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THE DIRECT-PERTURBATION METHOD VERSUS THE DISCRETIZATION-PERTURBATION METHOD: LINEAR SYSTEMS

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1. INTRODUCTION

Vibrations of continuous systems are modelled in the form of a partial differential equation system. In seeking approximate analytical solutions of these systems, one common choice is to discretize the partial differential equation system and then to apply perturbation methods to the resulting ordinary differential system (the discretization-perturbation method). An alternative approach is to seek approximate solutions of the original partial differential system. In this approach, perturbation methods are applied directly to the partial differential system (the direct-perturbation method). Comparisons of these two methods have appeared in the literature for various mathematical models [1–7]. Some of the work has addressed the comparisons for finite mode truncations [1–4] and some others for infinite modes [4–7]. It is shown that, while both methods produce identical results for infinite number of modes, the direct-perturbation method produces more accurate results for finite mode truncations. This is because the spatial functions appearing at higher orders of approximation represent the real system better in the case of the direct-perturbation method. It is shown that [5], for finite mode truncations, the spatial functions are the converged functions obtained by using the infinite series of eigenfunctions calculated at the first level of approximation. However, for the discretization-perturbation method, the spatial functions appearing at higher orders of approximations are only approximate, reducing the accuracy of the overall system.

In all of the previous work [1–7], comparisons have been made for non-linear systems, especially for systems having quadratic and cubic non-linearities. No discussions are presented for linear systems. This may lead one to conclude that the differences in results occur due to the non-linearities. However, such a conclusion would be wrong. Differences in results arise even for linear systems. As an illustration, we present here a case of a linear parametrically excited system. Instead of treating a specific problem, the formalism given in references [2] and [5] is followed and solutions are presented for arbitrary spatial operators. The algorithms developed are then applied to a specific problem. Differences in results occur if a higher order perturbation scheme is employed and if the boundary value problems appearing at higher orders of approximations yield different solutions in the case of direct treatment.

It is to be noted that results of the discretization-perturbation method would converge to those of the direct-perturbation method if the number of modes taken into consideration

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are increased. The convergence property mainly depends on the specific choice of eigenfunctions. For some cases, the convergence might be very poor, requiring a large number of modes, which is usually impractical. Alternatively, one may choose to attack the same problem using direct perturbation method with fewer modes.

2. EQUATION OF MOTION

We treat the following parametrically excited, linear, non-dimensional partial differential system:

$$\ddot{w} + \hat{\mu}\dot{w} + \mathbf{L}_0(w) + \varepsilon(F(x)\cos\Omega t)\mathbf{L}_1(w) = 0, \tag{1}$$

$$\mathbf{B}_1(w) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(w) = 0 \quad \text{at } x = 1, \tag{2}$$

where $w(x, t)$ is the response, $\hat{\mu}$ is the damping coefficient, ε is a small dimensionless parameter, and $F(x)$ and Ω are the parametric excitation amplitude and frequency, respectively. $\mathbf{L}_0$, $\mathbf{L}_1$, $\mathbf{B}_1$ and $\mathbf{B}_2$ are arbitrary linear spatial differential and/or integral operators. The dot denotes differentiation with respect to time t and the prime denotes differentiation with respect to the spatial variable x.

Defining a new time variable

$$T = (\Omega/2)t \tag{3}$$

we transform equation (1) to the following more convenient form for approximate analysis:

$$\frac{\Omega^2}{4}\ddot{w} + \bar{\mu}\dot{w} + \mathbf{L}_0(w) + \varepsilon(F(x)\cos 2T)\mathbf{L}_1(w) = 0, \tag{4}$$

where

$$\bar{\mu} = \hat{\mu}(\Omega/2). \tag{5}$$

We will investigate the steady state solutions of equation (4) for the case of principal parametric resonances ($\Omega \simeq 2\omega$, ω being the natural frequency of the unperturbed system). Solutions will be presented using both the direct-perturbation method and the discretization-perturbation method.

