{\delta_0}.$$

Expressing the flexural rigidity $D(\bar{x}, \bar{y})$ in the form

$$D(x, y) = \frac{Eh_0^3}{12(1-\nu^2)} g^3(\bar{x}, \bar{y}) = D_0 g^3(\bar{x}, \bar{y}),$$

the governing functional turns out to be Hamilton's form of the governing equation of motion with the temporal variable transformed into the frequency domain,

$$J(W) = D_0 \iint_{\bar{P}} g^3[(W_{\bar{x}\bar{x}} + W_{\bar{y}\bar{y}})^2 - 2(1-\nu)(W_{\bar{x}\bar{x}} W_{\bar{y}\bar{y}} - W_{\bar{x}\bar{y}}^2)] \, dx \, d\bar{y}$$

$$- \rho h_0 \omega^2 \iint_{\bar{P}} g W^2 \, d\bar{x} \, d\bar{y}. \tag{3}$$

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Introducing the dimensionless variables $x = \bar{x}/a$ and $y = \bar{y}/b$, equation (3) becomes

$$\frac{\lambda a^2}{D_0} J(W) = \iint_P g^3[(W_{xx} + \lambda^2 W_{yy})^2 - 2(1-\nu)\lambda^2(W_{xx}W_{yy} - W_{xy}^2)]\,\mathrm{d}x\,\mathrm{d}y$$

$$- \Omega^2 \iint_P g W^2\,\mathrm{d}x\,\mathrm{d}y, \qquad (4)$$

where $\Omega^2 = \rho h_0 a^4 \omega^2 / D_0$.

In the case of the structural system shown in Figure 1(a), one has

$$g(x, y) = \begin{cases} 1, & (x, y) \in P_0, \\ \eta, & (x, y) \in P_1; \end{cases}$$

while in the cases of Figure 1(b), $g(x, y)$ is given by

$$g(x, y) = \begin{cases} 1 & (x, y) \in P_0, \\ \dfrac{2(1-\eta)}{\delta_0 - \delta_1}\sqrt{(x - 1/2)^2 + \left(\dfrac{y - 1/2}{\lambda}\right)^2} + \dfrac{\eta - \delta_r}{1 - \delta_r}, & (x, y) \in P_1. \end{cases}$$

The fundamental mode shape will be now expressed in the form [2]

$$W_\alpha = \sum_{j=1}^{3} C_j \varphi_j(x, y), \qquad (5)$$

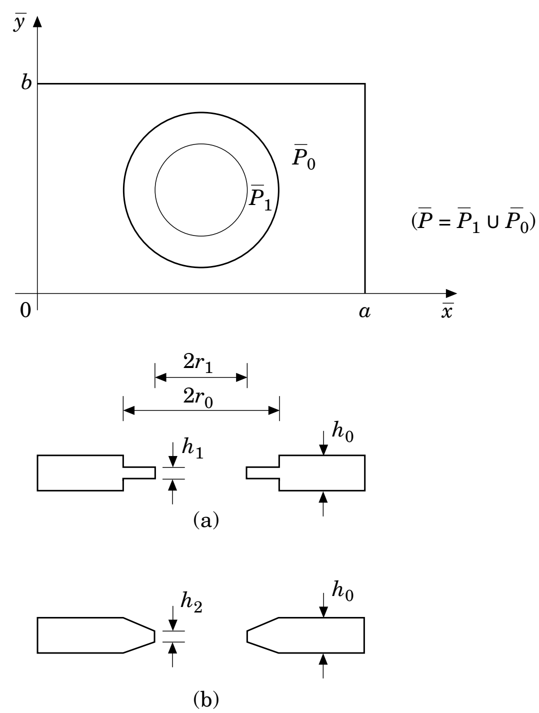

Figure 1. A mechanical system executing transverse vibrations: (a) step thickness variation; (b) gradual thickness variation.

where

$$\varphi_j(x, y) = (x^{p+j-1} + \alpha_{j3}x^3 + \alpha_{j2}x^2 + \alpha_{j1}x)(y^{p+j-1} + \beta_{j3}y^3 + \beta_{j2}y^2 + \beta_{j1}y),$$

and where p is the Rayleigh's optimization parameter, and the α's and β's are constants determined by substituting each co-ordinate function in the governing boundary conditions at the outer edge.

