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ON THE LARGE AMPLITUDE FREE VIBRATIONS OF A RESTRAINED UNIFORM BEAM CARRYING AN INTERMEDIATE LUMPED MASS

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The non-linear period, for each of the first four modes, of planar, flexural large amplitude free vibrations of a slender, inextensible cantilever beam with a flexible root and carrying a lumped mass at an intermediate position along its span is investigated theoretically. With shear deformation and rotary inertia assumed to be negligible, but with account taken of axial inertia and non-linear curvature, two different, simple, approaches—for comparison purposes—are used to formulate the equation of motion. In the first approach, the governing partial differential field equation of motion is obtained by using Hamilton's principle, following closely the analysis presented in reference [1], which does not take into account the inextensibility condition. By retaining non-linear terms up to order five and using the single mode approximation in conjunction with the Rayleigh–Ritz method, the field equation is reduced to a non-linear, single mode, Duffing type temporal problem. In the second approach, an assumed single mode Lagrangian method, with account taken of the inextensibility condition, is used to form directly the fifth order non-linear unimodal temporal problem. Because of the particular non-linear terms in the temporal problem in both formulations, the time transformation approach [2] is used to obtain an approximate solution to the period of oscillation. Results in non-dimensional forms are presented graphically for the effects of the base stiffness, position and magnitude of lumped mass on the variation of period of oscillation with amplitude. Comparison of the present models with some of the existing ones, and comparison of the time transformation results with those of the harmonic balance, and existing ones are presented.

1. INTRODUCTION

Many engineering systems can be modelled as slender cantilever beams with a flexible root and carrying an intermediate lumped mass. Such a beam, being flexible, often undergoes large amplitude vibrations—as, for example, when subjected to parametric excitation [3–5]—which are not adequately described using linear vibration theory. It is of interest in the analysis of a non-linear vibrating system to determine the free vibration frequency–amplitude dependence, as this allows one to establish the general qualitative behavior, i.e., the backbone curve, of the steady state forced response. Many theoretical and experimental investigations into non-linear vibrations of beam elements have appeared over the years. Reviews on this subject have been presented by, for example, Rosenberg [6], Nayfeh and Mook [7], Crespo da Silva and Glynn [8] and Eisley [9], among others. In general, these non-linearities may be classified into geometric, inertial or material [9]. Geometric non-linearities may be caused by axial stretching of the median line in axially restrained beams or by large beam slopes, so that it is no longer possible to use the small

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angle assumption which reduces the non-linear curvature to the simple linear form [10–14]. Non-linear inertia effects may be caused by the presence of concentrated or distributed inertia elements, and by non-planar or parametric motions [4, 7, 8, 15]. Material non-linearities occur whenever the stresses are non-linear functions of the strains. The non-linearities may also appear in the boundary conditions [16, 17].

Due to the diversity of ways by which non-linearities may enter the equations of motion, various simplifying assumptions regarding the effects of different kinds of non-linearities are usually introduced in order to derive approximate, manageable and meaningful models, depending on the nature of the problem and the objectives of the analysis. For example, the non-linear free and forced vibrations of a uniform beam with axially restrained edges to produce mid-plane stretching non-linearity, i.e., extensional beams, have been studied in references [9–14, 18–29], to mention a few. The studies in reference [9–13, 18–25] neglected axial inertia, used linear curvature and accounted for large deformations through the use of Green's strain measure in the longitudinal direction. Lewandowski [14] and Sevin [26] included the effect of axial inertia but used linear curvature, while Eringen [27], McIvor [28] and Atluri [29] included the effects of axial inertia and non-linear curvature.

The free and forced vibrations of inextensible beams, i.e., beams for which the length of the neutral axis remains constant during the motion, have been studied in [1, 3-5, 7, 15, 30–36], among others. Wagner [1] included axial inertia and non-linear curvature in the analysis of large amplitude, planar flexural free vibrations of an initially straight uniform beam having free–free or fixed–free end conditions, with neglibible shear and rotary inertia effects. A combination of Hamilton's principle and an assumed single linear mode initial deflection, Buhnov's method, was used to obtain a unimodal non-linear temporal problem which was solved by using Atkinson's superposition procedure. The variation of the non-linear period with the dimensionless amplitude a ($a = b/l$, where l is the beam length and b is the deflection at the free end and at $l/2$ for the clamped–free and free–free cases, respectively) was shown only for the first mode and for values of a up to 0·5. These results, which included the effects of polynomial non-linearities up to 13th order, showed non-linear characteristics of the hardening type; i.e., the period decreases monotonically with an increasing amplitude of motion. It is to be noted that although in the analysis in reference [1] the beam was assumed to be inextensible, the inextensibility condition was not taken into account in the derivation of the non-linear field partial differential equation of motion. In fact, and as will be shown later, the non-linear terms in this equation are all of the hardening type due to potential energy stored in bending and none is of the inertia (softening) type. Thus it is expected that the variation of period with amplitude obtained using the analysis in reference [1] will exhibit hardening characteristics for all modes of vibration. The above analysis, however, predicts the correct behavior for the fundamental period but not for the second and higher modes of an inextensible beam is due to the fact that for the lower modes the non-linear curvature effects dominate, while for the higher modes the inertia (softening) effects dominate [7, 15]. Nageswara Rao and Venkateswara Rao [30] accounted for axial inertia and the inextensibility condition, and used Newton's moment law and Euler–Bernoulli beam theory to derive a set of differential field equations for the planar flexural free motion of a clamped–free or a free–free beam. Assuming the time variation of the beam response to be simple harmonic, they reduced the field equations of motion to a two-point non-linear boundary value problem, which they solved using an iterative numerical method for the period of oscillation and the corresponding non-linear mode shape. They presented results only for the first and second modes and for amplitude ratio a (a is defined in reference [1]) up to 0·5 and up to 0·3 for the first and second modes, respectively. For the first mode their results were shown to be in good qualitative and quantitative agreement with those presented in reference [1]; for the second mode their results displayed a softening type non-linear behavior, while those obtained using the analysis in reference [1] displayed a hardening behavior. Takahashi [31] used a simplified version of the equations derived by Yoshimura and Uemura [32] to describe the planar free vibrations of inextensible free–free and clamped–free beams similar to those in references [1, 30]. In this case, Galerkin's method and a two-term harmonic balance (HB) method were used to analyze the free motions of these beams, with non-linear terms retained up to fifth order. He presented results of frequency–amplitude variation only for the first mode of each of these beams, and for an amplitude ratio a (a defined as in references [1, 30]) up to 0·5. These results showed that the influence of the large amplitude motion on the frequency of the first mode of each of these beams is similar to that of a weakly softening spring, in contradiction with those presented in references [1, 30]. The vibrations of inextensible beams, i.e., beams with inertial non-linearities, have also been studied in [4, 7, 15]. Haight and King [15] included inertial non-linearities and neglected those due to curvature and torsion in their study of the stability of in-plane motion of a slender inextensible cantilever rod with nearly equal principal moments of inertia of the transverse cross-sectional area and subjected to lateral harmonic base motion. Crespo da Silva and Glynn [8] used the extended Hamilton's principle to derive a set of fairly general non-linear partial integro-differential field equations of motion for an inextensible beam, which include third order non-linear contributions from inertia, curvature and torsion. A number of simplified versions of these equations were used by different authors to study the non-linear response of inextensional beam elements with various end conditions for different cases of base and other excitations [5, 33–36]. Zavodney and Nayfeh [4] considered axial inertia and the inextensibility condition and used Newton's moment law and Euler–Bernoulli beam theory to derive a non-linear partial integro-differential equation describing the transverse planar motion of a slender, vertically mounted cantilever beam undergoing harmonic vertical base motion and carrying a lumped mass and rotary inertia at an intermediate point along its span. They argued that their equation, which includes non-linear effects arising from curvature and inertia, can be made, after introducing certain assumptions regarding the magnitudes of some terms, to agree with a simplified planar version of the general equations in reference [7]. An objective of the present work, as will be discussed later, is to use a simple approach, but not as general as the above-mentioned ones, to the formulation of the large amplitude planar flexural motion of an inextensible beam which includes the effects of non-linear curvature and inertia.

Closed form solutions for non-linear vibrations of continuous one-dimensional beam systems are in general not possible. On the whole, two methods have been employed in obtaining approximate solutions to such non-linear problems. These methods are based on the assumption that the beam deflection is separable in space and time. In the first approach, one assumes a solution in the form of series combinations of n specified spatial co-ordinate admissible functions with n unspecified time dependent coefficients (generalized co-ordinates). When the assumed admissible spatial functions are the eigenfunctions of the associated linear beam problem, the procedure is known as Galerkin's method, whereby upon using the orthogonality of the beam eigenfunctions the non-linear partial differential equation of motion is converted into a set of n coupled non-linear ordinary differential equations in the generalized co-ordinate describing the temporal behavior of the system. When $n = 1$ the procedure is known as the single mode approach, and the temporal problem is reduced to a single unimodal, usually Duffing type, non-linear oscillator. Approximate analytic solutions to the equivalent non-linear temporal problem are usually obtained using perturbation methods; i.e., the method of multiple scales (MMS), [4, 9, 33–36], or HB method [5, 13, 18–23, 28, 31] or the averaged method [7]. With the exception of the multi-term HB method, the techniques are limited to weakly non-linear systems; i.e., these techniques lose their usefulness as the amplitude of vibration becomes relatively large [37, 38].

An obvious limitation of the single mode approach in comparison with the multi-mode one is that it does not allow for any interaction between the different modes of motion; i.e., it cannot be used to study internal and combination resonances in planar motions of beams. Recent experimental and theoretical studies [3, 39] have shown that non-linear modal interaction can occur in the planar response of a flexible cantilever, i.e., parametrically excited, beam even between modes of non-commensurate and widely spaced natural frequencies. An important aspect of the assumed mode(s) method is the assumption that the modes of vibration of the non-linear systems remain self-similar during the motion and are unchanged by the non-linearities; i.e., only the frequencies are allowed to be altered by the non-linearities. Theoretical and experimental studies indicate, however, that the vibration modes as well as the frequencies can deviate significantly from the associated linear problem [16, 24, 40–44]. Therefore, one would expect the assumed mode(s) method to lose its usefulness as the amplitude of motion becomes large. Despite its above inherent limitations, the assumed mode(s) method has been used to obtain meaningful results in the analysis of free and forced non-linear vibrations of a wide class of continuous systems [1, 4, 7–13, 18–23, 25–29]. This is due to the fact that modal subspaces are invariant and can capture solutions of the beam full partial non-linear differential equation of motion [45].

The second approach commonly used to analyze the non-linear vibrations of continuous beams assumes the time variation to be simple harmonic of unknown frequency, and then application of the harmonic balance method leads to a non-linear boundary value problem defining, approximately, the non-linear mode shapes and frequency [16, 24, 30]. Although this approach accounts for the variation of mode shape(s) with frequency, it suffers from the drawback that the consideration of the effects of higher harmonics on the frequency of a given mode is in general mathematically prohibitive, and from the fact that a non-linear boundary value problem is in general more difficult to treat analytically than the associated non-linear initial value problem. It is to be noted that direct application of perturbation and HB methods to non-linear partial differential equations and boundary conditions and the invariant manifold approach have recently been used in the vibration analysis of non-linear continuous systems [40–42].