3. DIRECT-PERTURBATION METHOD

A higher order perturbation scheme will be applied to the partial differential system governed by equations (4) and (2). The version of the method of multiple scales first proposed by Rahman and Burton [8] will be employed, since this method is more accurate in predicting the steady state response compared to the usual method of reconstitution. We assume expansions of the form

$$w(x, T; \varepsilon) = w_0(x, T_0, T_1, T_2) + \varepsilon w_1(x, T_0, T_1, T_2) + \varepsilon^2 w_2(x, T_0, T_1, T_2) + \cdots, \tag{6}$$

$$\Omega^2 = 4(\omega^2 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \cdots), \qquad \bar{\mu} = \varepsilon\mu_1 + \varepsilon^2\mu_2 + \cdots, \tag{7, 8}$$

where $T_0 = T$ is the usual fast-time scale, and $T_1 = \varepsilon T$ and $T_2 = \varepsilon^2 T$ are the slow-time scales. Time derivatives are represented as

$$\mathrm{d}/\mathrm{d}T = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \qquad \mathrm{d}^2/\mathrm{d}T^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots. \tag{9}$$

Substituting equations (6)–(9) into equations (4) and (2), and separating terms at each order of ε, we have the following:

Order 1: $\omega^2 D_0^2 w_0 + \mathbf{L}_0(w_0) = 0, \quad \mathbf{B}_1(w_0) = 0 \quad \text{at } x = 0, \quad \mathbf{B}_2(w_0) = 0 \quad \text{at } x = 1;$ (10, 11)

Order ε: $\omega^2 D_0^2 w_1 + \mathbf{L}_0(w_1) = -2\omega^2 D_0 D_1 w_0 - \sigma_1 D_0^2 w_0 - \mu_1 D_0 w_0 - (F \cos 2T_0)\mathbf{L}_1(w_0),$ (12)

$$\mathbf{B}_1(w_1) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(w_1) = 0 \quad \text{at } x = 1; \tag{13}$$

Order ε^2:

$$\begin{aligned}\omega^2 D_0^2 w_2 + \mathbf{L}_0(w_2) = {} & -2\omega^2 D_0 D_1 w_1 - \omega^2(D_1^2 + 2D_0 D_2)w_0 \\ & - \sigma_1 D_0^2 w_1 - 2\sigma_1 D_0 D_1 w_0 - \sigma_2 D_0^2 w_0 - \mu_1 D_0 w_1 \\ & - \mu_1 D_1 w_0 - \mu_2 D_0 w_0 - (F \cos 2T_0)\mathbf{L}_1(w_1),\end{aligned} \tag{14}$$

$$\mathbf{B}_1(w_2) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(w_2) = 0 \quad \text{at} \quad x = 1. \tag{15}$$

At order 1, the solution is

$$w_0 = (A(T_1, T_2)\,\mathrm{e}^{\mathrm{i}T_0} + cc)Y(x), \tag{16}$$

where cc denotes the complex conjugate of the preceding terms. The $Y(x)$ functions satisfy the differential system

$$\mathbf{L}_0(Y) - \omega^2 Y = 0, \qquad \mathbf{B}_1(Y) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(Y) = 0 \quad \text{at } x = 1, \tag{17}$$

and are normalized such that $\int_0^1 Y^2\,\mathrm{d}x = 1$. The above problem is an eigenvalue–eigenfunction problem.

Substituting equation (16) into the right side of equation (12), assuming a solution for w_1 of the form

$$w_1 = \varphi_1(x, T_1, T_2)\,\mathrm{e}^{\mathrm{i}T_0} + W_1(x, T_0, T_1, T_2) + cc, \tag{18}$$

we obtain

$$\mathbf{L}_0(\varphi_1) - \omega^2\varphi_1 = -2\mathrm{i}\omega^2 D_1 A Y + (\sigma_1 - \mu_1 \mathrm{i})AY - \frac{F}{2}\mathbf{L}_1(Y)\bar{A}, \tag{19}$$

$$\mathbf{B}_1(\varphi_1) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(\varphi_1) = 0 \quad \text{at } x = 1, \tag{20}$$

$$\omega^2 D_0^2 W_1 + \mathbf{L}_0(W_1) = -\frac{F}{2}\mathbf{L}_1(Y)(A\,\mathrm{e}^{3\mathrm{i}T_0} + cc), \tag{21}$$

$$\mathbf{B}_1(W_1) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(W_1) = 0 \quad \text{at } x = 1. \tag{22}$$

