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LETTERS TO THE EDITOR

COMMENTS ON "ZONES OF CHAOTIC BEHAVIOUR IN THE PARAMETRICALLY EXCITED PENDULUM"

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In reference [1], Bishop and Clifford studied the global stability of three types of chaotic motion occurring in the parametrically excited pendulum where the pivot point is subjected to a vertical periodic driving. The equation of motion is written in the form [2]

$$\ddot{\theta} + \beta\dot{\theta} + (1 + p\cos\omega t)\sin\theta = 0, \tag{1}$$

where θ represents the angular displacement. Several technical applications of this equation are concerned with non-linear oscillations in engineering and physics. With the damping coefficient $\beta = 0{\cdot}1$ chaotic motions of *oscillatory* and *rotatory* type appear only in very narrow zones in the space of the parameters ω and p. Therefore, they are very difficult to find both numerically and experimentally. However, *tumbling* chaos, in which the pendulum completes an apparently random number of clockwise rotations before changing direction—such changes also include a number of oscillations about the vertical position—occurs over a large range of parameters. As a consequence this type of chaotic response is robust with respect to small variations in the system parameters. The reader may see from Figure 5(a) in reference [1] the broad zone of tumbling chaos in the space of the parameters ω and p.

The different types of chaotic behaviour were illustrated in reference [1] with the associated figures of the sampled attractor in the phase plane, the time history of the angular velocity and the orbit in the phase plane. In addition, the bifurcation diagram of the sampled angular displacement has been shown.

Table 1

The Liapounov dimension d_L in the space of the parameters ω and p (blanco zones or gaps indicate periodic motion or convergence to the origin in the phase plane)

	d_L for $\omega =$										
	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1	2·2	2·3
$p = 3{\cdot}0$								1·73	1·74	1·74	
$p = 2{\cdot}8$							1·75	1·73	1·73	1·73	
$p = 2{\cdot}6$					1·73	1·75	1·75		1·73	1·66	
$p = 2{\cdot}4$				1·72	1·75	1·76	1·72	1·73	1·73		
$p = 2{\cdot}2$		1·71	1·73	1·76	1·75	1·75		1·73	1·71		
$p = 2{\cdot}0$	1·71	1·73	1·75	1·74	1·74	1·73	1·74	1·73			
$p = 1{\cdot}8$	1·71	1·72	1·75	1·73	1·73		1·73				
$p = 1{\cdot}6$		1·72	1·73	1·73	1·71	1·72	1·72				
$p = 1{\cdot}4$		1·72	1·72	1·72		1·72					
$p = 1{\cdot}2$			1·71	1·69	1·72						
$p = 1{\cdot}0$				1·70							
$p = 0{\cdot}8$											

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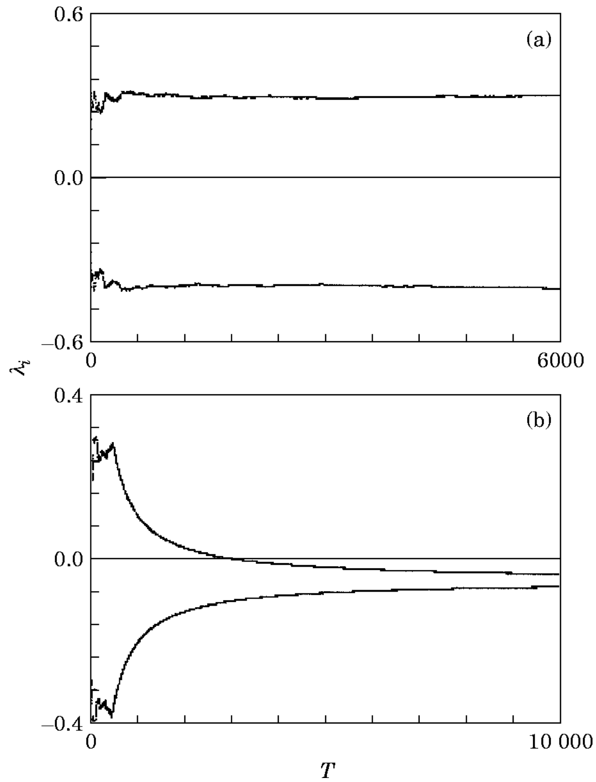

Figure 1. The stabilization of the Liapounov exponents over long times. (a) The chaotic case with $\omega = 1{\cdot}8$ and $p = 2{\cdot}4$; (b) the $8P$ periodic case with $\omega = 2$ and $p = 2{\cdot}6$ in the zone of tumbling chaos.

The aim of this letter is to confirm the robust feature of the chaotic solutions of tumbling type in the relevant zone in the parameter space *by a numerical computation of the Liapounov dimension*, which is a measure of the fractal nature of the corresponding attractor. Indeed, it will be shown that the Liapounov dimension changes only slightly over the broad zone of tumbling chaos in the parameter space.

