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ON THE PERIOD OF LARGE AMPLITUDE FREE VIBRATION OF CONSERVATIVE AUTONOMOUS OSCILLATORS WITH STATIC AND INERTIA TYPE CUBIC NON-LINEARITIES

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The concern of this paper is the large amplitude free vibration of strongly non-linear oscillators $\ddot{u} + mu + \varepsilon_1 u^2\ddot{u} + \varepsilon_2 u\dot{u}^2 + \varepsilon_2 u^3 = 0$, where $m = 1, 0$, or -1, ε_1 and ε_2 are positive parameters which may be arbitrarily large, and $u(t)$ may be of order unity. Approximate analytical solutions for the period of free motion are obtained, for comparison purposes, by using the single-term harmonic balance (SHB) method, the two-terms harmonic balance (2THB) method, and the two-term time transformation (2TT) method described in reference [1]. Parametric studies on the effects of m, ε_1 and ε_2 on the period-amplitude behaviour are presented as obtained by using the above three analytical methods. The results of these three methods are compared with each other and with those obtained numerically. For convenience, the results are displayed graphically. It is shown that for the case $m = 1$, a qualitative failure of the SHB method ocurs when ε_1 and ε_2 are in the range $1{\cdot}5 < \varepsilon_1/\varepsilon_2 < 1{\cdot}8$. It is also shown that for $m = 0$, or -1, the period–amplitude behavior is of hardening type regardless of the value of ε_1 relative to ε_2. In all cases $m = 1$, 0, or -1, the period becomes nearly constant independent of motion amplitude when this amplitude is relatively large. It is also shown that the period becomes a constant independent of motion amplitude and is equal to the linear period when $\varepsilon_1 \sim 1{\cdot}6\varepsilon_2$. © 1997 Academic Press Limited

1. INTRODUCTION

This work is concerned with the free vibrations of autonomous conservative oscillators with inertia and static type cubic non-linearities governed by a dimensionless equation of motion of the form

$$\ddot{u} + mu + \varepsilon_1 u^2\ddot{u} + \varepsilon_1 u\dot{u}^2 + \varepsilon_2 u^3 = 0, \tag{1}$$

where dots denote time derivatives, m is an integer which may take on the values $m = 1$, 0 or -1 in order to allow results to be obtained for which the associated linear oscillator is, respectively, statically stable, neutrally stable or statically unstable, ε_1 and ε_2 are positive parameters which may not be small, and the displacement $u(t)$ is of order unity. In equation (1), the first of the non-linear terms is a softening inertia type, the second non-linear term is a hardening inertia type, while that last non-linear term is a hardening static type. For $\varepsilon_2 < 0$ the last non-linear term in equation (1) becomes a softening static type, and such a case will not be considered in this work. When $\varepsilon_1 = 0$, equation (1) reduces

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to the usual Duffing oscillator

$$\ddot{u} + mu + \varepsilon_2 u^3 = 0. \tag{2}$$

Phenomena governed by an equation similar to equation (2) have been the subject of many investigations over the years see, e.g., references [2–11], in which numerical and various well established approximate analytical methods have been used by various authors to obtain the period of free oscillations for both the weakly non-linear ($\varepsilon_2 \ll 1$) and the strongly non-linear ($\varepsilon_2 > 1$) cases. On the other hand, studies dealing with oscillators of the type modelled by equation (1) are less abundant.

The oscillators in equation (1) and those in equation (2) have the same potential energy $V(u)$, defined as $V(u) = 0{\cdot}5mu^2 + 0{\cdot}25\varepsilon_2 u^4$. For the cases $m = 0$ and $m = 1$, $V(0) = 0$ and $V(u) > 0$ for all $u \neq 0$. Thus the origin of the phase space $(u(0) = 0, \dot{u} = 0)$ is a stable equilibrium point, and periodic motions centered at the origin will occur for all non-zero initial conditions; i.e., $u(0) \neq 0$ and/or $\dot{u} \neq 0$. For simplicity, and without loss of generality, the initial conditions in this work will be taken as $u(0) = A$, $\dot{u}(0) = 0$, where A is the oscillation amplitude. For an oscillator with only hardening static cubic non-linearity, such as the one given by equation (2) with $\varepsilon_2 > 0$, for which the associated linear oscillator is statically stable ($m = 1$) or neutrally stable ($m = 0$), the period of free oscillation decreases monotonically as the motion amplitude A is increased and approaches zero as A approaches infinity. For an oscillator with only softening static cubic non-linearity, i.e., equation (2) with $\varepsilon_2 < 0$, and which is linearly stable ($m = 1$), periodic motions about the origin are possible only over a limited range of amplitude A (with $\dot{u} = 0$), where $|A| \leqslant 1/\sqrt{-\varepsilon_2}$. In this case the period of motion increases monotonically with increasing amplitude A, but becomes infinite as $|A| \to 1/\sqrt{-\varepsilon_2}$, at which point beyond the oscillator becomes statically unstable and periodic motions cease to exist. For a linearly stable ($m = 1$) oscillator with inertia softening and hardening non-linearities and static hardening non-linearity, i.e., equation (1) with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, the period of oscillation variation with amplitude of motion may exhibit a softening or hardening behavior depending, respectively, on whether the non-linear inertia is stronger than ($\varepsilon_1 \gg \varepsilon_2$) the static hardening non-linearity. Hamdan and Shabaneh [12] have recently indicated that in either of these two cases the period of free motion of a system with inertia and hardening static non-linearities is amplitude dependent only for small moderate values of motion amplitude A and eventually becomes independent of amplitude A as A becomes relatively large. It is shown in this work that for the cases $m = 0$ and $m = -1$ the period–amplitude behavior of the oscillators in equation (1) is of hardening type regardless of the relative strength of the inertia non-linearity with respect to the static hardening non-linearity, and that the period of these oscillators is independent of amplitude A at moderate and large values of A. It is also shown that, for the case $m = 1$, the period of the oscillator in equation (1) becomes a constant, independent of A, and equals the linear period whenever $\varepsilon_1 \sim 1{\cdot}6\varepsilon_2$.

