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ON THE CONTINUITY OF THE BOUNDARY VALUE PROBLEM FOR VIBRATING FREE–FREE STRAIGHT BEAMS UNDER AXIAL LOADS

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1. Introduction

A straight beam, essentially characterized by its mass and bending stiffness, represents one of the most fundamental modelling artefacts in structural engineering, and it remains a basic building block of the more complex matricial formulations of large structures so commonly used today as powerful computers have become available. A great variety of structural problems may be treated with beams as long as boundary conditions are appropriately defined. Free–free boundary conditions, where the ends of the beams are free from shear forces and bending moments, are probably less commonly used than more restrictive boundary conditions such as pinned, clamped etc. They do occur in some engineering applications, however, such as the towing of large and stiff marine pipelines (see, e.g., Nihous [1]). In such cases, a fairly large axial tension is also necessary to ensure an optimal alignment between pipeline, current and waves.

Investigation of the behavior of beams, and the determination of vibratory mode shapes, has been textbook material for some time (see, e.g., Timoshenko *et al*. [2], Gorman [3] and Blevins [4, 5]). The particular study of free–free beams under axial loads appears to have been initiated by Shaker [6], followed by more work by Bokaian [7, 8], until the recent investigation by Liu *et al*. [9]. These last authors numerically solved the characteristic equation of the problem for a wide range of arbitrary axial loads, and discovered that the lowest non-zero natural frequency had been hitherto overlooked in the case of tensile loads.

The intent of this note is to establish that the problem of a free–free beam under axial loads is continuous in the two well-known limits of a "pure" beam (no axial load) and a "pure" cable (no bending stiffness or, equivalently, infinitely large tension). This continuity analysis is meant to provide physical arguments to explain previous results, such as the discovery by Liu *et al*. [9], and to derive approximation formulas more physically grounded than mere curve fits, for engineering purposes. A first section briefly formulates the complete modal boundary value problem (BVP). The limiting BVPs of a "pure" free–free straight beam and a "pure" free–free straight cable are then presented, with their well-known solutions. The study then proceeds to establish that the solutions of the complete BVP converge, in the two limits of small and large axial loads, to the solutions of the limiting BVPs. Finally, the simple approximation formulas obtained in the continuity analysis are evaluated for accuracy.

2. Basic mathematical formulation

The full boundary value problem for the transverse motions $y(x, t)$ of a free–free straight beam under axial tension, BVP (I), is well-known (see, e.g., Blevins [5]), and

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is expressed below:

$$m\frac{\partial^2 y}{\partial t^2}+\frac{EI}{L^4}\frac{\partial^4 y}{\partial x^4}-\frac{T}{L^2}\frac{\partial^2 y}{\partial x^2}=0,\qquad \frac{\partial^2 y}{\partial x^2}(0)=\frac{\partial^2 y}{\partial x^2}(1)=0, \tag{1a}$$

$$\frac{\partial^3 y}{\partial x^3}(0)-\frac{TL^2}{EI}\frac{\partial y}{\partial x}(0)=\frac{\partial^3 y}{\partial x^3}(1)-\frac{TL^2}{EI}\frac{\partial y}{\partial x}(1)=0, \tag{1b}$$

The beam length L has been used to non-dimensionalize the axial co-ordinate x, and transverse displacement function $y(x, t)$; m represents the mass per unit length, EI the bending stiffness and T a uniform axial load taken positive when tensile. Evidently, t is the time variable.

With a zero right side, the above partial differential equation is valid for "free vibrations", i.e., in the absence of any "external" loading, or "forcing function", other than the specified axial force T (here, the meaning of "free" must be distinguished from that used in defining free end conditions!). The first set of end conditions (1a) represents vanishing bending moments, whereas the second set (1b) is the expression of vanishing shear forces.

The traditional method of solution is to separate variables x and t by seeking solutions of the form $y(x, t) = y(x)f(t)$, where the notation y has been extended to the spatial function, or mode shape, with no loss of generality. The mode shape equation thus obtained is

$$\frac{\partial^4 y}{\partial x^4}-\pi^2\gamma\frac{\partial^2 y}{\partial x^2}-Cy=0, \tag{1c}$$

where γ is a non-dimensional load parameter defined as $TL^2/EI\pi^2$. C is a real constant, and y must also satisfy boundary conditions (1a) and (1b).

The temporal function $f(t)$ is vibratory, with its corresponding (natural) frequency ω, if C is strictly positive; in those cases, C is written as β^4, where $\beta^4 = m\omega^2L^4/EI$. The definition of β^4 is an expression of the relationship between temporal and spatial components of the complete solution. Although the literature primarily focuses on vibratory mode shapes because of their number and physical importance, the widely used solution technique known as modal expansion, described for example in Timoshenko *et al*. [2] or Blevins [3], must use the complete set of mode shapes, including those corresponding to non-vibratory time functions if they exist.

Thus, when $C = 0$, $f(t)$ is a linear function of time; yet, it is well-known that the mode shape BVP admits the normalized solutions $y_c = 1$ for all values of γ, and the buckling function $y(x) = \sin(\pi x)$ for $\gamma = -1$.