According to Wolff's procedure [3–5] the Liapounov dimension is computed as follows. With the definitions $x_1 = \theta$, $x_2 = \dot{\theta}$ and $x_3 = \omega t$, the equation of motion is written as an autonomous system and the corresponding system of its first variational equations is derived. Following the Gram–Schmidt method, one can re-orthonormalize the solutions to the system of the first variational equations with respect to the reference solution. As the integration method the Runge–Kutta–Hŭta method of order six is used (see references [6, 7]). If $N_i(t)$ with $i = 1, 2, 3$ characterize the lengths of the solution vectors in the Gram–Schmidt procedure, then the Liapounov exponents λ_i are defined by the following averages over long times:

$$\lambda_i = \lim_{T\to\infty}[\ln N_i(t)]/T. \tag{2}$$

One of the Liapounov exponents, λ_3, is always zero. According to the Kaplan–Yorke relation the Liapounov dimension d_L is given by

$$d_L = 1 - \lambda_1/\lambda_2. \tag{3}$$

The numerical results for the Liapounov dimension d_L are listed in Table 1. The averaged values in relation (2) have been calculated over long times in order to assure the stabilization of the computation of the Liapounov exponents. In general $T = 6000$ has been taken, which was an efficient choice in order to obtain the desired stabilization for the Liapounov exponents although in some cases a higher value had to be chosen. The blanco zones or gaps in Table 1 indicate regular periodic motion, or in some cases attraction towards the stable origin in the phase plane. From this table it is seen that the Liapounov dimension fluctuates only slightly about the value $d_L = 1{\cdot}73$ in the large zone of tumbling chaos in the parameter space. Its maximum value, i.e., $d_L = 1{\cdot}76$ is attained in the case with $\omega = 1{\cdot}8$, $p = 2{\cdot}4$ and also for $\omega = 1{\cdot}6$, $p = 2{\cdot}2$. It is emphasized that the same numerical results were obtained by using different initial conditions for the reference orbit. For larger values of ω ($2{\cdot}3 \leqslant \omega \leqslant 3$) and (or) for smaller values of p ($0 \leqslant p \leqslant 0{\cdot}8$) periodic solutions or convergence towards the origin in the phase plane were found. The chaotic region is not extended to the left part in Table 1 at lower values of ω, although it is not excluded that chaotic response may occur.

These numerical results based on the computation of the Liapounov dimension confirm that tumbling chaos is robust with respect to small changes in the system parameters ω and p. The zone in the parameter space where tumbling chaos is found, as is seen from Table 1, nearly coincides with the zone indicated in Figure 5(a) in reference [1] and is further commented on at the end of page 146 in the cited reference.

The stabilization of the Liapounov exponents over long times is shown in Figure 1(a) for the case with $\omega = 1{\cdot}8$ and $p = 2{\cdot}4$. The final time has been taken as $T = 6000$, which

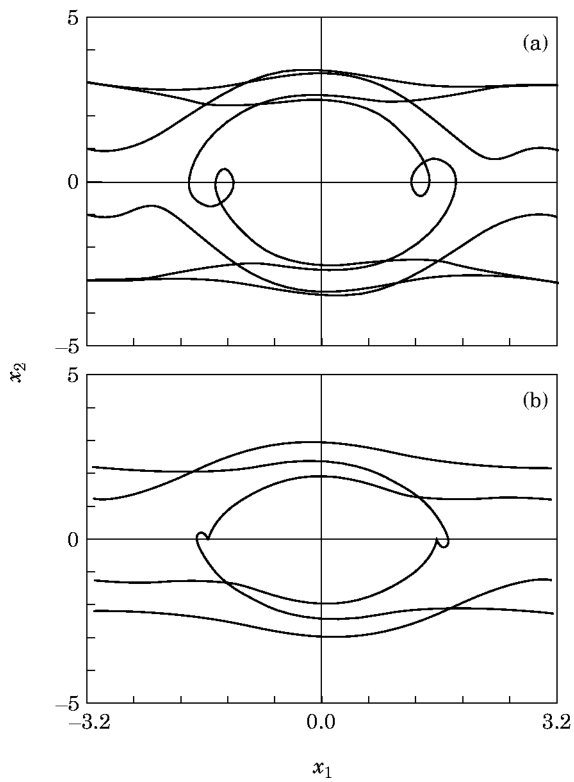

Figure 2. Periodic orbits in the phase plane in the zone of tumbling chaos. (a) $8P$ solution for $\omega = 2$, $p = 2{\cdot}6$; (b) $6P$ solution for $\omega = 1{\cdot}7$ and $p = 1{\cdot}4$.