In the present work, the non-linear, large amplitude free vibrations of a slender, inextensible cantilever beam with a rotationally flexible root and carrying a lumped mass at an intermediate position along its span are considered. The shear deformation and rotary inertia effects are assumed to be negligible, and the beam is assumed to be undergoing planar flexural vibrations. By Taking into account axial inertia and non-linear curvature, two different, simple, approaches—for comparison purposes—are used to formulate the equation of motion. In the first approach, the governing partial differential field equation of motion is obtained using Hamilton's principle, following closely the analysis presented in reference [1], which does not account for the inextensibility condition. Retaining non-linear terms up to fifth order and using the single mode approximation in conjunction with the Rayleigh–Ritz method, the field equation is reduced to a non-linear, unimodal, Duffing type temporal problem. This will be referred to as formulation I. In the second approach, an assumed single mode Lagrangian method, taking into account the inextensibility condition, is used to directly form the fifth order non-linear unimodal temporal problem. This will be referred to as formulation II. Because of the particular non-linear terms in the temporal problem in both formulations, the time transformation approach [2] is used to obtain an approximate solution to the period of oscillation. In addition to its generality and relative simplicity, the time transformation method offers the advantage over other approximate analytic methods in its applicability to strongly non-linear oscillators [2]; that is, it imposes no restrictions on the magnitudes of the non-linearities, so that relatively larger amplitudes of vibration than previously considered by other authors can be treated more accurately, especially for the higher modes where the linear and non-linear terms become of comparable magnitude at relatively low amplitude values. Results, in non-dimensional form, for each of the first four modes, are presented graphically for the effects of the base stiffness and the position and magnitude of the intermediate lumped mass on the variation of period of oscillation with amplitude. Comparison of the present models with some of the existing ones, and comparison of the time transformation results with those of the harmonic balance, and existing ones are presented. Hamdan and Jubran [46], Hamdan and Latif [47] and others (see, e.g., references [48, 49]), have shown that the natural frequencies of relevant linear problems may be considerably lower when the roots are flexible and when the beam carries a lumped mass. Studies of relevant non-linear problems, however, are less abundant.

2. ASSUMPTIONS AND EQUATION OF MOTION

2.1. SYSTEM DESCRIPTION AND ASSUMPTIONS

A schematic of the beam under study is shown in Figure 1. The beam is considered to be uniform of constant length l and mass m per unit length, hinged at the base to a rotational spring of stiffness K_r, and carries a lumped mass M at an arbitrary intermediate point $s = d$ along the beam span. The thickness of the beam is assumed to be small compared with the beam length, so that the effects of rotary inertia and shearing deformation can be ignored. Provided that the lumped mass M is placed symmetrically with respect to the beam length and the beam is relatively short, e.g., the ratio of the beam length to width is <30, the beam transverse motion can be considered to be purely planar [4]. It is further assumed that the amplitude of vibration may reach any large value, but remains below the limiting value for which the slope θ of the elastica $|\theta| = 90°$; also the beam is assumed to be conservative. These assumptions are the same as those used by Wagner [1] in studying the planar non-linear free vibration of a cantilever beam which was similar to the present one but with $K_r = \infty$ and $M = 0$. In this section, the governing equation of motion is formulated via Hamilton's principle, following closely the analysis in reference [1]. The result of this approach, referred to here as formulation I, is shown

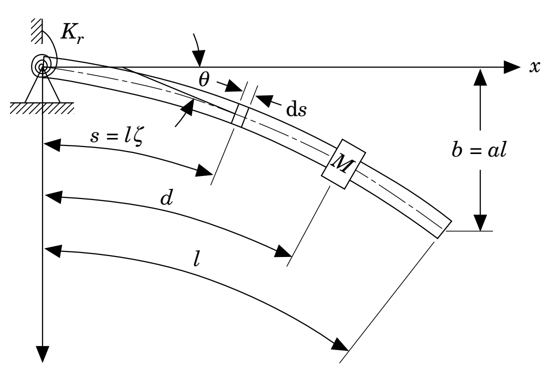

Figure 1. A sketch of the beam system under study.

to lead to incorrect prediction of the period–amplitude variation for the second and higher modes of the inextensible beam. In section 3 an alternative formulation, referred to as formulation II, which takes into account the inextensibility of the beam is derived by using a Rayleigh–Ritz–Lagrangian approach.

2.2. EQUATION OF MOTION: FORMULATION I

In terms of the co-ordinate system shown in Figure 1, the kinetic energy KE of the beam is

$$KE = (ml/2)\int_0^1 [1 + \mu\delta(\xi - \eta)](\dot{x}^2 + \dot{y}^2)\,\mathrm{d}\xi, \tag{1}$$

where $\xi = s/l$ is a dimensionless arc length, $\eta = d/l$ is the dimensionless relative position of the lumped mass M, $\delta(\xi - \eta)$ is Dirac's function, $\mu = M/ml$ is a dimensionless mass ratio parameter and a dot denotes a derivative with respect to time t. The potential energy V, due to bending, is given as

$$V = (EIl/2)\int_0^1 R^2(\xi, t)\,\mathrm{d}\xi, \tag{2}$$

where EI is the modulus of flexural rigidity and $R(\xi, t)$ is the radius of curvature of the neutral axis of the beam. In the present work the exact expression for the radius of curvature R will be used which, in terms of the variables x and y, takes the form [1]

$$R = \lambda^3(x'y'' - x''y'), \tag{3}$$

where $\lambda = 1/l$ and a prime denotes differentiation with respect to the dimensionless arc length ξ. Note that the potential energy expression in equation (2) does not include the potential energy stored in the rotational spring K_r at the base of the beam. The effect of this spring will, however, be taken into consideration in selecting the mode shape function, carried out in the next subsection, as this procedure was shown, in references [46, 47, 50], to lead to fairly good accuracy in the analysis of relevant linear vibration problems. Substituting for equations (1), (2) and (3) into the Lagrangian function L,

$$L = KE - V = (ml/2)\int_0^1 \{[1 + \mu\delta(\xi - \eta)](\dot{x}^2 + \dot{y}^2) - (EI\lambda^6/m)(x'y'' - x''y')^2\}\,\mathrm{d}\xi \tag{4}$$

and, applying Hamilton's principle,

$$\delta\int_{t_1}^{t_2} L\,\mathrm{d}t = 0 \tag{5}$$

leads to the variational problem

$$\int_{t_1}^{t_2}\int_0^1 F(\xi, t, x, y, \dot{x}, \dot{y}, \ldots, x'', y'')\,\mathrm{d}\xi\,\mathrm{d}t = \min. \tag{6}$$

From the calculus of variations, the minimization of equation (6) is equivalent to satisfying the Euler–Lagrange equations

$$\frac{\partial F}{\partial x}-\frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{x}}-\frac{\partial}{\partial \xi}\frac{\partial F}{\partial x'}+\frac{\partial^2}{\partial \xi^2}\frac{\partial F}{\partial x''}+\frac{\partial^2}{\partial \xi\,\partial t}\frac{\partial F}{\partial \dot{x}'}=0, \tag{7}$$

$$\frac{\partial F}{\partial y}-\frac{\partial}{\partial t}\frac{\partial F}{\partial \dot{y}}-\frac{\partial}{\partial \xi}\frac{\partial F}{\partial y'}+\frac{\partial^2}{\partial \xi^2}\frac{\partial F}{\partial y''}+\frac{\partial^2}{\partial \xi\,\partial t}\frac{\partial F}{\partial \dot{y}'}=0. \tag{8}$$

In addition to equations (7) and (8), the variables x and y are related by the subsidiary relation

$$x'^2+y'^2=l^2. \tag{9}$$

Upon carrying out the necessary differentiations in equation (8) and using equation (9) and its derivatives to eliminate the variable x, one obtains the following non-linear partial differential equation for the transverse vibration of the beam:

$$[1+\mu\delta(\xi-\eta)]\ddot{y}+\beta^2 y''''=\beta^2\lambda^2(1-\lambda^2 y'^2)^{-2}[2y'y''y'''+y''^3+\lambda^2(y'^2y''^3-2y'^3y''y''')], \tag{10}$$

where $\beta^2=EI/ml^4$. It is pointed out that an independent equation for x similar to equation (10) can be derived by using equations (7) and (9). It is also noted that for $\mu=0$ equation (10) becomes identical to that in reference [1]. The highly non-linear equation (10) may be simplified by noting that $|\lambda|<1$. Thus, expanding the term $(1-\lambda^2y'^2)^{-2}$ into a power series and retaining non-linear terms up to an order of five, one obtains, after some algebraic manipulations,

$$[1+\mu\delta(\xi-\eta)]\ddot{y}+\beta^2 y''''=\beta^2\lambda^2(2y'y''y'''+y''^3)+\beta^2\lambda^4(2y'^3y''y'''+3y''^3y'^2). \tag{11}$$

In order to compare equation (11) with that obtained by Takahashi [31], one sets $\mu=0$, changes the differentiation with respect to ξ to a differentiation with respect to s, and obtains

$$\ddot{y}=(EI/m)y''''=-(EI/m)[2y'y''y'''+y''^3+2y'^3y''y'''+3y''^3y'^2], \tag{12}$$

which is the same equation as that obtained in reference [31] except that the non-linear (bracketed) term in equation (12) is the negative of that obtained in reference [31]. Consequently, as will be shown in the next two subsections and as was shown in reference [1], the model in equation (11), or equation (12), predicts that the first mode of a clamped–free, or free–free [1], beam has a period–amplitude characteristic of the hardening type, contrary to the softening type predicted in reference [31]. The difference between the two predictions is due to the difference in the sign, rather than in the nature, of the non-linearities in these two models. Note that the non-linear terms in equation (11) are all the static, hardening type due to potential energy stored in bending, and none is of the inertia (softening) type. Thus, as will be shown in the next subsection, for a clamped–free beam, equation (11) yields period–amplitude characteristics of the hardening type regardless of the mode number. It is to be noted that, although the beam in equation (11) was assumed to be inextensible, the derivation of this equation, which took into account the axial inertia and non-linear curvature, however, did not account for the inextensibility condition. On the other hand, the simplified version, used in reference [5], of the equations derived by Crespo da Silva and Glynn [8] and that derived by Zavodney and Nayfeh [4] (see e.g., references [5, 9] for details), which account for the inextensibility of the beam, include non-linear inertia softening terms in addition to the static hardening terms. The inextensibility condition, which induced a force tangent to the neutral axis of the beam [33], will be included in formulation II in section 3. For comparison purposes, however, in the next section examples are presented of the period–amplitude variation for the first few modes of the cantilever beam in Figure 1, obtained by using the model in equation (11).