The homogenous problem of equations (19) and (20) possesses a non-trivial solution. For the non-homogenous problem to possess a solution, a solvability condition should be satisfied [9]. The condition is

$$2\mathrm{i}\omega^2 D_1 A = (\sigma_1 - \mu_1 \mathrm{i})A - \alpha\bar{A}, \tag{23}$$

where

$$\alpha = \tfrac{1}{2}\int_0^1 FY\mathbf{L}_1(Y)\,\mathrm{d}x. \tag{24}$$

For steady state solutions, $D_1A = 0$ and, substituting the polar form $A = 1/2a\,e^{i\beta}$ in equation (23), we find the first correction to the frequency,

$$\sigma_1 = \mp\sqrt{\alpha^2 - \mu_1^2}. \tag{25}$$

Having eliminated the secular terms, we are left with equations (21) and (22). A solution may be written of the form

$$W_1 = \tfrac{1}{2}(A\,e^{3iT_0} + cc)\phi(x), \tag{26}$$

where $\phi(x)$ satisfy the boundary value problem

$$\mathbf{L}_0(\phi) - 9\omega^2\phi = -F\mathbf{L}_1(Y), \qquad \mathbf{B}_1(\phi) = 0 \quad \text{at } x = 0, \qquad \mathbf{B}_2(\phi) = 0 \quad \text{at } x = 1. \tag{27}$$

The boundary value problem at the first order (equation (17)) and the boundary value problem at the second order (equation (27)) would yield different spatial functions in general. It is shown in reference [5] that the $\phi(x)$ functions are the converged functions of the infinite sum of eigenfunctions Y and hence represent the spatial variation of the real system better at this order of approximation. In contrast to this, in the discretization-perturbation method, the spatial variations would be assumed to be represented by the same $Y(x)$ functions at this order of approximation, leading to inaccuracy.

The solvability condition at order ε^2 is

$$2i\omega^2 D_2A = (\sigma_2 - \mu_2 i)A - \gamma A, \tag{28}$$

where

$$\gamma = \tfrac{1}{4}\int_0^1 FY\mathbf{L}_1(\phi)\,dx. \tag{29}$$

For steady state solutions, $D_2A = 0$ and the second correction to the frequency and damping are

$$\sigma_2 = \gamma, \qquad \mu_2 = 0. \tag{30}$$

Hence the excitation frequency is found to be

$$\Omega^2 = 4(\omega^2 \mp \varepsilon\sqrt{\alpha^2 - \mu_1^2} + \varepsilon^2\gamma + \cdots). \tag{31}$$

Using the polar form for the complex amplitudes, changing back to the original time variable t, we find the approximate response:

$$w(x, t; \varepsilon) = a\cos\left(\frac{\Omega}{2}t + \beta\right)Y(x) + \varepsilon\frac{a}{2}\cos\left(\frac{3\Omega}{2}t + \beta\right)\phi(x) + O(\varepsilon^2). \tag{32}$$

In the next section, we find the approximate frequency and response function using discretization-perturbation method.

4. DISCRETIZATION-PERTURBATION METHOD

In this section, we solve the same problem, this time by first discretizing the equations and then by applying perturbations to the resulting equations. Similarly to the previous analysis, we assume a single-mode discretization of the form

$$w(x, T) = q(T)Y(x), \tag{33}$$

where the $Y(x)$ satisfy equation (17). Substituting equation (33) into equation (4), multiplying the equation by $Y(x)$, and integrating over the domain, we have

$$\frac{\Omega^2}{4}\ddot{q} + \bar{\mu}\dot{q} + \omega^2 q + \varepsilon(\cos 2T)2\alpha q = 0, \tag{34}$$

where α is defined in equation (24). Assuming the expansion

$$q(T;\varepsilon) = q_0(T_0, T_1, T_2) + \varepsilon q_1(T_0, T_1, T_2) + \varepsilon^2 q_2(T_0, T_1, T_2) + \cdots \tag{35}$$

and substituting equation (35) together with equations (7)–(9) into equation (34), we obtain the following:

Order 1: $$\omega^2 D_0^2 q_0 + \omega^2 q_0 = 0; \tag{36}$$

Order ε: $$\omega^2 D_0^2 q_1 + \omega^2 q_1 = -2\omega^2 D_0 D_1 q_0 - \sigma_1 D_0^2 q_0 - \mu_1 D_0 q_0 - (2\alpha\cos 2T_0)q_0; \tag{37}$$

Order ε^2: $$\begin{aligned}\omega^2 D_0^2 q_2 + \omega^2 q_2 = {} & -2\omega^2 D_0 D_1 q_1 - \omega^2(D_1^2 + 2D_0 D_2)q_0 - \sigma_1 D_0^2 q_1 \\ & - 2\sigma_1 D_0 D_1 q_0 - \sigma_2 D_0^2 q_0 - \mu_1 D_0 q_1 - \mu_1 D_1 q_0 \\ & - \mu_2 D_0 q_0 - (2\alpha\cos 2T_0)q_1.\end{aligned} \tag{38}$$

The solution at order 1 is

$$q_0 = A(T_1, T_2)\,e^{iT_0} + cc. \tag{39}$$

Substituting this solution into the right side of equation (37), and eliminating the secular terms yields

$$2i\omega^2 D_1 A = (\sigma_1 - \mu_1 i)A - \alpha\bar{A}. \tag{40}$$

Comparing equation (40) with equation (23), we see that the results are identical up to this order of approximation. The first correction to the frequency is the same as given in equation (25).