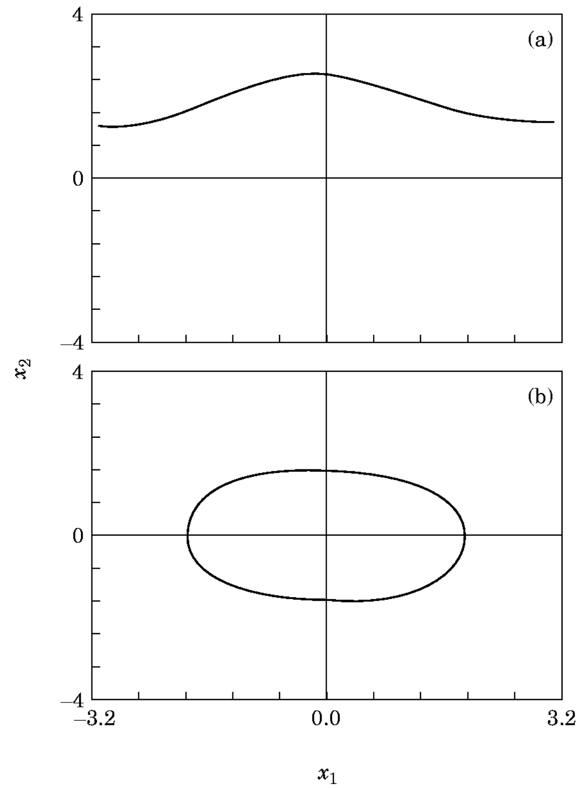

Figure 3. Coexisting orbits in the phase plane for $\omega = 1{\cdot}8$ and $p = 0{\cdot}8$. (a) $1P$ solution with winding number $W = 1$; (b) $2P$ solution with winding number $W = 0$.

corresponds to about 1720 cycles of the forcing period. The Liapounov exponents are given by $\lambda_1 = 0{\cdot}309$ and $\lambda_2 = -0{\cdot}409$. The Liapounov dimension is $d_L = 1{\cdot}76$: i.e., its maximum value in the zone of the parameter space under consideration. One of the Liapounov exponents is positive in this chaotic case.

As is seen from Table 1, several *periodic windows were found in the zone of tumbling chaos*: e.g., for $\omega = 1{\cdot}7$, $p = 1{\cdot}4$ ($6P$ solution with $P = 2\pi/\omega$); for $\omega = 1{\cdot}8$, $p = 1{\cdot}8$ ($8P$ solution); for $\omega = 1{\cdot}9$, $p = 2{\cdot}2$ ($8P$ solution) and for $\omega = 2$, $p = 2{\cdot}6$ ($8P$ solution). In these cases long chaotic transients have been noticed before they settle into a stable periodic solution. The behaviour of the Liapounov exponents for the case of the $8P$ solution with $\omega = 2$ and $p = 2{\cdot}6$ in Figure 1(b) is illustrated. The Liapounov exponents are both negative: $\lambda_1 = -0{\cdot}0342$ and $\lambda_2 = -0{\cdot}0658$. In Figure 2(a) is shown the phase portrait for this $8P$ solution with $x_1 = 0{\cdot}848561$ and $x_2 = 2{\cdot}883434$ at $t = 0$. The $6P$ solution for the case $\omega = 1{\cdot}7$, $p = 1{\cdot}4$ is illustrated in Figure 2(b), where $x_1 = 1{\cdot}454203$ and $x_2 = -2{\cdot}389148$ at $t = 0$. It is emphasized—without mentioning details—that periodic windows in the zone of tumbling chaos have already been found by Bishop and Clifford (see reference [1], page 146).

In certain cases *coexisting* $1P$ and $2P$ solutions have been detected. For example, with $\omega = 1{\cdot}8$, $p = 0{\cdot}8$ the $1P$ solution with $x_1 = -0{\cdot}675471$ and $x_2 = 2{\cdot}443927$ at $t = 0$ was found (see Figure 3(a) representing its orbit in the phase plane) and the $2P$ solution with $x_1 = 1{\cdot}878260$ and $x_2 = -0{\cdot}253306$ at $t = 0$ illustrated in Figure 3(b). For the $1P$ solution the winding number W defined by

$$W = \lim_{n\to\infty}(x_1^{(n)} - x_1^{(0)})/2\pi n, \tag{4}$$

is found to be $W = 1$. This reveals the rotatory type of this solution characterized by one complete rotation each forcing period. In the case of the $2P$ solution the winding number is $W = 0$, indicating the oscillatory nature of this solution.

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AUTHORS' REPLY

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In the letter by R. Van Dooren, the Liapunov dimension was calculated for the tumbling chaotic zone. A value of 1·73 was obtained over a wide region of parameter space, with narrow fluctuations in windows of period response. The fact that the Liapunov dimension remains roughly constant implies that the attractor does not alter its form (the attractor does not undergo any major crises; see reference [1]). This result, confirmed by our own findings on the Liapunov dimension, coupled with the fact that the motion is stable over a wide region in parameter space, reinforces the conclusion that the tumbling motion is robust. Moreover, in a large portion of the zone of stability it is the *only* stable solution.

The Liapunov dimension for the oscillating chaos is approximately 1·25, while for the rotating chaos was roughly 1·17. However these cannot be verified over a broad range of parameters since the attractor is stable only over a very narrow parameter range. To the naked eye these motions are almost indistinguishable from the periodic oscillating and rotating motions that precede it while the tumbling motions explore much of the phase space.

In a subsequent work [2], a pseudo-Anasov orbit of the pendulum was located which proved the existence of a horseshoe which extends much of the earlier numerical investigation of the chaotic response.

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