Substituting equation (5) equation (4) and minimizing with respect to the C_i's, one obtains

$$\frac{1}{2}\frac{\lambda a^2}{D_0}\frac{\partial J}{\partial C_i} = \sum_{j=1}^{3}\left\{\iint_P g^3[(\varphi_{jxx} + \lambda^2\varphi_{jyy})(\varphi_{ixx} + \lambda^2\varphi_{iyy}) - (1-v)\lambda^2(\varphi_{jyy}\varphi_{ixx} + \varphi_{jxx}\varphi_{iyy} - 2\varphi_{jxy}\varphi_{ixy})]\,\mathrm{d}x\,\mathrm{d}y - \Omega^2\iint_P g\rho_j\varphi_i\,\mathrm{d}x\,\mathrm{d}y\right\}C_j = 0, \qquad i = 1, 2, 3. \tag{6}$$

Equation (6) yields an homogeneous, linear system of equations in the C_i's. The non-triviality condition yields a determinantal equation the lowest root of which is the fundamental frequency coefficient Ω_1. Since

$$\Omega_1 = \Omega_1(p), \tag{7}$$

by requiring

$$\mathrm{d}\Omega_1/\mathrm{d}p = 0 \tag{8}$$

one obtains an optimized value of Ω_1. The determination of the value of p which yields a minimum value of Ω_1 is accomplished numerically.

All calculations have been performed for a Poisson ratio (v) equal to 0·30.

2. FINITE ELEMENT DETERMINATIONS

The numerical results have been obtained using the SAMCEF finite element code using hybrid elements of triangular and rectangular shape (elements type 55 and 56 of the SAMCEF Library). The number of elements varied in accordance with the ratio $2r_1/a$ (for $2r_1/a = 0{\cdot}1$, the mesh of half of the plate contained 661 elements).

3. NUMERICAL RESULTS

The fundamental frequency coefficients of a simply supported square plate are presented in Table 1.

In general, the finite element results (presumably more accurate) are lower than the analytical predictions. The maximum differences are of the order of 6·5%, which are acceptable from a practical viewpoint, especially if one considers the simplicity of the analytical approach.

The eigenvalues in the case of a square plate, with two opposite edges simply supported while the remaining ones are clamped, are depicted in Table 2. Similar conclusions as in the previous case apply, and are also applicable when dealing with the fully clamped square plate (Table 3).

In the case of a simply supported rectangular plate ($a/b = 1{\cdot}5$), the maximum differences are slightly larger (7%); see Table 4. This is also the case with edges $\bar{x} = 0$, $\bar{a}$ clamped, while the other two edges are simply supported (Table 5). On the other

TABLE 1

Case SS–SS, SS–SS, $\lambda = 1$

			Thickness variation				
			Discontinuous			Continuous	
δ_1	δ_0		$\eta = 1$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$
0·1	0·2	(1)†	19·83	19·71	19·79	19·76	19·73
		(2)	*19·53*	*19·26*	*18·96*	*19·38*	*19·14*
	0·3	(1)	19·83	19·50	19·65	19·67	19·56
		(2)	*19·52*	*19·00*	*18·47*	*19·25*	*18·75*
	0·4	(1)	19·83	19·19	19·34	19·54	19·28
		(2)	*19·50*	*18·74*	*18·17*	–	*18·39*
0·2	0·3	(1)	20·15	19·93	20·05	20·03	19·96
		(2)	*19·28*	*18·93*	*18·80*	*19·07*	*18·85*
	0·4	(1)	20·15	19·60	19·81	19·87	19·68
		(2)	*19·27*	*18·67*	*18·56*	–	*18·50*
0·3	0·4	(1)	20·70	20·35	20·57	20·51	20·43
		(2)	*19·48*	*19·17*	*19·35*	*19·29*	*19·18*

†(1) Analytical solution; *(2) numerical solution (finite element method).*

hand, the agreement is considerably better when $\bar{x} = 0$, a are simply supported and $\bar{y} = 0$, b are clamped; see Table 6.