Now solving equation (37), we obtain

$$q_1 = \frac{\alpha}{8\omega^2}(A\,e^{3iT_0} + cc). \tag{41}$$

In this approach, the spatial variation is again assumed to be represented by $Y(x)$. However, in the previous section, it is shown that the spatial variation is represented by a different function, $\phi(x)$. This difference will affect the results at the next order of approximation.

Substituting equations (39) and (41) into the right side of equation (38), and eliminating the secular terms, we obtain

$$2i\omega^2 D_2 A = (\sigma_2 - \mu_2 i)A - \frac{\alpha^2}{8\omega^2}A. \tag{42}$$

Requiring $D_2 A = 0$ for steady state solutions, substituting the polar form, and separating the real and imaginary parts, we obtain

$$\sigma_2 = \frac{\alpha^2}{8\omega^2}, \qquad \mu_2 = 0. \tag{43}$$

Hence, the frequency is determined up to $O(\varepsilon^2)$:

$$\Omega^2 = 4(\omega^2 \mp \varepsilon\sqrt{\alpha^2 - \mu_1^2} + \varepsilon^2(\alpha^2/8\omega^2) + \cdots). \tag{44}$$

Comparing equations (44) and (31), we see that results differ at $O(\varepsilon^2)$. In general, γ is not equal to $\alpha^2/8\omega^2$. The approximate response is

$$w(x, t; \varepsilon) = a \cos\left(\frac{\Omega}{2} t + \beta\right) Y(x) + \varepsilon \frac{\alpha a}{8\omega^2} \cos\left(\frac{3\Omega}{2} t + \beta\right) Y(x) + O(\varepsilon^2). \tag{45}$$

Comparing equations (45) and (32), we see that the responses differ at $O(\varepsilon)$, due to the spatial variations. Note that all of the results in this section, as well as those in the previous section, can be retrieved using the Lindstedt–Poincaré technique.

5. AN EXAMPLE

In this section, the algorithms developed in sections 3 and 4 are applied to a specific problem. We consider the following simple equation so that the integrals are easier to handle:

$$\ddot{w} + \hat{\mu}\dot{w} - w'' + \varepsilon(F \cos \Omega t)(b_1 w + b_2 w') = 0, \qquad w(0, t) = w(1, t) = 0. \tag{46, 47}$$

The specific operators are

$$\mathbf{L}_0(w) = -w'', \qquad \mathbf{L}_1(w) = b_1 w + b_2 w'. \tag{48}$$

Since solutions are presented for the general case in the previous sections, all we need is to calculate $Y(x)$, ω, α, $\phi(x)$ and γ. $Y(x)$ is determined by the boundary value problem in equation (17) or

$$Y'' + \omega^2 Y = 0, \qquad Y(0) = Y(1) = 0. \tag{49}$$

The solution is

$$Y = \sqrt{2} \sin n\pi x, \quad \omega = n\pi, \qquad n = 1, 2, \ldots . \tag{50}$$

Assuming F to be constant, α is calculated from equation (24):

$$\alpha = \tfrac{1}{2} F b_1. \tag{51}$$

$\phi(x)$ can now be calculated from the boundary value problem given in equation (27)

$$\phi'' + 9\omega^2 \phi = F(b_1 Y + b_2 Y'), \qquad \phi(0) = \phi(1) = 0. \tag{52}$$

The solution is

$$\phi(x) = \frac{F\sqrt{2}}{8n^2\pi^2} [b_2 n\pi(\cos n\pi x - \cos 3n\pi x) + b_1 \sin n\pi x]. \tag{53}$$

γ is calculated from equation (29):

$$\gamma = \frac{F^2}{32n^2\pi^2} (b_1^2 - b_2^2 n^2 \pi^2). \tag{54}$$

Hence the frequencies for the two cases are as follows:

Direct-perturbation method,

$$\Omega^2 = 4\left(n^2\pi^2 \mp \varepsilon \sqrt{\frac{F^2 b_1^2}{4} - \mu_1^2} + \varepsilon^2 \frac{F^2}{32n^2\pi^2} (b_1^2 - b_2^2 n^2 \pi^2) + \cdots\right). \tag{55}$$

Discretization-perturbation method,

$$\Omega^2 = 4\left(n^2\pi^2 \mp \varepsilon\sqrt{\frac{F^2b_1^2}{4} - \mu_1^2} + \varepsilon^2\frac{F^2b_1^2}{32n^2\pi^2} + \cdots\right). \tag{56}$$

Note that the information regarding b_2 is lost at $O(\varepsilon^2)$ in the discretization-perturbation method.