A physical example of a system described by equation (1) is the large amplitude planar flexural free vibrations of an inextensible slender cantilever beam element with or without intermediate inertia element [12–14]. In this case equation (1) represents the scaled, single-mode, equivalent temporal problem in which the parameters ε_1 and ε_2 are defined by integrals involving products of an assumed mode shape deflection and its derivatives. It is known that the large amplitude fundamental mode response of such a beam is dominated by the static non-linearity, arising from potential energy stored in bending, where $\varepsilon_2 > \varepsilon_1$. At the higher modes the response becomes dominated by the non-linear inertia softening effects, arising from the kinetic energy of axial motion, where ε_1 becomes much larger than ε_2 and increases, while ε_2 decreases, as the mode number is increased. Examples of calculated values of ε_1 and ε_2 for such a cantilever beam system show that, depending on the scaling procedure, $\varepsilon_2 \sim 1$, $\varepsilon_1 \sim 0{\cdot}1$ for the fundamental mode while the fourth mode, for example, $\varepsilon_2 \sim 0{\cdot}1$, $\varepsilon_1 \sim 10$ or higher, depending on the scaling procedure, and the magnitude and location of the attached inertia element, if any [12–14]. In other words, ε_1 and ε_2 in equation (1) are not in this case small compared to 1, as is usually assumed in perturbation theory. Furthermore, such a beam system, being flexible, when subjected to a direct and/or parametric (i.e., base) excitation usually undergoes large amplitude stable resonance motions where peak amplitudes of the order of one quarter of the beam length may be anticipated [13–15] and thus the non-linear terms, taking into account the fact that ε_1 and/or ε_2 are not small compared to 1, become an order of magnitude comparable to or even greater than that of the linear ones. It is of great interest in such cases, and in forced vibration analysis of non-linear systems in general, to determine the free vibration frequency-amplitude dependence, as this defines the so-called "backbone" curve of the resonance response, which allows one to establish the qualitative behavior of the resonance response.

Available analytical techniques for the analysis of non-linear conservative oscillators such as the ones modelled by equation (1) or (2) can only provide approximate solutions to the actual response of non-linear oscillators. Classical analytical methods which have been widely used for the linearly stable, weakly non-linear oscillators (i.e., $m = 1$, $\varepsilon_1 \ll 1$ and $\varepsilon_2 \ll 1$) include perturbation methods, such as the Linstedt–Poincaré (L–P) and multiple time scales (MMS) methods, and the generalized averaging method of Krylov–Bogoliubov–Mitropolski (KBM). These methods are described, for example, by Nayfeh [7], Nayfeh and Mook [9] and Minorsky [4]. In the perturbation methods one seeks asymptotically valid, usually low order approximations to the response by expanding the dependent variable and system parameters in a small positive gauge parameter to convert the non-linear differential equation to an "equivalent" system of linear differential equations. In the MMS a number of time scales are used and the resulting equivalent linear system is a set of linear partial differential equations, while in the L–P method a single time scale is used and the equivalent system is a set of linear ordinary differential equations in the defined (transformed) time scale. The KBM method also uses a number of time scales and power series expansion of the dependent variable and system parameters. In principle, the equivalent system of linear differential equations can be solved, in sequence, to any desired order of approximation; however, in practice, the solutions to these equations are usually limited to the first or second order approximations as the algebraic manipulations, although straightforward, become increasingly laborious as the order of approximation is increased. Furthermore, the assumed series expansions are neither unique nor convergent, and thus carrying the calculations to second or higher order may not always improve the solution; i.e., second or higher order solutions do not always represent, as they should do, small corrections to the lower order solutions and thus violate the ordering requirement of the perturbation method [16–19]. Furthermore, the small positive gauge parameter used in the assumed power series expansions is usually naturally present in the equation of motion; otherwise, this parameter is intentionally introduced by "arbitrarily" scaling the equation of motion and/or re-ordering the appropriate terms in the equation of motion based on the relative importance of these terms in the anticipated response [9–14]. Thus, the range of system parameters and amplitudes over which the predicted perturbation solution is satisfactory is fixed in advance by the ordering and scaling schemes; however, this range is usually left unspecified. Furthermore, the amplitudes of different harmonics of the predicted approximate periodic response are assumed to satisfy the ordering scheme which determines in advance the relative importance of each of these harmonics and assumes the rapid attenuation of the higher ones for the assumed weakly non-linear system.

Classical methods which have been used for both the weakly and strongly non-linear oscillators for which the associated linear oscillator may or may not be statically stable (i.e., m may be $1, 0$, or -1) include the harmonic balance (HB) [8], equivalent linearization [5] and describing functions methods [6]. In these methods, the periodic solution to the non-linear problem is specified in advance, and usually these methods work well provided that the filter hypothesis is satisfied; that is, higher harmonics output by the non-linearities remain small compared to the assumed lower ones. In these methods, the arbitrary re-ordering of various terms in the non-linear differential equation of motion, employed in the perturbation methods, is not necessary. In the HB method, the more commonly used of these methods, a periodic solution of the dependent variable is assumed in the form of a Fourier series, mostly truncated to only a few leading harmonics which are assumed to be dominant and of equal level of importance over the full range of system parameters and motion amplitude. Upon substituting the asusmed series solution in the equation of motion, and equating the coefficients of different harmonics to zero, one obtains a set of coupled non-linear algebraic equations in the coefficients of the assumed harmonics and frequency of motion. These coupled non-linear equations, along with an equation (when a cosine or sine series with no phase shift is used), obtained by imposing the initial conditions, are then solved simultaneously for a given motion amplitude to obtain an approximation to the non-linear response. The number of these coupled non-linear equations which need to be solved simultaneously is equal to the number of harmonics in the assumed series solution. Therefore, the use of a sufficiently large number of harmonics to improve accuracy results in a messy non-linear algebraic problem. Furthermore, one intuitively expects the assumed HB solution to converge to the actual solution as the number of harmonics in the assumed solution is increased. However, this is generally true provided that the harmonics in the assumed truncated series solution are the dominant ones and the neglected harmonics are small compared to the retained ones [20]. Furthermore, since the retained harmonics, as indicated above, enjoy an equal level of importance and are allowed to interact with each other in a non-linear way, the problem of selecting the "right" combination of these leading harmonics, which will lead to the correct qualitative, and hopefully quantitative behavior of the response, becomes a difficult task especially when the non-linearities are strong [20]. In fact, there are many examples in the literature in which an HB solution in which only a few leading harmonics are used can fail not only quantitatively but also qualitatively [20]. Therefore, confidence in the accuracy of the HB solution cannot always be established without comparing it with the results of numerical and other approximate analytical methods [20].