2.3. THE NON-LINEAR TEMPORAL PROBLEM

The governing non-linear partial differential equation (11) does not admit a closed form solution. However, this equation may be reduced to an ordinary differential equation in time by applying any of the variational methods, such as the Galerkin or the Rayleigh–Ritz method. Accordingly, an approximate unimodal solution is assumed to be of the form

$$y(\xi, t) = \phi(\xi)u(t), \tag{13}$$

where $u(t)$ is an unknown function of time and $\phi(\xi)$ is a normalized eigenfunction of the associated linear problem. In this work, the Rayleigh–Ritz method which, unlike the Galerkin method, does require that $\phi(\xi)$ of the true eigenfunction of the linear problem, is used for reasons of simplicity and computational effort, but at the expense of reduced accuracy, which is small for a small attached mass to beam mass ratio μ, as will be illustrated later in this section. In this case, $\phi(\xi)$ is taken to be the eigenfunction of the linear base beam corresponding to the beam under consideration (the same, assumed linear, beam as in Figure 1, but without the attached lumped mass), governed by

$$\ddot{y} + \beta^2 y'''' = 0, \tag{14}$$

subject to the boundary conditions

$$y(0, t) = y''(1, t) = y'''(1, t) = 0, \qquad EIy''(0, t) = K_r l y'(0, t), \tag{15}$$

with the restriction $\eta < 1$: that is, the lumped mass M cannot be placed at the tip of the beam. Note that the boundary conditions in equation (15) are formulated in terms of the dimensionless arc length ξ. The first three of these boundary conditions are the same as those formulated in terms of the variable x in the classical linear beam theory [1]. The fourth boundary condition, however, may be assumed to be the same as that formulated in terms of x in the linear theory, provided that the rotation at the clamped end of the beam is small. Also note that the linear problem given by equation (14) does not take into account the presence of the lumped mass M. Thus the eigenfunctions $\phi(\xi)$ obtained for this problem are not the true linear ones. The true eigenfunctions, which take complicated forms, however, can be obtained by solving the linear problem [47, 48]

$$[1 + \mu\delta(\xi - \eta)]\ddot{y} + \beta^2 y'''' = 0, \tag{16}$$

subject to the boundary conditions in equation (15). The solution of equation (16), by using the Laplace transform method, has been obtained by Liu and Huang [48], and for the sake of simplicity will not be used in the present work. This is also justified by the fact that, for small μ, the eigenfunction of each of the first few modes of the base beam in equation (14) closely resembles that of beam with attached mass in equation (16) [47].

Using the standard method of separation of variables to solve equation (14), one obtains the transcendental frequency equation [46, 49]

$$(S/p)[1 + \cos p \cosh p] + \cos p \sinh p - \sin p \cosh p = 0 \tag{17}$$

and the eigenfunctions

$$\phi(\xi) = (1/r)[\sin p\xi - A \sinh p\xi - B(\cos p\xi - \cosh p\xi)], \tag{18}$$

where $r = \phi(1)$ is a scaling factor chosen so that $|\phi(\xi)|_{\max} = 1$, $S = K_r l/EI$ is a dimensionless stiffness parameter, $p^4 = m\omega^2 l^4/EI$ is a dimensionless frequency parameter, ω is the natural frequency of the base beam, and A and B are weighting constants associated with each mode, defined as

$$A = \frac{(S/2p)[\cos p + \cosh p] - \sin p}{(S/2p)[\cos p + \cosh p] + \sinh p}, \qquad B = \frac{\sin p + A \sinh p}{\cos p + \cosh p}. \tag{19, 20}$$

Note that as $S \to \infty$ and $A \to 1$, the beam becomes rigidly clamped, and equations (17) and (18) reduce to those that one obtains for a rigidly clamped–free beam [46, 50]. Equations (17)–(20) will be solved numerically, as explained later in this section, to obtain $\phi(\xi)$ and p of each of the first four modes for given S. Note that in the present formulation the attached mass M is treated as an applied inertial load which is accounted for in the equation of motion, and thus is allowed to affect the frequency but not the mode shape of the linear problem. In order to see the effect of the present procedure on the linear frequency, one substitutes equation (13), with $\phi(\xi)$ as given by equation (18), into equation (16), multiplies equation (16) by the same $\phi(\xi)$, integrates the results with respect to ξ from 0 to 1 and applies the boundary conditions in equation (15). Then, upon assuming simple harmonic motion with frequency ω, one obtains the dimensionless linear frequency parameter $\Omega = \omega/\beta = (\alpha_2/\alpha_1)^{1/2}$, where α_1 and α_2 are given by equation (22). In Table 1 is shown a comparison of the dimensionless linear frequency parameters $\Omega^{1/2}$ of equation (16) calculated in this way, i.e., by using the present Rayleigh–Ritz method and a numerical procedure explained later on, and that obtained in reference [48] by solving equation (16) by using the Laplace transform method for selected values of S, μ and η. As can be seen from these results, the present procedure, in addition to its simplicity, yields reasonably accurate results for the first few modes even when the attached mass to beam mass ratio μ is not too small.

TABLE 1

The linear frequency parameter $\Omega^{1/2} = (2\pi/v_0)^{1/2}$, *for the beam shown in Figure 1*

S	μ	η	Mode	Present equation (32)	Reference [48]
0·10	0·5	0·5	1	0·67990	0·679870
0·10	0·5	0·5	2	3·45429	3·551608
0·10	0·5	0·5	3	6·86493	6·957619
0·10	0·5	0·5	4	8·74384	9·363541
0·10	2·0	0·5	1	0·58595	0·585824
0·10	2·0	0·5	2	2·82837	3·266057
0·10	2·0	0·5	3	6·37858	6·885321
0·10	2·0	0·5	4	7·03278	9·040730
∞	0·5	0·5	1	1·78033	1·778434
∞	0·5	0·5	2	3·93810	4·032716
∞	0·5	0·5	3	7·85324	7·853989
∞	0·5	0·5	4	9·24614	9·981989
∞	0·5	0·3	1	1·85804	1·857729
∞	0·5	0·3	2	4·20450	4·174915
∞	0·5	0·3	3	6·49137	6·877625
∞	0·5	0·3	4	10·15139	10·668627

Substituting equation (13) into equation (11), multiplying by $\phi(\xi)$, integrating with respect to ξ from 0 to 1, and using the boundary conditions in equation (15), one obtains the unimodal Duffing type temporal problem [1],

$$\alpha_1 \ddot{u} + \beta^2(\alpha_2 u + \alpha_3 \lambda^2 u^3 + \alpha_4 \lambda^4 u^5) = 0, \tag{21}$$

where

$$\alpha_1 = \int_0^1 \phi^2\,\mathrm{d}\xi + \mu\phi^2(\eta), \qquad \alpha_2 = \int_0^1 \phi\phi''''\,\mathrm{d}\xi = p^4 \int_0^1 \phi^2\,\mathrm{d}\xi,$$

$$\alpha_3 = \int_0^1 \phi'^2\phi''^2\,\mathrm{d}\xi, \qquad \alpha_4 = \int_0^1 \phi'^4\phi''^2\,\mathrm{d}\xi. \tag{22}$$

Note that the integrals in equations (22) can be evaluated analytically; however, the evaluation involves laborious mathematical operations. Therefore, these integrals were evaluated numerically by using Simpson's rule, in which an integration step size of $\Delta\xi = 0{\cdot}001$ was found to yield good accuracy. A bisection method was used to evaluate the first four roots p_i, to 10^{-7} accuracy, of equation (17) for various values of stiffness parameter S. The following are the results of samples of these calculations. For the case $S = 1000$, $\mu = 0{\cdot}5$, $\eta = 0{\cdot}5$: $p_1 = 1{\cdot}873233$, $A = 0{\cdot}9949165$, $B = 1{\cdot}3568937$, $\alpha_1 = 0{\cdot}3080842$, $\alpha_2 = 3{\cdot}081346$, $\alpha_3 = 1{\cdot}2574844$, $\alpha_4 = 1{\cdot}1076840$; $p_4 = 10{\cdot}9846758$, $A = 0{\cdot}9785036$, $B = 0{\cdot}9784702$, $\alpha_1 = 0{\cdot}5008872$, $\alpha_2 = 3647{\cdot}586$, $\alpha_3 = 60077{\cdot}86$, $\alpha_4 = 2377061{\cdot}3$. For the case $S = 10$, $\mu = 0{\cdot}4$, $\eta = 0{\cdot}8$: $p_1 = 1{\cdot}7227415$, $A = 0{\cdot}6528884$, $B = 1{\cdot}0074396$, $\alpha_1 = 0{\cdot}4918255$, $\alpha_2 = 2{\cdot}3736932$, $\alpha_3 = 0{\cdot}8338772$, $\alpha_4 = 0{\cdot}6435092$; $p_4 = 10{\cdot}5217848$, $A = 0{\cdot}3221550$, $B = 0{\cdot}3221150$, $\alpha_1 = 0{\cdot}4104304$; $\alpha_2 = 3121{\cdot}8115$, $\alpha_3 = 47551{\cdot}04$, $\alpha_4 = 1762706{\cdot}4$. These results, and others not shown, indicate that the coefficients α_i in equation (21) are positive regardless of the mode number of values of S, μ and η. This fact is also evident from equation (22), since each α_i is given as the integral of even powered, or products of even powered, functions. Thus, the non-linearities in equation (21) for given S, μ and η are of the hardening type regardless of the mode number considered. Nonetheless, for comparison purposes, some selected examples of period–amplitude variations for various modes obtained by using a dimensionless version of equation (21), derived in this section, will be presented in the next subsection.

Note that the numerical values of the coefficients α_3 and α_4 of the non-linear terms in equation (21), in general, increase sharply and attain large values as the mode number for a given beam is increased. For convenience, equation (21) is converted to the dimensionless form

$$\ddot{q} + q + \varepsilon_1 q^3 + \varepsilon_2 q^5 = 0, \tag{23}$$

where

$$\varepsilon_1 = \alpha_3/(p^2\alpha_2), \qquad \varepsilon_2 = \alpha_4/(p^4\alpha_2). \tag{24}$$

In equation (23), dots denote derivatives with respect to the new dimensionless time $t^* = (\alpha_2/\alpha_1)^{1/2}[EI/(ml^4)]^{1/2}t$, and $q = pu/l$ is a dimensionless beam displacement. Note that in the above scaling procedure, which is similar to that used in reference [5], no prior assumption is made regarding the relative order of magnitude of various terms in equation (21). In particular, the arbitrary use of the dimensionless frequency parameter p in the definition of the displacement scaling factor is done for numerical convenience, so that the coefficients ε_1 and ε_2 in equation (23) would not attain large values for the higher modes;

i.e., this does not change the relative order of magnitude of various terms in the original equation of motion. The following are examples of the results calculated by using equation (24). For the case $S = 1000$, $\mu = 0{\cdot}5$, $\eta = 0{\cdot}5$: $\varepsilon_1 = 0{\cdot}1162996$, $\varepsilon_2 = 0{\cdot}0291950$ for $p = p_1$, and $\varepsilon_1 = 0{\cdot}1365006$, $\varepsilon_2 = 0{\cdot}0447596$ for $p = p_4$; for the case $S = 10$, $\mu = 0{\cdot}4$, $\eta = 0{\cdot}8$: $\varepsilon_1 = 0{\cdot}1183688$, $\varepsilon_2 = 0{\cdot}0307786$ for $p = p_1$, and $\varepsilon_1 = 0{\cdot}1375861$, $\varepsilon_2 = 0{\cdot}0460697$, for $p = p_4$. The analytical solution of equation (23) is carried out in the next section.