The fundamental frequency coefficients for a fully clamped rectangular plate ($a/b = 1{\cdot}5$) are depicted in Table 7. In this case, the maximum difference is of the order of 10% (discontinuous variation: $\delta_1 = 0{\cdot}1$; $\delta_0 = 0{\cdot}4$ and $\eta = 0{\cdot}4$).

TABLE 2

Case C–C, SS–SS, $\lambda = 1$

			Thickness variation				
			Discontinuous			Continuous	
δ_1	δ_0		$\eta = 1$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$
0·1	0·2	(1)†	29·18	29·07	29·34	29·11	29·14
		(2)	*28·70*	*28·40*	*28·22*	*28·53*	*28·29*
	0·3	(1)	–	28·92	29·61	29·02	29·06
		(2)	–	*28·19*	*28·16*	*28·38*	*27·94*
	0·4	(1)	–	28·74	30·11	28·92	28·96
		(2)	–	*28·05*	*28·56*	*28·27*	*27·73*
0·2	0·3	(1)	30·01	29·88	30·38	29·92	30·00
		(2)	*28·89*	*28·64*	*28·97*	*28·73*	*28·65*
	0·4	(1)	–	29·73	31·06	29·82	29·98
		(2)	–	*28·56*	*29·59*	*28·62*	*28·57*
0·3	0·4	(1)	31·63	31·56	32·62	31·55	31·81
		(2)	*30·35*	*30·35*	*31·36*	*30·30*	*30·52*

†(1) Analytical solution; *(2) numerical solution (finite element method).*

TABLE 3

Case C–C, C–C, $\lambda = 1$

			Thickness variation				
			Discontinuous			Continuous	
δ_1	δ_0		$\eta = 1$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$
0·1	0·2	(1)†	36·67	36·24	36·68	36·28	36·35
		(2)	*35·67*	*35·31*	*35·21*	*35·45*	*35·17*
	0·3	(1)	–	36·07	37·25	36·18	36·29
		(2)	–	*35·11*	*35·50*	*35·29*	*34·85*
	0·4	(1)	–	35·93	38·34	36·07	36·25
		(2)	–	*35·05*	*36·42*	–	*34·73*
0·2	0·3	(1)	37·75	37·63	38·53	37·65	37·84
		(2)	*36·30*	*36·11*	*36·90*	*36·15*	*36·20*
	0·4	(1)	–	37·56	40·00	37·56	37·99
(2)			–	*36·15*	*38·31*	*36·08*	*36·37*
0·3	0·4	(1)	40·56	40·69	42·70	40·56	41·15
		(2)	*39·14*	*39·42*	*41·41*	*39·22*	*39·81*

†(1) Analytical solution; *(2) numerical solution (finite element method).*

Admittedly, the analytical approach (the optimized Rayleigh–Ritz method) constitutes a classical and well known methodology. On the other hand, it is rather remarkable that polynomial co-ordinate functions which satisfy the essential boundary conditions at the

TABLE 4

Case SS–SS, SS–SS, $\lambda = 1{\cdot}5$

			Thickness variation				
			Discontinuous			Continuous	
δ_1	δ_0		$\eta = 1$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$
0·1	0·2	(1)†	32·32	31·99	32·10	32·14	32·01
		(2)	*31·58*	*30·94*	*30·37*	*31·23*	*30·70*
	0·3	(1)	–	31·40	31·53	31·88	31·48
		(2)	–	*30·37*	*29·49*	*30·92*	*29·88*
	0·4	(1)	–	30·69	30·7	31·55	30·74
		(2)	–	*29·82*	*28·93*	*30·61*	*29·14*
0·2	0·3	(1)	32·92	32·29	32·48	32·57	32·35
		(2)	*31·24*	*30·53*	*30·50*	*30·83*	*30·43*
	0·4	(1)	–	31·53	31·83	32·17	31·59
		(2)	–	*30·00*	*30·15*	*30·50*	*29·79*
0·3	0·4	(1)	34·16	33·37	33·90	33·71	33·52
		(2)	*32·27*	*31·73*	*32·41*	*31·94*	*31·84*