Refinements of existing approximate analytical techniques, as well as new methods with various restrictions placed on the type and strength of the non-linearities, have also appeared over the years [1, 21–27]. Surveys of such methods are presented, i.e., in references [1, 21], and are briefly summarized in what follows. For example, Sinha and Srinivasan [22], Anderson [23] and others, have used various sets of orthogonal polynomial expansions of non-linear terms, often in conjunction with the K–B averaging method. Helleman and Montroll [24] and Eminhizer, Hellman and Montroll [25] presented a perturbation procedure free of secular terms. Einuada [26] presented a perturbation procedure free of secular terms. Einuada [26] presented a successive approximation method which also does not involve secular terms. Jones [27] presented a perturbation procedure which involves the formulation of a new small expansion parameter, in terms of the original small parameter, and a linear transformation of time. His technique was shown [27] to yield a fairly accurate approximation to the period of the strongly non-linear Duffing oscillator. Burton [21] has also used the definition of a new small parameter, based on the results of series expansion for the period presented in reference [1] which are discussed subsequently, and a linear transformation of time to extende the L–P perturbation method to the analysis of strongly non-linear Duffing oscillators. His method, like that in reference [1], is also applicable to the cases in which the associated linear oscillator of the strongly non-linear oscillator is neutrally stable ($m = 0$), or statically unstable ($m = -1$), which cannot be analyzed using standard perturbation theory. Burton and Hamdan [1] presented a time transformation (TT) method for the analysis of a generalized version of the non-linear conservative autonomous oscillators in equation (2). Their method, which does not involve perturbation, places no restriction on the strength of the non-linearity. It has the advantage over other approximate analytical methods of yielding an exact series expansion for the period of the non-linear oscillator in terms of the Fourier coefficients of the non-linearity, from which the period may be calculated to the desired accuracy with comparatively less computational effort. It also has the advantage of being applicable to situations in which other methods may fail: i.e., when the non-linearity is strong and when the associated linear oscillator is neutrally stable ($m = 0$) or statically unstable ($m = -1$). The procedural steps of this TT method are described in detail in reference [1] and are summarized in section 2.3. In this work, because of the strength of the non-linearities considered, the TT procedure [1] will be used to obtain an approximation of the period of the non-linear oscillator modelled by equation (1) for the case in which ε_1 and/or ε_2 are not small and the displacement $u(t)$ is of order unity. It is to be noted that the TT method in reference [1] has been applied for the cases in which the non-linear oscillator has only static non-linearity; therefore, the present work attempts to extend this method to the cases in which the non-linear oscillator has static and inertia non-linearities. For comparison purposes, the TT results will be compared with those obtained by using the HB method as well as with those obtained by numerical integration. The emphasis of the present work is on the effect of large amplitude motion on the quantitative as well as qualitative behavior of the non-linear period of free motion and on the effect of the relative strength of the inertia and the hardening static non-linearities on this behavior. It is also of interest to know the extent to which variations in the system parameters ε_1 and ε_2 can have a significant effect on the quantitative and qualitative behavior of the period with motion amplitude. For example, as indicated above, in the case of the free vibration of an inextensible cantilever beam the values of ε_1 and ε_2 depend on the arbitrarily selected mode shape; thus one is usually interested to know the effect of different selected mode shape on the accuracy of the obtained results. For convenience, the period–amplitude relations for various selected values of ε_1 and ε_2 are presented in graphical form. The analytical solutions obtained by using single-term HB, two-term HB and two–term TT methods are presented in this section. The results obtained using these approximate analytical solutions as well as the results of numerical integration are presented and discussed in this section. As indicated above, most of the published studies on the free vibration of conservative autonomous oscillators are concerned with oscillators having only hardening or softening static non-linearities and, to the authors knowledge, despite its practical importance, studies dealing with the free vibration of oscillators having inertia and hardening static non-linearities of the type modelled by equation (1) are not commonly available.

2. ANALYSIS

In this section, approximate analytical solutions of the conservative autonomous non-linear oscillator in equation (1) are obtained by using the single-term HB method, the two-term HB method and the TT method described in reference [1]. Without loss of generality, the initial conditions are taken to be $u(0) = A$, $\dot{u}(0) = 0$, where A is the amplitude of motion.