2.4. A TIME TRANSFORMATION SOLUTION

The sample calculations of the parameters α_i in equation (21) and ε_i in equation (23), presented in the previous section, and examination of the various terms in these equations indicates that, for the range of amplitudes to be considered in this work (u/l up to 0·7 for the first mode and up to 0·5 for the second and higher modes, $q = pu/l$), the non-linear oscillator described by either of these equations is in general strongly non-linear, especially for the second and higher modes. Therefore, an approximate analytic solution for this oscillator given by using perturbation methods will not be adequate for large amplitudes of the vibrations, as these methods are restricted to the solution of weakly non-linear oscillators: e.g., when the amplitude of vibration is restricted to values for which the non-linear terms in equation (23) remain small, (in this case less than unity), compared to the linear ones. In the present work an approximate solution of equation (23) is obtained by using the time transformation method described in detail in reference [2]. This technique differs from other approximate analytic methods in that the oscillation period τ may be obtained to any desired degree of accuracy for the strongly non-linear case with relatively less computational effort. According to this method, a single valued transformation $T(t^*)$ is sought between the time t^* and a new time T, such that in the new time domain T the solution of equation (23) is simple harmonic with period equal to 2π: i.e., $q(T) = c\cos(T)$, where $T(0) = 0$ and c is the amplitude of vibration. Writing equation (23) with T as the independent variable and substituting for $q(T) = c\cos(T)$ in the result, one obtains [2]:

$$(1 - f^2)\cos T - ff'\sin T + \varepsilon_1 c^2(\cos T)^3 + \varepsilon_2 c^4(\cos T)^5 = 0, \tag{25}$$

where primes denote differentiation with respect to T and $f = \mathrm{d}T/\mathrm{d}t^*$. Upon using the trigonometric identities $\cos^3 T = \frac{1}{4}(3\cos T + \frac{1}{4}\cos 3T)$ and $\cos^5 T = \frac{1}{16}(10\cos T + 5\cos 3T + \cos 5T)$, equation (25) becomes

$$(1 - f^2)\cos T - ff'\sin T + (\tfrac{3}{4}\varepsilon_1 c^2 + \tfrac{5}{8}\varepsilon_2 c^4)\cos T + \left(\frac{\varepsilon_1 c^2}{4} + \tfrac{5}{16}\varepsilon_2 c^4\right)\cos 3T + \frac{\varepsilon_2 c^4}{16}\cos 5T = 0. \tag{26}$$

A periodic solution of period 2π may be obtained for $f(T)$ by substituting the series [2]

$$f^2(T) = \sum_{n=0,2}^{\infty} G_n \cos(nT) \tag{27}$$

into equation (26) and equating the coefficients of $\sin(nT)$ and $\cos(nT)$ to zero: i.e., by using the HB method. Taking the square root of the result, using the relation $f = \mathrm{d}T/\mathrm{d}t^*$, integrating from 0 to 2π in T and noting that the period τ in T is 2π, one obtains

$$\tau = (1 + \tfrac{3}{4}\varepsilon_1 c^2 + \tfrac{5}{8}\varepsilon_2 c^4)^{-1/2}\int_0^{2\pi} [1 + H_2\cos 2T + H_4\cos 4T]^{-1/2}\,\mathrm{d}T, \tag{28}$$

where

$$H_2 = \frac{\varepsilon_1 c^2/4 + \varepsilon_2 c^4/3}{1 + \frac{3}{4}\varepsilon_1 c^2 + \frac{5}{8}\varepsilon_2 c^4}, \qquad H_4 = \frac{\varepsilon_2 c^4/24}{1 + \frac{3}{4}\varepsilon_1 c^2 + \frac{5}{8}\varepsilon_2 c^4}. \tag{29, 30}$$

Note that $H_2 < 1/3$ and $H_4 < 8/15$ for all $\varepsilon_1 c^2$ and $\varepsilon_2 c^4$, so that $H_2 \cos 2T + H_4 \cos 4T < 1$ for all T. Thus the bracketed radical in equation (28) may be expanded in a convergent power series and then integrated term by term. This leads to the period v, in the dimensionless time $t\beta$ [2],

$$v = 2\pi(\alpha_1/\alpha_2)^{1/2}(1 + \tfrac{3}{4}\varepsilon_1 c^2 + \tfrac{5}{8}\varepsilon_2 c^4)^{-1/2}[1 + \tfrac{3}{16}(H_2^2 + H_4^2) - \tfrac{15}{64}(H_2^2 H_4) + \tfrac{105}{1024}(H_2^4 + 4H_2^2 H_4^2 + H_4^4) + \cdots]. \tag{31}$$

Note that the leading term in equation (31) represents the effect of the fundamental harmonic of the non-linearities, and is the result that one obtains by using the single term harmonic balance method or the classical perturbation methods, while the bracketed (correction) term represents a measure of the importance of the higher harmonics. Since H_2 and H_4 are less than unity for all $\varepsilon_1 c^2$ and $\varepsilon_2 c^4$, the series (bracketed term) in equation (31) is a rapidly convergent one, and only the leading terms in this series are needed to obtain an accurate approximation for the period. The linear period parameter of the beam v_0 may be obtained by setting all of the non-linear terms in equation (31) to zero; that is,

$$v_0 = 2\pi(\alpha_1/\alpha_2)^{1/2}. \tag{32}$$

2.5. NUMERICAL CALCULATIONS AND RESULTS

With the aid of equations (17)–(20) and equation (22), the parameters ε_1 and ε_2 in equation (24) have been evaluated for various values of S, μ and η. In each case the first four roots of equation (17) were found by using a bisection algorithm. All of the numerical computations were programmed in double precision on the VAX/VMS version 5 digital computer. Equations (29), (30) and (31) were then used to calculate the dimensionless period v for each selected value of the dimensionless displacement a ($a = c/p = b/l$; see Figure 1), in the range 0 to 0·7. Examples of the results of these calculations showing the typical general period–amplitude behavior obtained for each of the first four modes for various values of S, μ and η are displayed in Figures 2–4. As expected, because the non-linearities in equation (23) are all of the hardening type, as was shown in section 2.3, these results, and others shown in reference [51] for the first mode, as well as others not shown, indicate that the period–amplitude characteristics for each of the first four modes of the present beam model in equation (11) are of the hardening type regardless of the

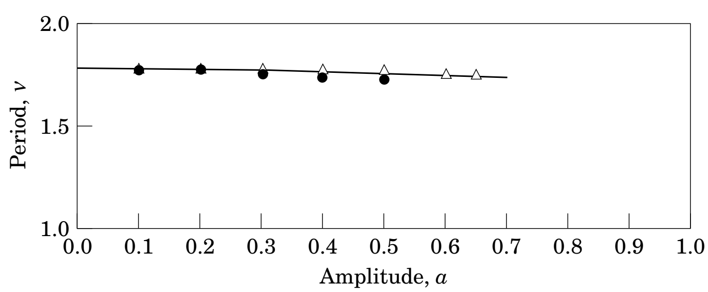

Figure 2. The non-linear period of first mode versus the non-dimensional amplitude for the case $\mu = 0{\cdot}0$, $S = 10^7$. ——, Formulation I, equation (31); ●, reference [1]; △, reference [30].

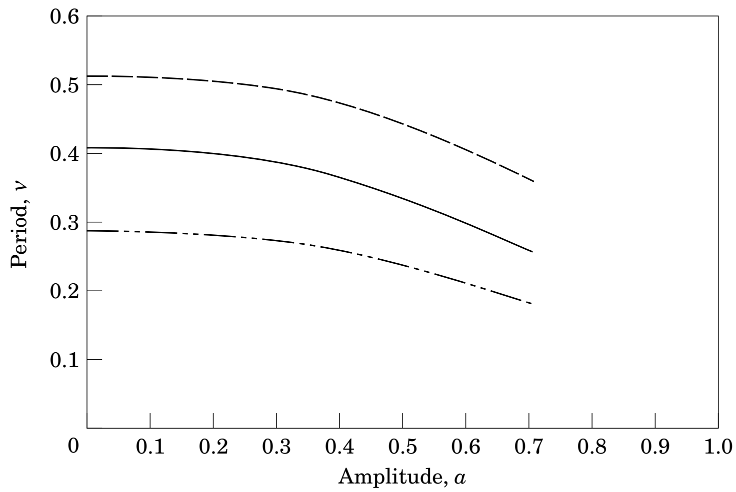

Figure 3. The non-linear period of the second mode versus the non-dimensional amplitude, given by formulation I, equation (31), for the case $\mu = 0\cdot5$. —· ·—, $S = 10^7$, $\mu = 0\cdot0$; ——, $S = 10^7$, $\mu = 0\cdot5$; —, $S = 1\cdot0$, $\mu = 0\cdot5$.

values of S, μ and η. In Figure 2 it is shown that for the first mode the TT results are in good agreement with those in reference [1], and are also in good agreement with those in reference [30] which include non-linear inertia terms. These results also indicate that the TT method is a viable technique for the solution of the non-linear conservative oscillator in equation (23). The reason why the present beam model in equation (11), which does not include non-linear inertia terms, predicts the correct qualitative as well as quantitative behavior for the first mode, and fails to predict the correct qualitative behavior of the second and higher modes of an inextensible beam, is due to the fact that—as will be shown in the next section—for such beams the curvature (hardening) non-linearities dominate in the lower modes, while inertia non–linearities dominate in the higher modes [8, 15]. The absence of inertia non-linearities in equation (11) may be explained by the fact that the

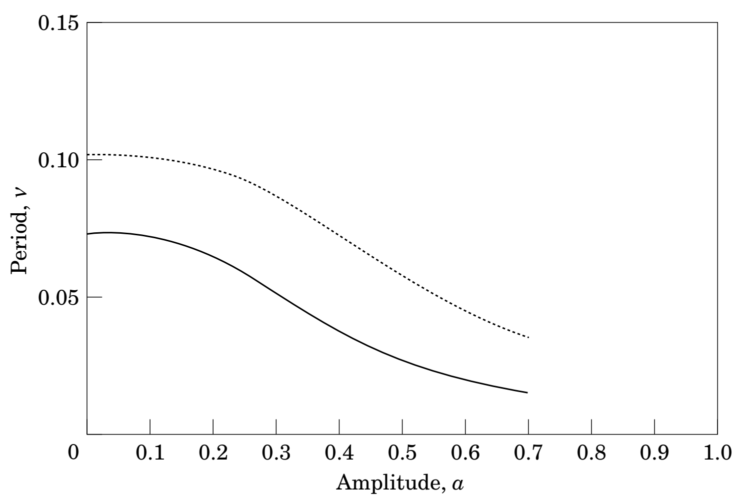

Figure 4. The non-linear period versus the dimensionless amplitude given by formulation I, equation (31) for the case $S = 10^7$, $\mu = 0\cdot5$, $\mu = 0\cdot5$, —··—·—, third mode; ——, fourth mode.

derivation of this equation did not take into account the inextensibility of the beam. A derivation of the equation of motion for the beam in Figure 1 which takes into account the condition of inextensibility of the beam is presented in the next section.