The responses are as follows:

Direct-perturbation method,

$$w(x, t; \varepsilon) = a\sqrt{2}\cos\left(\frac{\Omega}{2}t + \beta\right)\sin n\pi x + \varepsilon\frac{aF\sqrt{2}}{16n^2\pi^2}\cos\left(\frac{3\Omega}{2}t + \beta\right) \times [b_1 \sin n\pi x + b_2 n\pi(\cos n\pi x - \cos 3n\pi x)] + O(\varepsilon^2); \tag{57}$$

Discretization-perturbation method,

$$w(x, t; \varepsilon) = a\sqrt{2}\cos\left(\frac{\Omega}{2}t + \beta\right)\sin n\pi x + \varepsilon\frac{aF\sqrt{2}}{16n^2\pi^2}\cos\left(\frac{3\Omega}{2}t + \beta\right)[b_1 \sin n\pi x] + O(\varepsilon^2). \tag{58}$$

Comparing equations (57) and (58), we again conclude that information regarding b_2 is lost in the discretization-perturbation method. The spatial variation at $O(\varepsilon)$ in equation (57) is in fact the converged infinite sum of eigenfunctions multiplied by appropriate constants [5] and hence better represents the real system. Note that an infinite mode analysis would yield identical results.

6. CONCLUDING REMARKS

Depending on the analysis presented here as well as on the previous work [1–7], the following conclusions are of vital importance:

(1) The results of the direct-perturbation method and the discretization-perturbation method would be identical for infinite modes.

(2) The direct-perturbation method would yield better results compared to discretization-perturbation method for finite mode truncations if (a) a higher order perturbation scheme were used; (b) the boundary value problems were to yield different solutions at each order of approximation for the direct-perturbation method.

(3) The above conclusions are valid for both linear and non-linear systems.

If higher order perturbation schemes are not used, both methods yield identical results. However, the choice of orthogonal basis functions might not be so straightforward for some more involved cases and a transformation of equations to a convenient form may be needed. In such cases, employing the direct-perturbation method would be more straightforward, at the expense that the algebra might be more involved.

Finally, the conclusions are not restricted to vibrations of continuous systems, but are valid in general for any physical problem modelled in the form of partial differential equations.

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REFERENCES

1. A. H. NAYFEH, J. F. NAYFEH and D. T. MOOK 1992 *Nonlinear Dynamics* **3**, 145–162. On methods for continuous systems with quadratic and cubic nonlinearities.
2. M. PAKDEMIRLI 1994 *Mechanics Research Communications* **21**, 203–208. A comparison of two perturbation methods for vibrations of systems with quadratic and cubic nonlinearities.
3. M. PAKDEMIRLI, S. A. NAYFEH and A. H. NAYFEH 1995 *Nonlinear Dynamics* **8**, 65–83. Analysis of one-to-one autoparametric resonances in cables: discretization vs. direct treatment.
4. A. H. NAYFEH, S. A. NAYFEH and M. PAKDEMİRLİ 1995 in *Nonlinear Dynamics and Stochastic Mechanics* (N. S. Namachchivaya and W. Kliemann, editors), 175–200. Boca Raton, FL: CRC Press. On the discretization of weakly nonlinear spatially continuous systems.
5. M. PAKDEMİRLİ and H. BOYACI 1995 *Journal of Sound and Vibration* **186**, 837–845. Comparison of direct-perturbation methods with discretization-perturbation methods for non-linear vibrations.
6. A. H. NAYFEH and S. A. NAYFEH 1995 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **117**, 199–205. Nonlinear normal modes of a continuous system with quadratic nonlinearities.
7. A. H. NAYFEH and S. A. NAYFEH 1994 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration and Acoustics* **116**, 129–136. On nonlinear modes of continuous systems.
8. Z. RAHMAN and T. D. BURTON 1989 *Journal of Sound and Vibration* **133**, 369–379. On higher order methods of multiple scales in nonlinear oscillations-periodic steady state response.
9. A. H. NAYFEH 1981 *Introduction to Perturbation Techniques*. New York: John Wiley.