2.1. SINGLE-TERM HARMONIC BALANCE (SHB) SOLUTION

According to the SHB method, an approximate solution of equation (1), with $u(0) = A$, $\ddot{u}(0) = 0$, takes the form

$$u(t) = A \cos \omega t, \tag{3}$$

where A is the amplitude and ω is the frequency of motion. Substituting equation (3) and its time derivatives into equation (1), using the trigonometric identity $\cos^3 \omega t = \frac{1}{4}(3 \cos \omega t + \cos 3\omega t)$, and collecting the $\cos \omega t$ and $\cos 3\omega t$ terms, leads to

$$\left[m + \tfrac{3}{4}\varepsilon_2 A^2 - \omega^2\left(1 + \frac{\varepsilon_1}{2} A^2\right)\right] \cos \omega t + \left(\frac{\varepsilon^2}{4} A^2 - \frac{\varepsilon_1}{2} A^2\omega^2\right) \cos 3\omega t = 0. \tag{4}$$

Ignoring the effect of the third harmonic $\cos 3\omega t$, and equating the coefficient of $\cos \omega t$ to zero, one obtains the non-linear frequency–amplitude relation

$$\omega^2 = \left(m + \tfrac{3}{4}\,\varepsilon_2 A^2\right)\left(1 + \frac{\varepsilon_1}{2} A^2\right)^{-1}. \tag{5}$$

The period τ of the motion is obtained by substituting $\tau = 2\pi/\omega$ into equation (5), which yields the period–amplitude relation

$$\tau = 2\pi\left(1 + \frac{\varepsilon_1}{2} A^2\right)^{1/2} \left(m + \tfrac{3}{4}\,\varepsilon_2 A^2\right)^{-1/2}. \tag{6}$$

The period–amplitude relation in equation (6) represents the first order approximation which one also obtains when using classical perturbation methods. It can be seen from this equation that for the cases in which ε_1 and ε_2 are not small compared to 1, the period τ for the case $m = 1$ becomes nearly constant independent of amplitude A: i.e., $\tau \to 2\pi(1{\cdot}5\varepsilon_2/\varepsilon_1)^{-1/2}$, as the amplitude A becomes of order unity. The results obtained by using equation (6) for various selected values of ε_1 and ε_2 are presented and discussed, for convenience, in section 3.

2.2. TWO-TERM HARMONIC BALANCE (2THB) SOLUTION

Improvement of the accuracy of the SHB solution is sought by adding more harmonics in the assumed HB solution. When the number of different harmonics in the assumed solution is equal to two, the HB method is called the two-term harmonic balance (2THB) method. According to this method, with $u(0) = A$, and $\dot{u}(0) = 0$, an approximate solution of equation (1) takes the form

$$u(t) = A_1 \cos \omega t + A_3 \cos 3\omega t. \tag{7}$$

Application of the initial condition $u(0) = A$, yields

$$A = A_1 + A_3. \tag{8}$$

Equation (8) relates the total amplitude A of motion to the ampliudes A_1 and A_3 of the fundamental and third harmonics of the response, respectively. Substituting equation (7) and its time derivatives into equation (1), using trigonometric identities, retaining only the

$\cos \omega t$ and $\cos 3\omega t$ terms and then equating the coefficient of each of these two harmonics to zero, one obtains the following non-linear coupled algebraic equations:

$$A_3 = \frac{0{\cdot}25\varepsilon_1(A_1^3 + 3A_3^3) - 0{\cdot}5\varepsilon_1\omega^2(A_1^3 + 9A_3^3)}{9\omega^2 - m - 1{\cdot}5\varepsilon_2 A_1^2 + 5\varepsilon_1\omega^2 A_1^2} \tag{9}$$

$$\omega^2 = \frac{m + \varepsilon_2[0{\cdot}75A_1^2 + 0{\cdot}75A_1A_3 + 1{\cdot}5A_3^2]}{1 + \varepsilon_1[0{\cdot}5A_1^2 + 1{\cdot}5A_1A_3 + 5A_3^2]} \tag{10}$$

Equations (9) and (10), along with equation (8), define the amplitudes A_1 and A_3 of the fundamental and third harmonics, respectively, and the frequency ω of the assumed periodic motion. For a given amplitude A, these non-linear coupled equations are solved numerically by a direct iteration technique with 10^{-6} accuracy. The period τ of the motion is then calculated by using the relation

$$\tau = 2\pi/\omega. \tag{11}$$

The period τ calculated by using equations (8)–(11) is presented and discussed, for convenience, in the next section.

2.3. A TIME TRANSFORMATION (TT) SOLUTION

An approximate analytic solution for the period τ of the non-linear autonomous conservative oscillator in equation (1) can also be obtained, for comparison purpose by using the time transformation (TT) procedure described in detail in reference [1]. This method, like the HB method, places no restriction on the strength of the non-linearities and is applicable in each of the three cases $m = 1$, 0 or -1. However, it differs from the HB and the other approximate analytical methods in that it yields a series expansion for the period of τ of a non-linear autonomous conservative oscillator, in terms of the Fourier coefficients of the non-linearity, from which the period τ can be calculated to the desired degree of accuracy with relatively less computational effort [1]. The TT series solution in reference [1], however, is given only for an oscillator with general static non-linearity; a TT solution for a conservative oscillator of the type given by equation (1) which has both static and intertia non-linearities is carrried out in what follows.

According to the TT method [1], a simple valued transformation of time $T(t)$ is sought between the real time t and a new time T such that in the new time T domain the solution of equation (1) is simple harmonic with period $\tau = 2\pi$: i.e., with $\dot{u}(0) = 0$, $u(T)$ takes the form $u(T) = A \cos T$, where $T(0) = 0$ and A is, as before, the motion amplitude. Writing equation (1) with T as the independent variable and substituting for $u(t) = \cos T(t)$ in the result, one obtains, after dividing by A,

$$\begin{aligned}(m - f^2)\cos T - ff' \sin T - 2\varepsilon_1 A^2 f^2 \cos^3 T - \varepsilon_1 A^2 ff' \cos^2 T \sin T + \varepsilon_1 A^2 \cos T \\ + \varepsilon_2 A^2 \cos^3 T = 0\end{aligned} \tag{12}$$

where primes denotes differentiation with respect to T and $f = \mathrm{d}T/\mathrm{d}t$. Using the trigonometric identities $\sin^2 T = 1 - \cos^2 T$ and $\cos^3 T = \frac{1}{4}(3\cos T + \cos 3T)$ equation (12) becomes

$$\begin{aligned}[m + 0{\cdot}75\varepsilon_2 A^2 - f^2(1 + \varepsilon_1 A^2/2)]\cos T - ff'(1 + \varepsilon_1 A^2/4)\sin T \\ + (0{\cdot}25\varepsilon_2 A^2 - \varepsilon_1 A^2 f^2/2)\cos 3T - 0{\cdot}25\varepsilon_1 A^2 ff' \sin 3T = 0\end{aligned} \tag{13}$$