3. EQUATION OF MOTION: FORMULATION II

The beam in Figure 1 is again assumed to have the same geometric and material properties and undergoes large amplitude planar flexural free vibrations as described in section 2.1. Thus, the Lagrangian of this beam, which takes into account the axial inertia and non-linear curvature, is again given by equation (4). Using the subsidiary equation (9) and its derivatives to eliminate x' and x'', assuming that $(\lambda y)^2 \ll 1$, one finds that the beam Lagrangian in equation (4), with non-linear curvature terms retained up to fifth order, becomes

$$L = (ml/2) \int_0^1 \{[1 + \mu\delta(\xi - \eta)](\dot{x}^2 + \dot{y}^2) - (EI\lambda^4/2m)[y''^2 + (\lambda y' y'')^2 + \lambda^4 y'^4 y''^2]\} \, \mathrm{d}\xi. \quad (33)$$

Here, as in section 2, primes denote derivatives with respect to the dimensionless arc length $\xi = s/l$, and dots derivatives with respect to real time t. Assuming the beam to be inextensional implies that the length of the neutral axis of the beam remains constant. This leads to the constraint relation [8]

$$(1 + \lambda x')^2 + (\lambda y')^2 = 1. \quad (34)$$

Equation (34) may be rewritten as

$$1 + \lambda x' = [1 - (\lambda y')^2]^{1/2}. \quad (35)$$

Upon assuming $(\lambda y')^2 \ll 1$, expanding the right side of equation (35) in a power series and retaining terms up to fifth order, equation (35), when integrated from 0 to an arbitrary value of ξ, yields

$$x = -\frac{1}{2} \int_0^\xi (\lambda y'^2 + \tfrac{1}{4}\lambda^3 y'^4) \, \mathrm{d}\chi. \quad (36)$$

Differentiating equation (36) with respect to time t, one obtains

$$\dot{x} = -\frac{1}{2} \dot{\left[\int_0^\xi (\lambda y'^2 + \tfrac{1}{4}\lambda^3 y'^4) \, \mathrm{d}\chi \right]}. \quad (37)$$

Substituting equation (37) into equation (33), one obtains the one-dimensional beam Lagrangian

$$L = (ml/2) \int_0^1 [1 + \mu\delta(\xi - \eta)] \left\{ \dot{y}^2 + \tfrac{1}{4} \dot{\left[\left(\int_0^\xi \lambda y'^2 + \tfrac{1}{4}\lambda^3 y'^4 \right) \mathrm{d}\chi \right]}^2 \right\} \mathrm{d}\xi$$

$$- (EI\lambda^3/2) \int_0^1 [\, y''^2 + (\lambda y' y'')^2 + \lambda^4 y''^2 y'^4] \, \mathrm{d}\xi \quad (38)$$

It is to be noted that, if one wishes, one may apply the Euler–Lagrange equations (8), after integrating some of the terms in equation (38) and application of boundary conditions, to obtain an integro-differential field equation of motion of the beam and the corresponding boundary conditions. Since the interest here, however, is to obtain an equivalent single mode temporal problem, one can avoid the step of derivation of the integro-differential equation of motion and instead apply the assumed single linear mode Rayleigh–Ritz procedure, as in section 2.3. Accordingly, substituting a solution of the form of equation (13) into equation (38), one obtains the beam discrete Lagrangian, after carrying out some algebraic manipulations,

$$L = (ml/2)\{\alpha_1\dot{u}^2 + \alpha_3\lambda^2u^2\dot{u}^2 + \alpha_4\lambda^2u^2\dot{u}^2 + \alpha_5\lambda^4u^4\dot{u}^2 + \alpha_6\lambda^4u^4\dot{u}^2 - (EI\lambda^4/m)(\alpha_2u^2 + \alpha_7\lambda^2u^4 + \alpha_8\lambda^4u^6)\}, \quad (39)$$

where

$$\alpha_1 = \int_0^1 \phi^2\,\mathrm{d}\xi + \mu\phi^2(\eta), \qquad \alpha_2 = \int_0^1 \phi''^2\,\mathrm{d}\xi,$$

$$\alpha_3 = \int_0^1 \left(\int_0^\xi \phi'^2\,\mathrm{d}\chi\right)^2 \mathrm{d}\xi, \qquad \alpha_4 = \mu\left[\left(\int_0^\xi \phi'^2\,\mathrm{d}\chi\right)^2\right]_{\xi=\eta},$$

$$\alpha_5 = \int_0^1 \left[\left(\int_0^\xi \phi'^2\,\mathrm{d}\chi\right)\left(\int_0^\xi \phi'^4\,\mathrm{d}\chi\right)\right]\mathrm{d}\xi, \qquad \alpha_6 = \mu\left[\left(\int_0^\xi \phi'^2\,\mathrm{d}\chi\right)\left(\int_0^\xi \phi'^4\,\mathrm{d}\chi\right)\right]_{\xi=\eta}, \quad (40)$$

$$\alpha_7 = \int_0^1 \phi'^2\phi''^2\,\mathrm{d}\xi, \qquad \alpha_8 = \int_0^1 \phi''^2\phi'^4\,\mathrm{d}\xi.$$

Here, as in section 2.3, the assumed mode shape functions ϕ, given by equations (18)–(20), are those of the base beam described by equations (14) and (15) (see Figure 5). Upon application of the Euler–Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{u}}\right) - \partial L/\partial u = 0 \quad (41)$$

and following a scaling procedure similar to that used in section 2.4, one obtains the dimensionless unimodal temporal problem

$$\ddot{q} + q + \varepsilon_1 q^2\ddot{q} + \varepsilon_1 q\dot{q}^2 + \varepsilon_2 q^4\ddot{q} + 2\varepsilon_2 q^3\dot{q}^2 + \varepsilon_3 q^3 + \varepsilon_4 q^5 = 0, \quad (42)$$

where

$$\varepsilon_1 = \alpha_3\alpha_4/(p^4\alpha_1), \qquad \varepsilon_2 = \alpha_5\alpha_6/(p^4\alpha_1), \qquad \varepsilon_3 = 2\alpha_7/\alpha_2, \qquad \varepsilon_4 = 2\alpha_8/\alpha_2. \quad (43)$$

Here, dots are derivatives with respect to the dimensionless time $t^* = (EI\lambda^4/m)^{1/2}(\alpha_2/\alpha_1)^{1/2}t$ and $q = pu/l$ is, as in section 2.3, a dimensionless deflection at the beam tip. In equation (42) the first four non-linear terms are the result of kinetic energy of axial motion. The first and third of these non-linear terms are of the softening inertia type, while the second and fourth terms are of the hardening inertia type. These terms, which are absent in formulation I (see equation (23)), arise as a result of using the inextensibility condition, equation (37), in the present formulation II. The last two non-linear terms in equation (42), which are the same as those in equation (23), are static hardening non-linearities due to potential energy stored in bending. A numerical procedure, similar to that described in section 2.3, in conjunction with a symbolic manipulator program, was used to evaluate

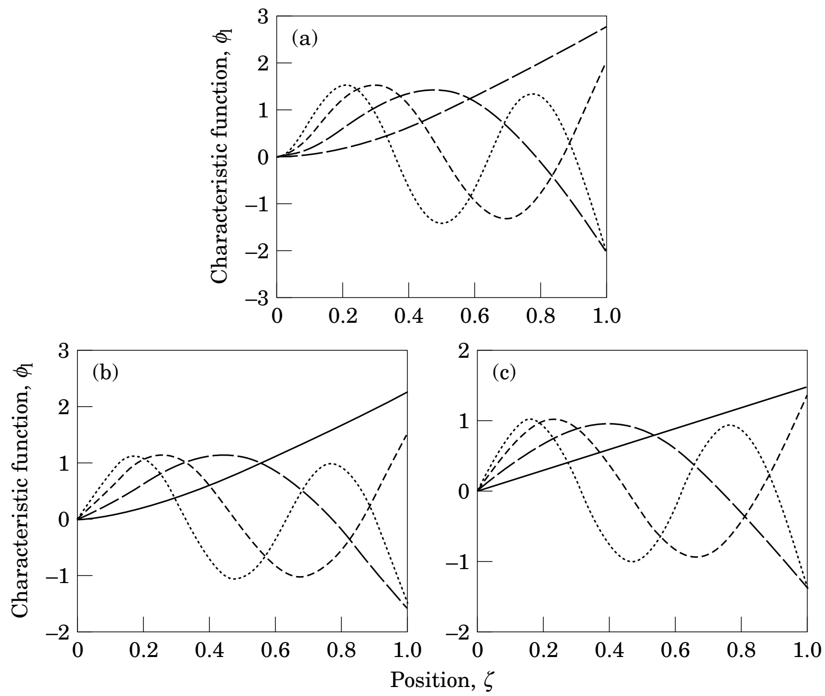

Figure 5. The assumed mode shapes of the vibrating beam system, from equations (18)–(20). (a) $S = 10^7$; (b) $S = 10$; (c) $S = 0{\cdot}5$. ——, ϕ_1; –––, ϕ_2; -----, ϕ_3; · · ·, ϕ_4.

the parameters ε_i in equation (42) by using equations (40) and (43) and equations (17)–(20), for various values of base stiffness S, mass ratio μ and position η of the intermediate mass M. Examples of the results of these calculations for some selected values of S, μ and η are shown in Table 2. These results, and others not shown, indicate that the non-linear inertia terms are dominant in the higher modes (i.e., $\varepsilon_1 > \varepsilon_3$ and $\varepsilon_2 > \varepsilon_4$), while the

TABLE 2

Values of dimensionless parameters ε_i in equation (42) *for various values of S, μ and η* (*p is the frequency parameter*)