†(1) Analytical solution; *(2) numerical solution (finite element method).*

TABLE 5

Case C–C, SS–SS, $\lambda = 1{\cdot}5$: $\bar{x} = 0$, a, *clamped*; $\bar{y} = 0$, b, *simply supported*

			Thickness variation				
			Discontinuous			Continuous	
δ_1	δ_0		$\eta = 1$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$
0·1	0·2	(1)†	39·63	39·33	39·71	39·46	39·40
		(2)	*38·51*	*37·83*	*37·39*	*38·13*	*37·59*
	0·3	(1)	–	38·79	39·72	39·21	39·01
		(2)	–	*37·32*	*37·04*	*37·82*	*36·84*
	0·4	(1)	–	38·19	39·81	38·9	38·46
		(2)	–	*36·89*	*37·17*	*37·53*	*36·27*
0·2	0·3	(1)	41·01	40·49	41·31	40·70	40·71
		(2)	*38·80*	*38·22*	*38·81*	*38·44*	*38·25*
	0·4	(1)	–	39·91	41·85	40·37	40·29
		(2)	–	*37·87*	*39·54*	*38·18*	*37·96*
0·3	0·4	(1)	44·13	43·72	45·64	43·85	44·18
		(2)	*41·74*	*41·62*	*43·66*	*41·61*	*42·03*

†(1) Analytical solution; *(2) numerical solution (finite element method).*

outer boundary, and do not possess a singularity at the plate center, turn out to yield good engineering approximations. The procedure is also applicable when dealing with orthotropic plates and other polygonal outer boundary shapes.

TABLE 6

Case SS–SS, C-C, $\lambda = 1{\cdot}5$: $\bar{x} = 0$, a, *simply supported*; $\bar{y} = 0$, b, *clamped*

			Thickness variation				
			Discontinuous			Continuous	
δ_1	δ_0		$\eta = 1$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$
0·1	0·2	†(1)†	57·22	57·00	57·79	57·07	57·13
		(2)	*55·93*	*55·44*	*55·73*	*55·62*	*55·33*
	0·3	(1)	–	56·84	59·19	56·91	57·14
		(2)	–	*55·30*	*56·85*	*55·44*	*55·16*
	0·4	(1)	–	56·82	61·48	56·80	57·36
		(2)	–	*55·28*	*56·74*	*55·32*	*55·25*
0·2	0·3	(1)	59·67	59·62	61·50	59·56	60·02
		(2)	*58·01*	*58·02*	*59·78*	*57·94*	*58·36*
	0·4	(1)	–	59·75	64·48	59·53	60·66
		(2)	–	*58·14*	*61·11*	*57·95*	*58·94*
0·3	0·4	(1)	65·42	65·99	69·74	65·62	66·90
		(2)	*63·64*	*63·96*	*65·95*	*63·77*	*64·46*

†(1) Analytical solution; *(2) numerical solution (finite element method).*

TABLE 7

Case C–C, C–C, $\lambda = 1{\cdot}5$

			Thickness variation				
			Discontinuous			Continuous	
δ_1	δ_0		$\eta = 1$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$	$\eta = 0{\cdot}8$	$\eta = 0{\cdot}4$
0·1	0·2	(1)†	62·04	61·78	62·87	61·86	61·96
		(2)	*60·27*	*59·69*	*60·15*	*59·90*	*59·56*
	0·3	(1)	–	61·56	64·85	61·66	62·00
		(2)	–	*59·57*	*61·77*	*59·68*	*59·41*
	0·4	(1)	–	61·58	68·14	61·53	62·33
		(2)	–	*59·58*	*61·39*	*59·55*	*59·59*
0·2	0·3	(1)	65·51	65·52	68·29	65·40	66·12
		(2)	*63·20*	*63·35*	*65·99*	*63·17*	*63·85*
	0·4	(1)	–	65·80	72·91	65·42	67·17
		(2)	–	*63·62*	*68·26*	*63·24*	*64·83*
0·3	0·4	(1)	74·16	75·34	81·48	74·65	76·81
		(2)	*71·64*	*72·52*	*76·32*	*72·03*	*73·43*

†(1) Analytical solution; *(2) numerical solution (finite element method).*

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