Equation (13) is an inhomogeneous first order linear differential equation in $f(T)$. For $u(T)$ to be a simple harmonic in time T, the time transformation $f(T)$ must satisfy this equation. Since the harmonics in this equation are odd, $f(T)$, and thus $f^2(T)$, will be periodic and of period π in T. Therefore, a periodic solution of period π may be obtained by substituting the assumed Fourier series [1]

$$f^2 = \sum_{n=0,2}^{\infty} G_n \cos nT, \tag{14}$$

with coefficients G_n to be determined, into equation (13) and equating the coefficients of the harmonics $\sin nt$ and $\cos nT$ to zero. This leads to the following collection of linear algebraic equations in the coefficients G_n:

$$(1 + \varepsilon_1 A^2/2)G_0 + (0{\cdot}25\varepsilon_1 A^2)G_2 = m + 0{\cdot}75\varepsilon_2 A^2,$$

$$(\varepsilon_1 A^2/2)G_0 + (1 + 3\varepsilon_1 A^2/8)G_2 = (1/2)G_4 + (\varepsilon_1 A^2/8)G_6 + \varepsilon_2 A^2/4$$

$$\left[\left(\frac{n+2}{16}\right)\varepsilon_1 A^2\right]G_{n+6} + \left[\frac{n+4}{4} + \frac{n+6}{16}\varepsilon_1 A^2\right]G_{n+4}$$

$$= \left(\frac{n+4}{16}\varepsilon_1 A^2\right)G_n + \left(\frac{n+4}{4} + \frac{n+6}{16}\varepsilon_1 A^2\right)G_{n+2}. \qquad n \geqslant 2. \tag{15}$$

Note that, for the case $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$, i.e., for an oscillator with static cubic non-linearity, it can be easily deduced from equations (15) that $G_n = 0$ for $n \geqslant 4$, so that only the first two of these equations need to be solved which in this case yield the exact G_0 and G_2. However, for the present case with ε_1 and ε_2 both non-zero, the G_n for $n \geqslant 4$ are not zero, and thus finding closed form solutions for all of the G_n becomes a difficult task unless one truncates the series in equation (14), and thus reduces the number of equations in the set of equations (15) needed to be solved, to an arbitrary finite number n. Therefore, in the present work, for the sake of simplicity, approximate closed form solutions for the G_n in equation (15) are sought by assuming that $G_n = 0$ for $n > 2$. Consequently, with $G_n = 0$ for $n > 2$, only the first three of the linear algebraic equations (15) do not vanish, which are then solved to yield.

$$G_0 = (R_1 b_2 - a_2 R_2)/\Delta, \qquad G_2 = (a_1 R_2 - R_1 b_1)/\Delta,$$

$$G_4 = -(3\varepsilon_1 A^2/8)/(1{\cdot}5 + \varepsilon_1 A^2/2), \qquad \Delta = a_1 b_2 - a_2 b_1, \tag{16}$$

where

$$a_1 = 1 + \varepsilon_1 A^2/2, \qquad a_2 = \varepsilon_1 A^2/4, \qquad R_1 = m + 0{\cdot}75\varepsilon_2 A^2$$

$$b_1 = \varepsilon_1 A^2/2, \qquad b_2 = 1 + \tfrac{3}{8}\varepsilon_1 A^2 + (3\varepsilon_1 A^2)/(24 + 8\varepsilon_1 A^2), \qquad R_2 = \varepsilon_2/4. \tag{17}$$

Next, upon taking the square root of equation (14), factoring out G_0, using the relation $f = \mathrm{d}T/\mathrm{d}t$, integrating the result from 0 to 2π in T and noting that the period τ in time T is 2π, one obtains

$$\tau = 2\pi(G_0)^{-1/2}\int_0^{2\pi} [1 + H_2 \cos 2T + H_4 \cos 4T]^{-1/2}\,\mathrm{d}T, \tag{18}$$

where

$$H_2 = G_2/G_0, \qquad H_4 = G_4/G_0. \tag{19}$$

Expanding the bracketed term on the right side of equation (18) into a power series, by noting that $H_2 + H_4$ is, in general, less than unity, and retaining terms only in H_2, one obtains [1]

$$\tau = 2\pi(G_0)^{-1/2}[1 + \tfrac{3}{15}H_2^2 + \frac{105}{1024}H_2^4 + \cdots]. \tag{20}$$

The period τ of the non-linear oscillator in equation (1) obtained by using equation (20) for various selected values of ε_1, ε_2 and m are presented and discussed in the next section. Because only the first two harmonics of the Fourier series expansion of the time transformation in equation (14) were used to arrive at equation (20), the results obtained by using this equation will be referred to as 2TT results.

Note that by setting $G_n = 0$ for $n \geqslant 2$, equations (16) yield $G_0 = (m + 0{\cdot}75\varepsilon^2 A^2)/(1 + \varepsilon_1 A^2)$ and the period τ from equation (20) becomes

$$\tau = 2\pi(m + 0{\cdot}75\varepsilon_2 A^2)^{-1/2}(1 + \varepsilon_1 A^2/2)^{1/2},$$

which is the same result in equation (6), obtained by the SHB method. Furthermore, in order to include the effect of higher harmonics on the period τ, one needs when using the HB method to solve a set of non-linear algebraic equations, whereas when using the TT method the effect of the higher harmonics can be examined, with comparatively less computational effort, by solving for additional G_n from a set of linear algebraic equations.