S	μ	η	Mode	p	ε_1	ε_2	ε_3	ε_4
10^7	0·00	0·00	1	1·875104	0·326845	0·129579	0·232598	0·087584
10^7	0·00	0·00	2	4·694091	1·642033	0·913055	0·313561	0·204297
10^7	0·00	0·00	3	7·854757	4·051486	1·665232	0·281418	0·149677
10^7	0·00	0·00	4	10·99554	8·205578	3·145368	0·272313	0·133708
10^7	0·50	0·50	1	1·875104	0·303844	0·115076	0·232598	0·087584
10^7	0·50	0·50	2	4·694091	0·893981	0·467399	0·313561	0·204297
10^7	0·50	0·50	3	7·854757	5·074636	1·971962	0·281418	0·149677
10^7	0·50	0·50	4	10·99554	5·371626	1·995820	0·272313	0·133708
10·0	0·20	0·60	1	1·722741	0·333861	0·131914	0·333105	0·129923
10·0	0·20	0·60	2	4·399523	1·512930	0·802700	0·379417	0·250283
10·0	0·20	0·60	3	7·451057	4·278991	1·715424	0·318201	0·173313
10·0	0·20	0·60	4	10·52178	10·015007	3·820495	0·297109	0·149231
10·0	0·40	0·60	1	1·722741	0·327231	0·127022	0·333105	0·129923
10·0	0·40	0·60	2	4·399523	1·356964	0·696043	0·379417	0·250283
10·0	0·40	0·60	3	7·451057	4·233389	1·657464	0·318201	0·173313
10·0	0·40	0·60	4	10·52178	11·178563	4·200250	0·297109	0·149231

non-linear static terms are dominant in the lower modes. These results also indicate that for the range of amplitude to be considered in this work, i.e., when $u = q/p$ is of order l, the non-linear oscillator in equation (42) is strongly non-linear. The TT procedure [2], with the same procedural steps as summarized in section 2.4 is, therefore, again chosen to obtain an approximate analytic solution for the period of the non-linear oscillator in equation (42). Accordingly, upon transforming equation (42) to a new time domain T, defining $f = \mathrm{d}T/\mathrm{d}t^*$, letting a periodic solution in T be $q = c \cos T$, and assuming a series solution for f^2 in the form of equation (27), i.e.,

$$f^2 = \sum_{n=0,2}^{\infty} G_n \cos nT, \tag{44}$$

equation (42) becomes, after using trigonometric identities to simplify some of the trigonometric terms,

$$\begin{aligned}
&\left(1 - \sum_{n=0,2}^{\infty} G_n \cos nT\right)\cos T + \frac{1}{2}\left(\sum_{n=2,4}^{\infty} nG_n \sin nT\right)\sin T \\
&\quad - 2\varepsilon_1 c^2 \cos^3 T\left(\sum_{n=0,2}^{\infty} G_n \cos nT\right) + \frac{1}{2}\,\varepsilon_1 c^2 \cos^2 T \sin T\left(\sum_{n=0,2}^{\infty} nG_n \sin nT\right) \\
&\quad + \varepsilon_1 c^2 \cos T\left(\sum_{n=0,2}^{\infty} G_n \cos nT\right) - 3\varepsilon_2 c^4 \cos^5 T\left(\sum_{n=0,2}^{\infty} G_n \cos nt\right) \\
&\quad + \tfrac{1}{2}\,\varepsilon_2 c^4 \cos^4 T \sin T\left(\sum_{n=2,4}^{\infty} nG_n \sin nt\right) + 2\varepsilon_2 c^4 \cos^3 T\left(\sum_{n=0,2}^{\infty} G_n \cos nT\right) \\
&\quad + \varepsilon_3 c^2 \cos^3 T + \varepsilon_4 c^4 \cos^5 T = 0.
\end{aligned} \tag{45}$$

Note that, in section 2.3, closed form solutions for all of the coefficients G_n for the oscillator in equation (23) were obtained from those given in reference [2] for a generalized, all static non-linearities, version of equation (23). For the present oscillator in equation (42), which has inertial as well as static non-linearities, finding closed form solutions for all of the G_n in equation (45), which are not available in reference [2], appears to be a difficult task which requires special attention. Therefore, in the present work, for the sake of simplicity, approximate closed form solutions for the coefficients G_n in equation (45) are sought be assuming $G_n = 0$ for $n > 4$. Consequently, upon application of the HB method, i.e., by equating the coefficient of each harmonic in equation (45) to zero, with $n = 4$, leads to the following set of the independent, simultaneous linear algebraic equations for the coefficients G_0, G_2 and G_4:

$$(1 + \tfrac{1}{2}\varepsilon_1 c^2 + \tfrac{3}{8}\varepsilon_2 c^4)G_0 + \tfrac{1}{4}(\varepsilon_1 c^2 + \varepsilon_2 c^4)G_2 + (\tfrac{1}{16}\varepsilon_2 c^4)G_4 = 1 + \tfrac{3}{4}\varepsilon_3 c^2 + \tfrac{5}{8}\varepsilon_4 c^4, \tag{46}$$

$$(\tfrac{1}{2}\varepsilon_1 c^2 + \tfrac{7}{16}\varepsilon_2 c^4)G_0 + (1 + \tfrac{3}{8}\varepsilon_1 c^2 + \tfrac{5}{16}\varepsilon_2 c^4)G_2 + (-1/2 + \tfrac{1}{16}\varepsilon_2 c^4)G_4 = \tfrac{1}{4}\varepsilon_3 c^2 + \tfrac{5}{16}\varepsilon_4 c^4, \tag{47}$$

$$(\tfrac{3}{16}\varepsilon_2 c^4)G_0 + (\tfrac{3}{8}\varepsilon_1 c^2 + \tfrac{5}{16}\varepsilon_2 c^4)G_2 + (3/2 + \tfrac{1}{2}\varepsilon_1 c^2 + \tfrac{5}{16}\varepsilon_2 c^4)G_4 = \tfrac{1}{16}\varepsilon_4 c^4. \tag{48}$$

These non-homogeneous, linear algebraic equations are solved in the Appendix for G_0, G_2 and G_4 by using the well known Cramer's rule.

Next, as in section 2.4, upon noting that the non-linear period in the time T

domain is 2π, equation (44), with known coefficients G_n, is integrated from 0 to 2π in T to obtain the non-linear period ν. The result, in the dimensionless time $t\beta$, is

$$\nu = 2\pi(G_0)^{-1/2}(\alpha_1/\alpha_2)^{1/2}[1 + \tfrac{3}{16}(H_2^2 + H_4^2) - \tfrac{15}{16}H_2H_4 + \tfrac{105}{1024}(H_2^4 + 4H_2^2H_4^2 + H_4^4) + \cdots], \tag{49}$$

where

$$H_2 = G_2/G_0, \qquad H_4 = G_4/G_0. \tag{50}$$

Note that, when G_2 and G_4 are set to zero, H_2 and H_4 become zero, and equations (46)–(48) yield

$$\nu = 2\pi\left[1 + \frac{\varepsilon_1 c^2}{2} + \tfrac{3}{8}\varepsilon_2 c^4\right]^{1/2}[1 + \tfrac{3}{4}\varepsilon_2 c^2 + \tfrac{3}{8}\varepsilon_4 c^4]^{-1/2}(\alpha_1/\alpha_2)^{1/2}, \tag{51}$$

which is the non-linear period that one obtains by solving equation (42) using the single term HB method. Examples of the results obtained by using equations (49) and (50) for various values of S, μ and η are presented and discussed in the next section.

3.1. RESULTS AND DISCUSSION

By using equations (40), (43), (A1), (A2) and (A3), the non-linear frequency parameter ν, for the inextensible beam in Figure 1, obtained by using the TT method, given in equation (49), and that obtained by using the single mode HB method, given in equation (51), were calculated for various values of the beam base stiffness S, mass ratio μ and position η of the attached mass M. These calculations were programmed on the VAX/VMS version 5 digital computer. Examples of the results of these calculations are shown in Figures 6–13, in which is displayed the variation of the non-linear period parameter ν with the beam tip displacement $a = b/l = c/p$ for different modes and various selected values of S, μ and η. A typical example of the effect on non-linear inertia of the beam on the variation of the non-linear period η of the first mode with amplitude a is shown in Figure 6. In this figure, the period calculated using formulation II, which includes non-linear inertia terms, and the TT method (see equation (49)), and that obtained using formulation II and single term HB method (see equation (51)), are compared to that obtained by using formulation I, which does not include non-linear inertia terms, and the TT method (see equation (31)). This figure shows a fairly good agreement between the three results and, as was pointed out before, indicates that the non-linear inertia has small effect on the period of the fundamental mode of the inextensible beam in Figure 1.

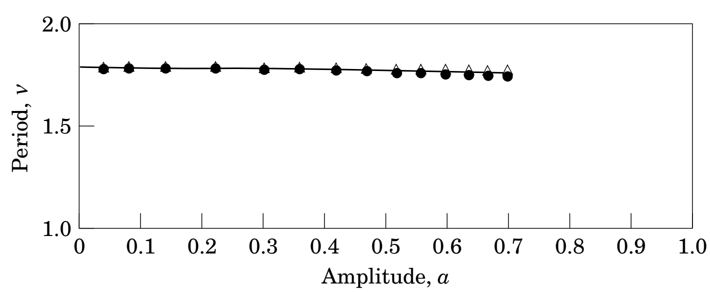

Figure 6. The effects of the vibration amplitude on the non-linear period of the first mode for the case $\mu = 0{\cdot}0$, $S = 10^7$. ——, Formulation II, TT method, equation (49); △, formulation II, single mode HB, equation (51); ●, formulation I, TT method, equation (31).

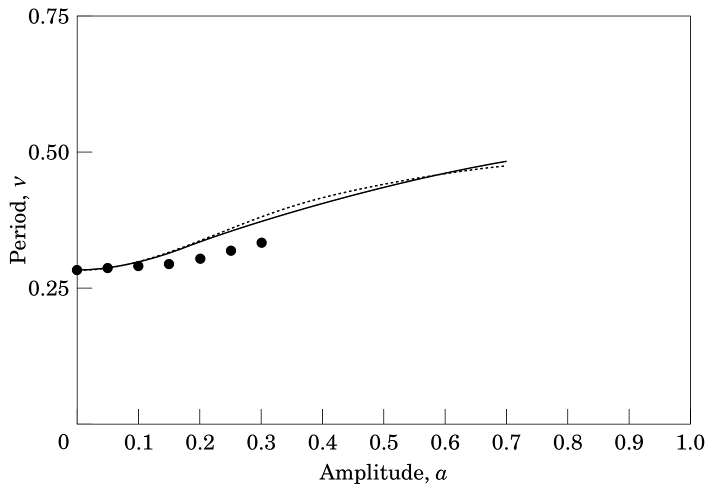

Figure 7. As Figure 6, but for the second mode. ——, Formulation II, TT method, equation (49); · · ·, single mode HB, formulation II; equation (51); ●, reference [30].

These results also establish confidence in the reliability of the TT results obtained from equation (49).