3. RESULTS AND DISCUSSION

The period τ of free vibration of the autonomous non-linear conservative oscillator modelled by equation (1) was calculated, for given values of the parameters ε_1, ε_2 and m and motion amplitude A, analytically by using the SHB, equation (6), the 2THB, equations (8)–(11), and the 2TT, equations (16), (17), (19) and (20), and numerically by using the

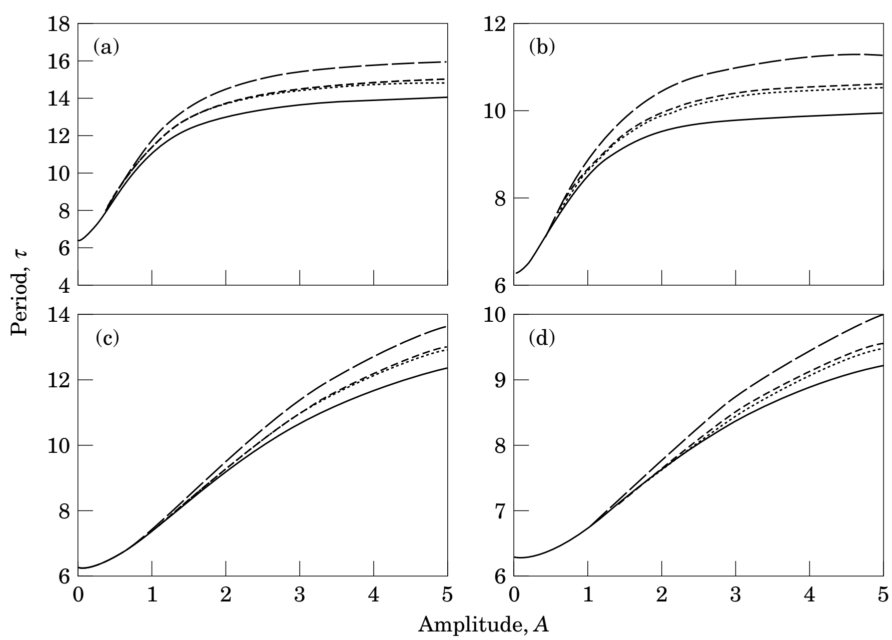

Figure 1. The variation of period τ with amplitude A for $m = 1$. ε_1, ε_2 values: (a) 10, 1·0; (b) 5, 1·0; (c) 1·0, 0·1; (d) 0·5, 0·1 —, Numerical; — —, SHB; ---, 2THB; ······, 2TT.

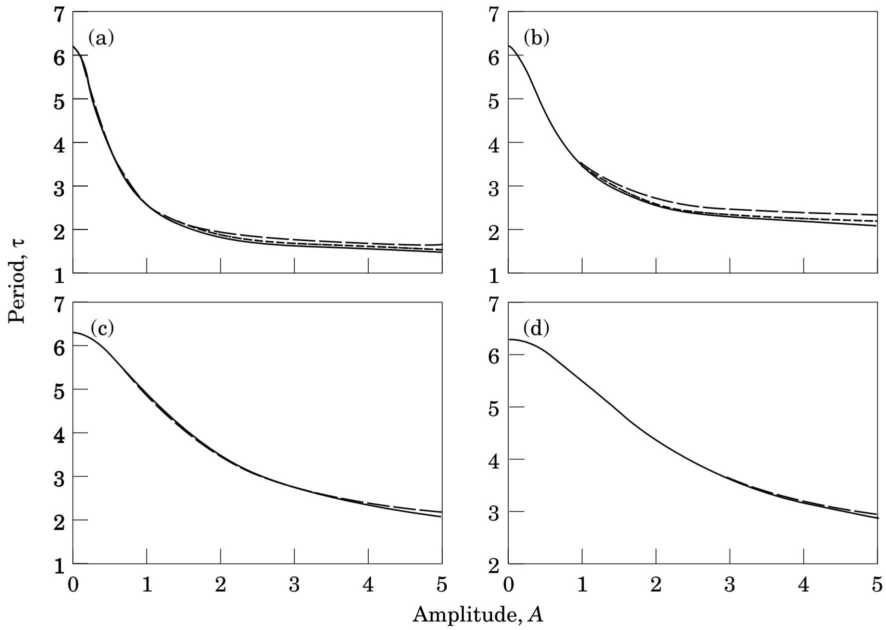

Figure 2. As Figure 1, but with the following ε_1, ε_2 values: (a) 1·0, 10; (b)1.0, 5.0; (c)0·1, 1·0; (d) 0·1, 0·5.

fourth order Runge–Kutta method with a 10^{-3} integration step size. The concern of this work is on the strongly non-linear cases; therefore, the period τ was calculated for cases in which ε_1 and/or ε_2 are not small compared to unity and the amplitude A of motion is of order unity. Examples of the results of these calculations for various selected cases of m, ε_1 and ε_2 are displayed in Figures 1–5.

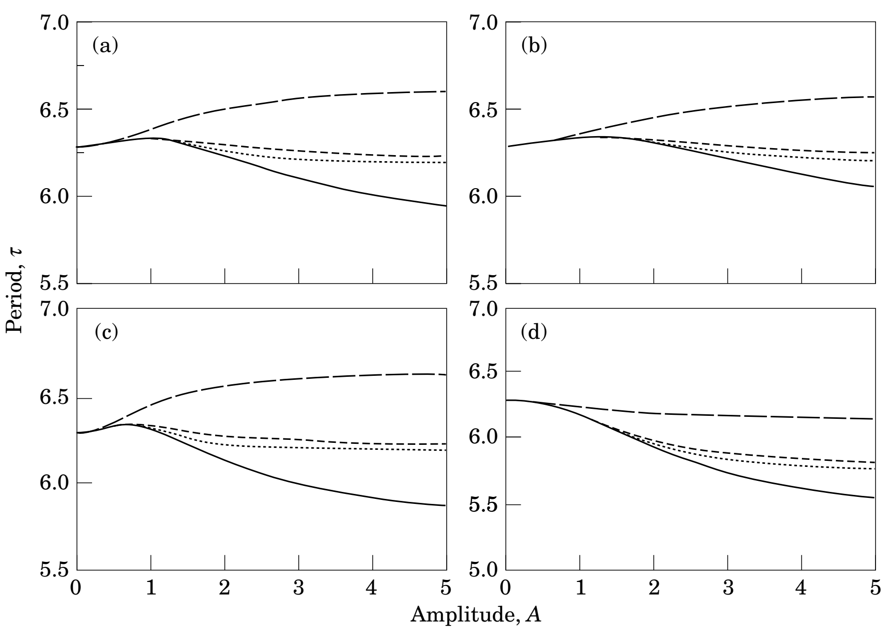

Figure 3. As Figure 1, but with the following ε_1, ε_2 values: (a) 0·1, 0·6; (b) 0·5, 0·3; (c) 2·0, 1·2; (d) 1·0, 0·7.

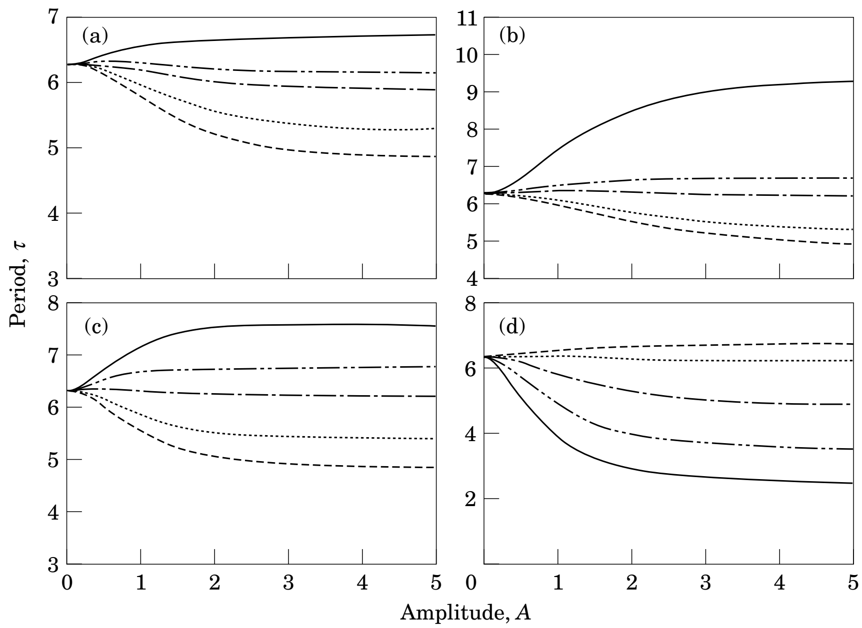

Figure 4. The variation of period I with amplitude A given by using the 2TT method, for $m = 1{\cdot}0$. (a) $\varepsilon_2 = 1{\cdot}0$; (b) $\varepsilon_2 = 0{\cdot}5$; (c) $\varepsilon_2 = 2{\cdot}0$; (d)$\varepsilon_1 = 1{\cdot}0$. (a) ε_1 values:, -----, 1·0; · · · · · , 1·2; ·--·, 1·5; ·—··, 1·65; —, 2·0, (b) ε_1 values: ----, 0· 5; · · · · · , 0·6; –-–, 0·85; —··—, 1·0; — 2·0. (c) ε_1 values: -----, 2·0; · · ·· · , 2·5; –·–, 3·35; —··—, 4·0; —, 5·0. (d) ε_2 values: · · · · ·, 0·5; ....., 0·6; —·—, 1·0; – – – – – – – – – , 2·0; —, 4·0.

In Figures 1 and 2, the period–amplitude (τ–A) variations for the statically stable ($m = 1$) oscillator obtained analytically by using the SHB, 2THB and 2TT methods are compared to those obtained numerically for the cases in which ε_1 is relatively large or small with respect to ε_2. The results in these figures indicate that when ε_1 is relatively large or small with respect to ε_2 then: (1) the accuracy of the SHB solution is good for small, and fair for relatively moderate values of the amplitude $A < 1$, but becomes poor, and eventually deteriorates, when A approaches, and becomes greater than 1; (2) the accuracy of the 2TT solutions is fairly good for moderate values of A, and becomes poor, but is appreciably better than that of the SHB solution, for relatively large A; (3) the accuracy of the 2TT solution at moderate and large values of A is better than that of the 2THB solution; (4) the accuracy of the three solutions, SHB, 2THB and 2TT, for moderate and large A is better when $\varepsilon_2 \gg \varepsilon_1$ than when $\varepsilon_1 \gg \varepsilon_2$; (5) all of the three analytic SHB, 2THB and 2TT solutions predict the correct qualitative τ–A behavior when $\varepsilon_1 \gg \varepsilon_2$ or $\varepsilon_1 \ll \varepsilon_2$; (6) the three analytic, SHB, 2THB, 2TT and numerical solutions show that the period of the oscillator in equation (1) for the above cases, i.e., $m = 1$ with $\varepsilon_1 \gg \varepsilon_2$ or $\varepsilon_1 \gg \varepsilon_2$, becomes constant nearly independent of amplitude A as A becomes relatively large.

As shown in Figure 3 examples of the qualitative failure of the SHB method for the case $m = 1$. These figures, and others not shown, indicate that when the inertia and static non-linearities of the statically stable ($m = 1$) oscillator are nearly of the same strength, the SHB method may fail to predict the correct qualitative behavior of the τ–A variation. For example, from Figure 3(a), it can be seen that, for $m = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 0{\cdot}6$, the SHB solution predicts a softening τ–A behavior, while the 2THB and 2TT as well as the numerical solutions predict that the τ–A behavior for this oscillator is of the hardening type. The results in Figure 3, as well as others not shown, indicate that when the ratio $\varepsilon_1/\varepsilon_2$, with $m = 1$, is, roughly, in the range $1{\cdot}5 < \varepsilon_1/\varepsilon_2 < 1{\cdot}8$, the SHB solution fails qualitatively, as it incorrectly predicts softening behavior of the τ–A variation for the oscillator in equation (1). It is to be noted that the 2THB and 2TT solutions can also fail and yield qualitatively wrong period–amplitude behavior for the above oscillator; however, this failure occurs over a smaller subrange of the above range in the SHB method. The results in Figure 3, like those in Figures 1, 2, also indicate that the period of the non-linear oscillator, for both the hardening and softening cases, becomes nearly constant, independent of motion amplitude A at relatively large values of A.