An example of the results obtained for the second mode is shown in Figure 7, where the TT (equation (49)), and the single mode HB (equation (51)) results of formulation II are compared with those obtained in reference [30], which include non-linear inertia effects. Note that the results obtained by using equations (49) and (51) are shown in this figure for an amplitude ratio a up to 0·7, while those obtained from reference [30] are shown, as in reference [30], for values of a up to 0·3. As can be seen from this figure, the present formulation II results obtained by using equations (49) and (51) are in fairly good qualitative—but not quantitative—agreement with those in reference [30]. That is, the present formulation II, as well as that in reference [30], predict a softening behavior of

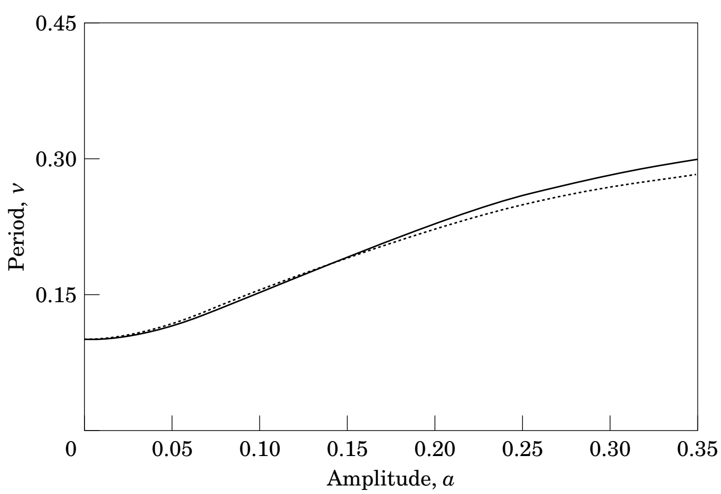

Figure 8. The effects of the vibration amplitude on the non-linear period of the third mode for the case $\mu = 0{\cdot}5$, $\mu = 0{\cdot}5$, $S = 1000$. ——, Formulation II, TT method, equation (49); · · ·, formulation II, single mode HB, equation (51).

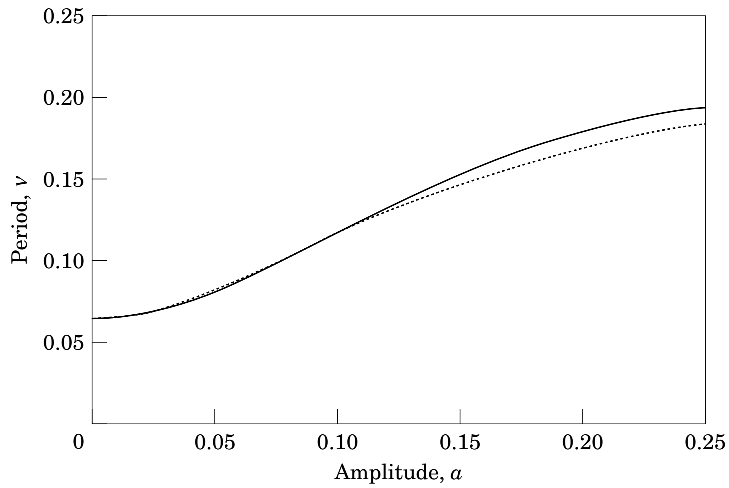

Figure 9. As Figure 8, but for the fourth mode.

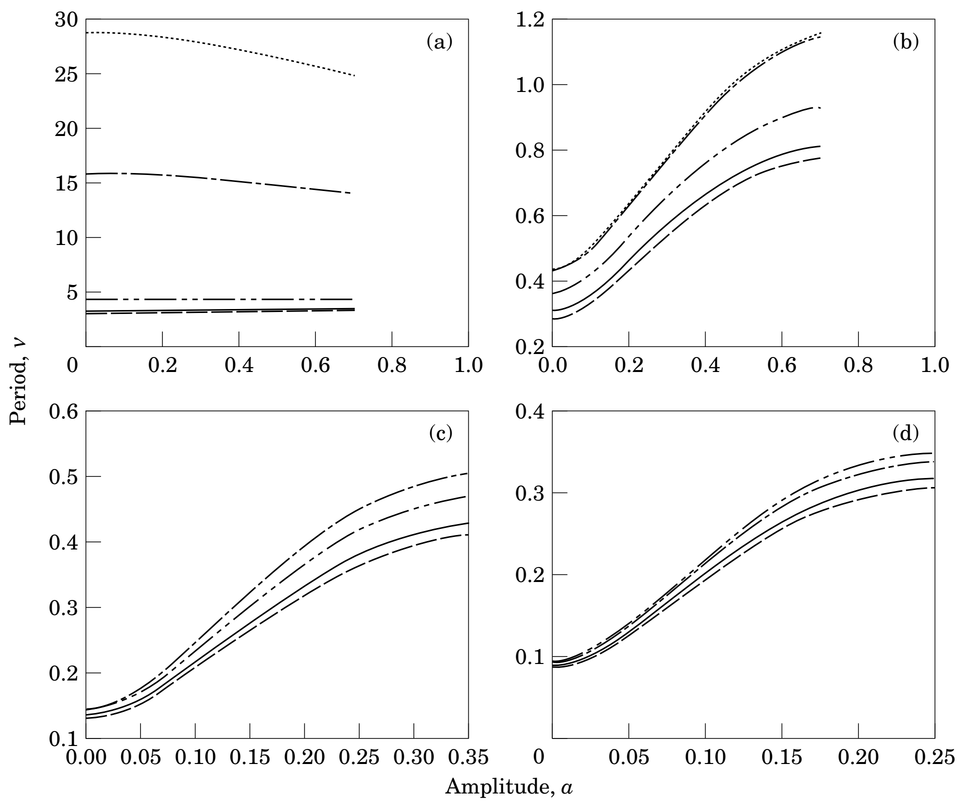

Figure 10. The effects of the base stiffness and amplitude of vibration on the period given by formulation II TT method, equation (49), for the case $\mu = 1{\cdot}0$ and $\mu = 0{\cdot}8$. (a) First mode; (b) second mode; (c) third mode; (d) fourth mode. · · · · · , $S = 0{\cdot}5$; $-\cdot-\cdot$, $S = 1$; $-\cdot\cdot-\cdot\cdot$, $S = 10$; ——, $S = 50$; $---$, $S = 1000$.

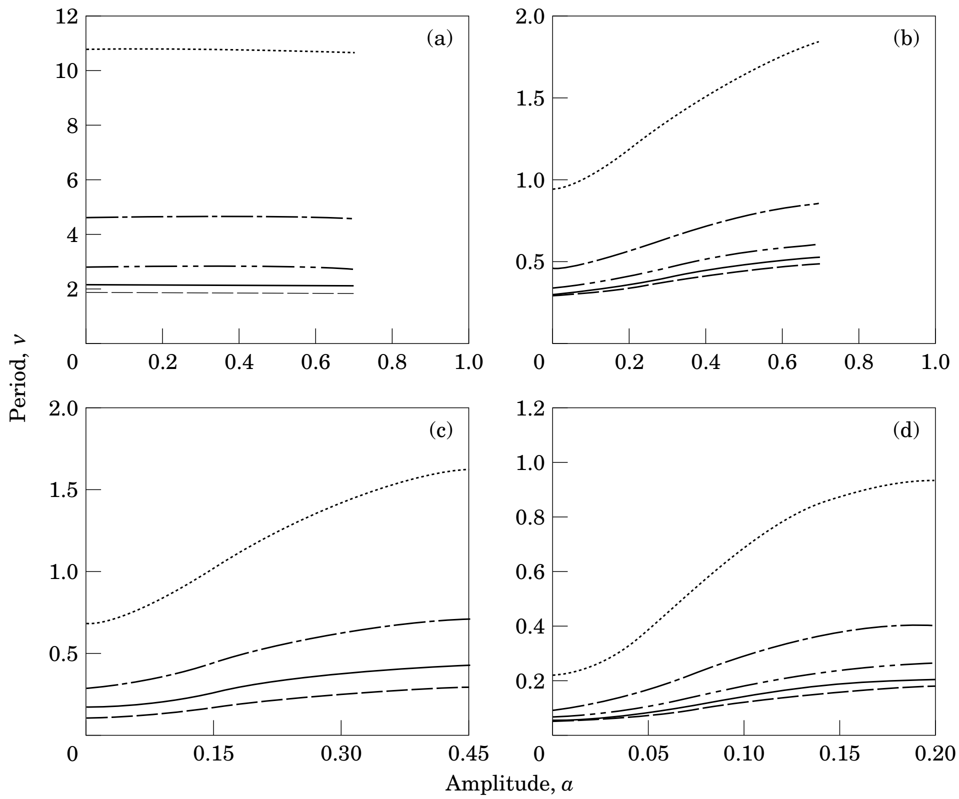

Figure 11. The effects of the lumped mass and amplitude of vibration on the period given by formulation II TT method, equation (49), for the case $S = 1000$, $\mu = 0{\cdot}7$. (a)–(d) First–fourth modes. μ values: · · · · , 5; – · – · , 2; – ·· – ·· , 1; ——, 0·5; – – –, 0·1.

the non-linear period of the second mode of the present (i.e., $\mu = 0$, $S = \infty$) inextensible beam. However, the results of equation (49), which are in fairly good agreement with those of equation (51), show a significant quantitative difference with those in reference [30] as the amplitude ratio a is increased.

Figures 8 and 9 are shown typical examples of the variation of the non-linear period parameter v, with amplitude a, of the third and fourth modes, respectively, obtained by using equations (49) and (51). It can be seen from these figures that for both the third and fourth modes, the period–amplitude variation is of the softening type. These figures also show that the TT results obtained by using equation (49) are in fairly good agreement with the single term HB method results obtained by using equation (51). Note that the results in Figures 7–9 indicate that the non-linear period v obtained by using equation (49) or (51) becomes nearly independent of the amplitude ratio a as a is increased beyond a certain value, which becomes smaller as the mode number is increased. This behavior of v with a can be easily deduced from equation (49) and Table 2, by noting that ε_1, as well as p, are, in general, larger than 1 for the second mode and increase relatively sharply as the mode number is increased. Thus as the amplitude ratio c ($a = c/p$) is increased, the terms $\varepsilon_1 c^2/2$ and $\frac{3}{4}\varepsilon_3 c^2$ in the bracketed terms in equation (51) become greater than 1 at a relatively small value of a, $a < 1$, which decreases as the mode number is increased. As a result, the period parameter v, according to equation (51), approaches a limiting value $(2\varepsilon_1/3\varepsilon_3)^{1/2}$. This behavior is also exhibited in these figures by the TT results obtained by using equation

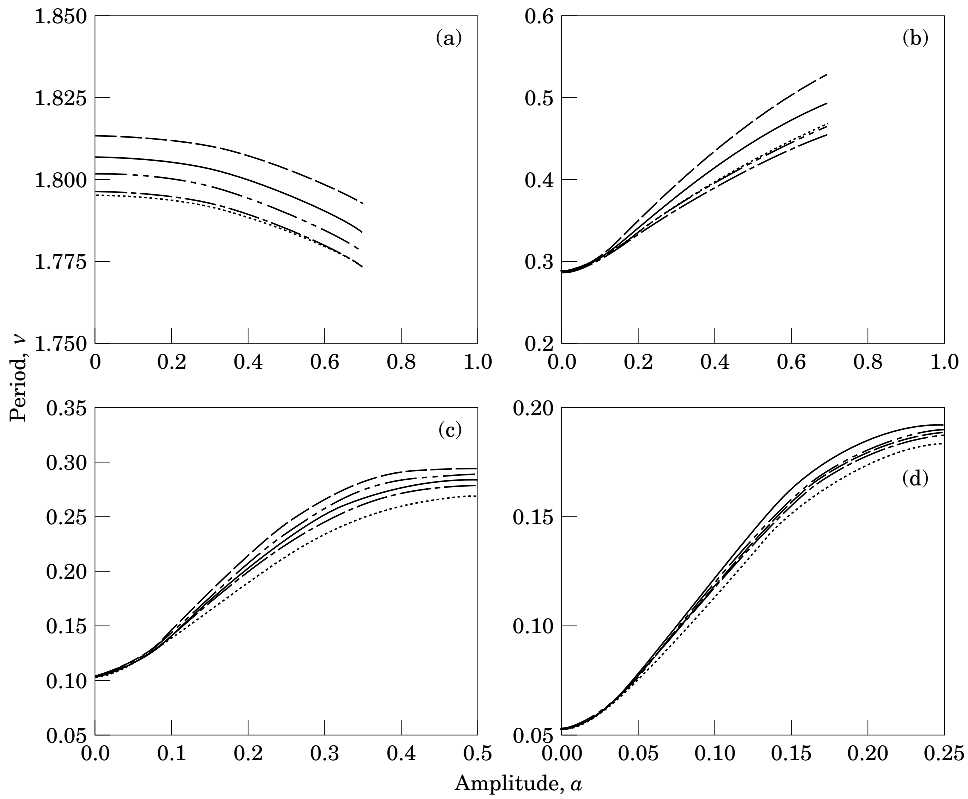

Figure 12. The effects of the lumped mass position and amplitude of vibration on the non-linear period given by formulation II TT method; equation (48) for the case $S = 1000$, $\mu = 0{\cdot}1$. (a)–(d) First–fourth modes. μ values: $\cdots\cdot$, 0·3; $-\cdot-\cdot$, 0·4; $-\cdot\cdot-\cdot\cdot$, 0·5; ——, 0·6; $---$, 0·8.

(49), but in this case the value of a at which the TT results show a near levelling of the $v - a$ curve appears to be larger, due to the correction produced by these results, than that predicted by the single term HB results, given by equation (51). This behavior of the $v - a$ curves at a large amplitude ratio a, predicted by equations (49) and (51), may not be accurate, as each of these two equations represents an approximate solution, which becomes inaccurate at large motion amplitude, of the strongly non-linear oscillator in equation (42). Therefore, in Figures 10–13 presented subsequently, only the results of using the TT method (equation (49)), are displayed, and the $v - a$ curve in each case is shown up to an amplitude ratio a below which v increases monotonically with a.

In Figure 10 is shown the effect of varying the beam base stiffness S on the period of each of the first four modes of the beam in Figure 1. It can be seen from each of these figures that as S decreases the period parameter v increases; i.e., the non-linear period v shows a general behavior with S similar to the linear one [46, 50]. These figures also indicate that, for small values of base stiffness S, small changes in S can lead to relatively large changes in v, especially for the first mode, and for the higher modes at large amplitudes where the relative effect of changing S becomes smaller as the mode number is increased.

In Figure 11 is shown the effect of increasing the mass ratio μ of the attached mass M to the beam mass. In general, as in linear theory [46–48], increasing μ leads to a decrease

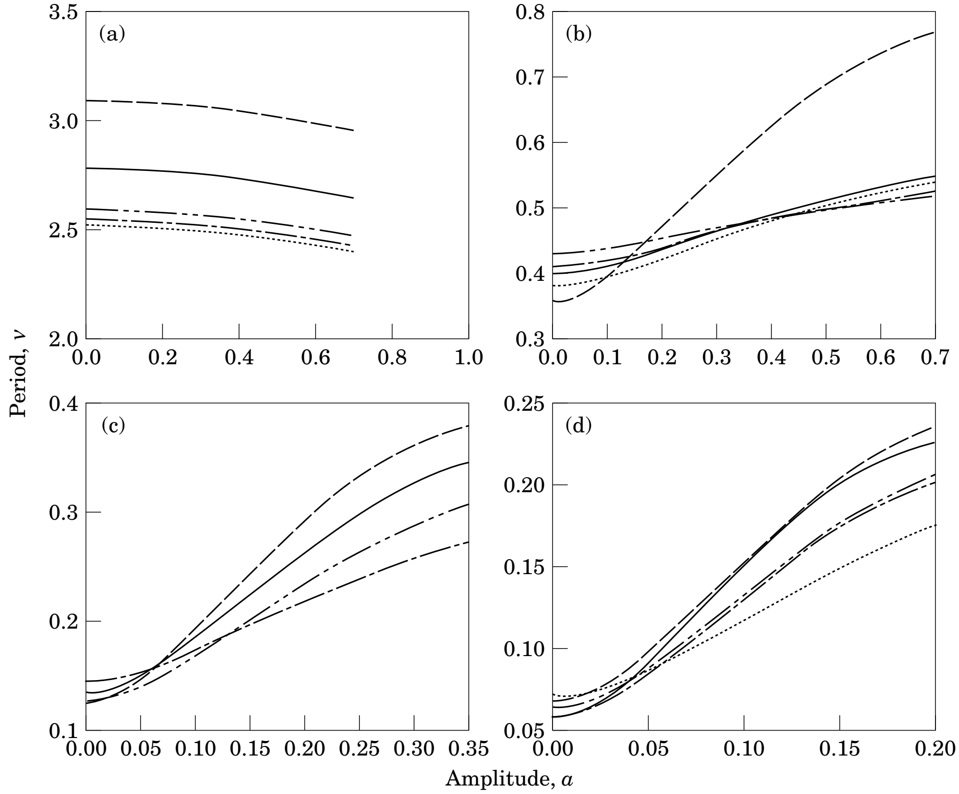

Figure 13. As Figure 12, but for $S = 10$, $\mu = 1{\cdot}0$ and the following μ values: $\cdots\cdot$, 0·2; $-\cdot-\cdot$, 0·3; $-\cdot\cdot-\cdot\cdot$, 0·4; ——, 0·6; – – –, 0·1.

in the period of motion, ν. These results also indicate that small changes in the mass ratio μ can lead to relatively large changes in the period of each of the first four modes, especially for the higher modes at large motion amplitudes. Note that in the present study the mode shape used is assumed to be unaffected by the attached mass M, even when the amplitude of motion is large. Therefore, one expects the present results accuracy to deteriorate for relatively large values of μ and a.

The effect of changing the relative position v of the attached mass M is shown in Figure 12. It can be seen from these figures that, in general, for large amplitude ratio a, moving the attached mass towards the clamped end of the beam tends to increase the period of the first mode and decrease that of the second, or higher, mode. However, as in linear theory [46–48], when the amplitude of motion is small, the period of the second, or higher, mode goes through regions of increasing and decreasing values as the attached mass is moved towards the clamped end of the beam. This behavior is more obvious in Figure 13, where the beam is assumed to have a smaller base stiffness S than that considered in Figure 12. Note that the above behavior is not obvious in Figure 12 due to the small value of μ and the large value of S used for the beam presented in these figures. In Figure 13 it is also indicated that, depending on the value of base stiffness S and mass ratio μ of the attached mass M, changing the positions of the attached mass can lead to relatively large changes in the period of free motion, especially when the amplitude of motion is large.

4. CONCLUSIONS

The objective of the present work is to provide a simple formulation, by using well established analytical techniques, of the problem of large amplitude, planar, flexural free vibrations of an inextensible beam carrying a small lumped mass element and attached to a flexible root, and to study the effects of base stiffness and attached mass magnitude and position on the non-linear period of motion of such a beam element. When the assumed mode(s) method is used in conjunction with Lagrange's method in the analysis of non-linear one-dimensional continuous systems, then one can avoid the derivation of the field integro-partial differential equation of motion and the associated boundary conditions. It is shown that inertia non-linearities arise in this case as a result of using the inextensibility of the beam. The present analysis is based on the assumption that the frequencies of beam, which are amplitude dependent, remain widely spaced, as do the linear ones, even when the amplitude of motion is relatively large. It is also assumed that the beam deflection during the motion resembles a linear mode shape of the base beam which is unaffected by the attached mass, and remains self-similar during the motion, even when the amplitude of motion is large. Although these assumptions simplify the calculations considerably, they may introduce significant errors at large amplitudes, especially when the ratio of the attached mass to the beam mass is not small. For example, one can see from Figures 10–13 that, even at relatively moderate values of motion amplitude, the period of the third mode may become equal to or greater than that of the second mode; similarly, the period of the fourth mode may also become equal to or greater than that of the third mode at relatively moderate values of motion amplitude. At such, and higher, amplitude values one may expect the beam vibration to occur at more than one mode simultaneously. The present results also show that the base stiffness, and the magnitude and position of the attached mass have similar effects on the period of the non-linear system as in linear theory when the amplitude of motion is small; but their effects are more pronounced than in linear theory when the amplitude of motion is relatively large.

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APPENDIX

The G_n coefficients in equation (43) are as follows:

$$G_0 = \frac{R_1(b_2c_3 - b_3c_2) - a_2(R_2c_3 - b_3R_3) + a_3(R_2c_2 - b_2R_3)}{\Delta}, \tag{A1}$$

$$G_2 = \frac{a_1(R_2c_3 - b_3R_3) - R_1(b_1c_3 - c_1b_3) + a_3(b_1R_3 - c_1R_2)}{\Delta}, \tag{A2}$$

$$G_4 = \frac{a_1(b_2R_3 - R_2c_2) - a_2(b_1R_3 - R_2c_1) + R_1(b_1c_2 - b_2c_1)}{\Delta}, \tag{A3}$$

where

$$\Delta = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - b_2c_1) \tag{A4}$$

and

$$a_1 = 1 + \tfrac{1}{2}\varepsilon_1c^2 + \tfrac{3}{8}\varepsilon_2c^4, \qquad a_2 = \tfrac{1}{4}(\varepsilon_1c^2 + \varepsilon_2c^4), \qquad a_3 = \tfrac{1}{16}\varepsilon_2c^4, \tag{A5–A7}$$

$$b_1 = \tfrac{1}{2}\varepsilon_1c^2 + \tfrac{7}{16}\varepsilon_2c^4, \qquad b_2 = 1 + \tfrac{3}{8}\varepsilon_1c^2 + \tfrac{5}{16}\varepsilon_2c^4, \qquad b_2 = -\tfrac{1}{2} + \tfrac{1}{16}\varepsilon_2c^4, \tag{A8–A10}$$

$$c_1 = \tfrac{3}{16}\varepsilon_2c^4, \qquad c_2 = \tfrac{3}{8}\varepsilon_1c^2 + \tfrac{5}{16}\varepsilon_2c^4, \qquad c_3 = \tfrac{3}{2} + \tfrac{1}{2}\varepsilon_1c^2 + \tfrac{5}{16}\varepsilon_2c^4, \tag{A11–A13}$$

$$R_1 = 1 + \tfrac{3}{4}\varepsilon_3c^2 + \tfrac{5}{8}\varepsilon_4c^4, \qquad R_2 = \tfrac{1}{4}\varepsilon_3c^2 + \tfrac{5}{16}\varepsilon_4c^4, \qquad R_3 = \tfrac{1}{16}\varepsilon_4c^4. \tag{A14–A16}$$