In Figure 4 are shown examples of the effects of variations in the relative strength of inertia non-linearity with respect to the hardening non-linearity, i.e., effects of variation in ε_1 with respect to ε_2, for the case $m = 1$ on the τ–A variations obtained by using the 2TT result. These results, as well as those in Figures 1(a)–3(d) and others not shown, indicate the following: (1) a small variation in the value of ε_1 and/or ε_2 can lead to relatively large variations, both quantitative and qualitative, in the calculated value of the period τ, especially at moderate and large values of A; (2) the τ–A variation exhibits a softening behavior when, roughly, $\varepsilon_1/\varepsilon_2 > 1{\cdot}6$, and a hardening behavior when $\varepsilon_1\varepsilon_2 < 1{\cdot}6$; (3) when $\varepsilon_1 \sim 1{\cdot}6\varepsilon_2$ the period of the non-linear oscillator in equation (1), with $m = 1$, becomes nearly constant, equal to the linear period 2π, independent of motion amplitude A for all values of A. Note that the SHB solution in equation (6) predicts that, by setting the right side of this equation equal to 2π, the period of the non-linear oscillator in equation (1) becomes a constant independent of motion amplitude A whenever $\varepsilon_1/\varepsilon_2$ is equal to 1·5.

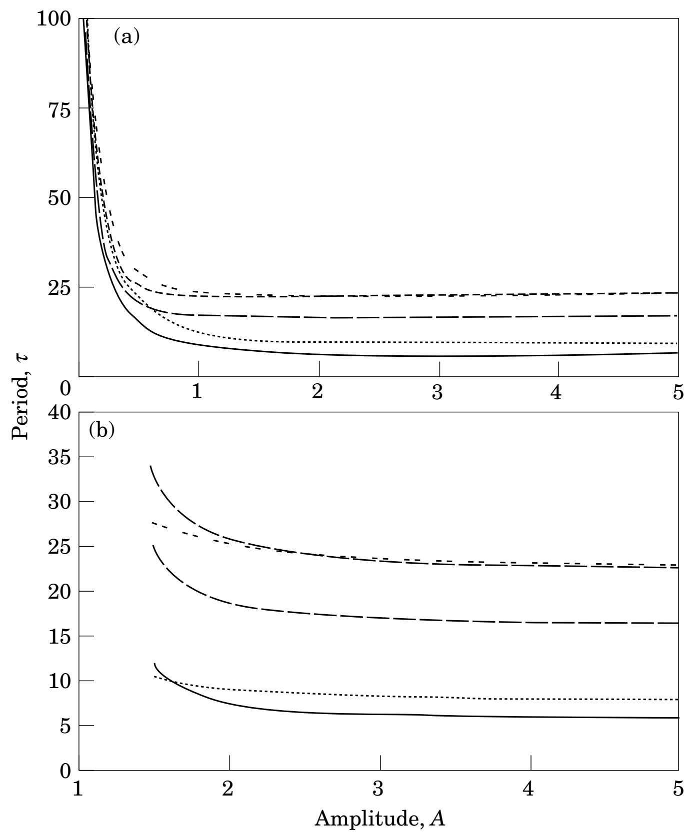

Figure 5. The variation of period τ with amplitude A obtained by using 2TT method. (a) $m = 0$; (b) $m = -1{\cdot}0$. ε_1, ε_2 values: ——, 1·0, 1·0; ———, 10, 1·0; – – – –, 20, 1·0; -----1·0, 0·5; · · · · ·, 10, 0·5.

In Figure 5 are shown examples of τ–A variations for the cases in which the associated linear oscillator in equation (1) is neutrally stable ($m = 0$), or statically unstable ($m = -1$), respectively. These results, and others not shown, indicate that the period τ of these oscillators exhibits a hardening behavior with amplitude A variations regardless of the relative strength of the inertia non-linearity with respect to the static hardening non-linearity: i.e., regardless of the value of ε_1 with respect to ε_2 with $\varepsilon_1 > 0$, and $\varepsilon_2 > 0$. These results also indicate that the period τ of these types of oscillators becomes nearly constant independent of amplitude A at relatively smaller values of A than those of the case $m = 1$ where the associated linear oscillator is statically stable.

4. CONCLUSIONS

On the basis of the results presented in this work, one may conclude that when equation (1) is used to model physical systems, such as the assumed single-mode planar large amplitude free vibration of an inextensible cantilever beam element which may or may not carry an intermediate inertia element, as indicated in section (1), one should be careful not only in choosing the appropriate solution method but also in evaluating the system parameters ε_1 and ε_2: i.e., in selecting the appropriate mode shape deflection. For example, the results of the present study show that a small variation in the value of ε_1 and/or ε_2 along with approximate first order solution, such as the SHB solution, can lead to appreciable, not only quantitative, but also qualitative errors in the predicted response. Therefore, in order to avoid errors, one should compare the approximate analytic solution of the method used with those of other approximate analytical methods and with numerical ones. The present results also indicate that when the strength of the inertia non-linearities is about 1·6 that of static hardening non-linearity; i.e., $\varepsilon_1 \sim \varepsilon_2$, the period of the cubically non-linear oscillator in equation (1) for which the associated linear oscillator is statically stable ($m = 1$) becomes nearly constant, equal to the linear period, independent of amplitude of motion A for all values of A. This implies that the resonance response in this case resembles that of the linear oscillator, and thus one expects in this case the resonance response of the non-linear oscillator not to exhibit forms of behaviors such as chaos and amplitude jumps.

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