On the other hand, little attention seems to have been given to possible mode shapes corresponding to strictly negative values of C, when $f(t)$ is an exponential function of time. It is probably anticipated that such mode shapes do not exist. A study is currently under way to investigate the question of non-vibratory mode shapes for free–free straight beams under axial loads. In the more traditional context of straight beams without axial loads, the proof that no mode exists for $C < 0$ is somewhat easier: the procedure for free–free boundary conditions is summarized in Appendix A.

In what follows, attention is restricted to the modal BVP, or some of its limiting forms, when $C \geqslant 0$. Since several authors, from Shaker [6] to Liu *et al*. [9], have presented the solution of this modal BVP in great detail, a superfluous repetition of the steps leading to that solution will be avoided henceforth, and only results will be recalled when necessary. In addition, the indexing conventions for natural frequencies and mode shapes are motivated by the continuity analysis, and may occasionally appear different from other authors' choice.

3. TWO LIMITING BOUNDARY VALUE PROBLEMS AND THEIR SOLUTIONS

3.1. *Free–free beam without axial load* (*BVP II*)

The reduced BVP obtained by setting $T = 0$ in BVP (I) has been textbook material for a long time (see, e.g., Blevins [5]). In this case, free–free boundary conditions are unique in as much as they result in the existence of two distinct rigid-body mode shapes, both corresponding to a zero natural frequency (i.e., rigid-body solutions are not vibratory); adopting the normalized convention $y(0) = 1$, these rigid body mode shapes are

$$y_c(x) = 1, \qquad y_0(x) = ax + 1 \quad (a \neq 0).$$

Because these two modes share the same natural frequency, they are not necessarily orthogonal; thus, in general:

$$\int_0^1 y_c(x) y_0(x)\, \mathrm{d}x \neq 0.$$

However, the arbitrary slope a may be chosen such that the above integral vanishes, although there is no mathematical requirement to do so; in this case:

$$y_0(x) = 1 - 2x.$$

The non-zero natural frequencies $\lambda_i (i = 1, 2, \ldots)$, corresponding to truly vibratory mode shapes ($\lambda^4 = m\omega^2 L^4/EI$), satisfy the characteristic equation [5]:

$$1 - \cosh(\lambda_i)\cos(\lambda_i) = 0, \qquad i = 1, 2, \ldots. \tag{2}$$

For the related mode shapes, one has

$$y_i(x) = \tfrac{1}{2}\{\cosh(\lambda_i x) + \cos(\lambda_i x) + B(\sinh(\lambda_i x) + \sin(\lambda_i x))\},$$

where the coefficient B satisfies

$$\{\cosh(\lambda_i) - \cos(\lambda_i)\} + 2B\{\sinh(\lambda_i) - \sin(\lambda_i)\} = 0.$$

3.2. *Free–free cable* (*BVP III*)

A cable is understood as a beam of vanishing bending stiffness or, equivalently, subjected to very large tensions. In this case, the differential equation (1) degenerates from the fourth to the second x-order. Therefore, the resulting BVP only requires two end boundary conditions. We argue here that the zero bending moment end conditions become irrelevant as bending stiffness vanishes, and that one should only retain

$$\frac{\partial y}{\partial x}(0) = \frac{\partial y}{\partial x}(1) = 0.$$

The straight cable problem thus obtained is also standard textbook material [5], except that solutions are usually given for the more practical fixed end conditions $y(0) = y(1) = 0$. The difference in end conditions does not affect the natural frequencies $\mu_i (i = 0, 1, \ldots,)$, such that $\mu^4 = m\omega^2 L^4/EI$, but the mode shapes are different, although still purely trigonometric. Aside from the non-vibratory solution corresponding to a constant mode shape, one can easily determine that

$$\mu_i = \gamma^{1/4}\pi\sqrt{i+1}, \quad y_i(x) = \cos\{(i+1)\pi x\}, \qquad i = 0, 1, \ldots .$$

4. LIMITING SOLUTIONS OF THE FULL BOUNDARY VALUE PROBLEM

The solution of the full boundary value problem BVP (I) was first attempted by Shaker [6]. Recently, Liu *et al*. [9] discovered that Shaker omitted the lowest non-zero natural frequency and its related mode shape. These and other authors numerically solved the characteristic equation; that is, the relationship between non-dimensional time frequencies β^2 and their corresponding non-dimensional modal "wavenumbers" α_1 and α_2. This equation in its full form is repeated below:†

$$2\beta^6\{1 - \cosh(\alpha_1)\cos(\alpha_2)\} - \gamma\pi^2(\gamma^2\pi^4 + 3\beta^4)\sinh(\alpha_1)\sin(\alpha_2) = 0, \tag{3}$$

where

$$\alpha_1 = \{\gamma\pi^2/2 + \sqrt{\gamma^2\pi^4/4 + \beta^4}\}^{1/2}, \qquad \alpha_2 = \{-\gamma\pi^2/2 + \sqrt{\gamma^2\pi^4/4 + \beta^4}\}^{1/2}.$$

Strictly speaking, the characteristic equation (sometimes called the transcendental equation) is defined for $\beta \neq 0$ and thus excludes rigid body modes, although it may coincidentally be satisfied by a rigid body mode.

Also recall that, for $\beta \neq 0$, the mode shapes are given (prior to normalization) by

$$y(x) = A(\cosh(\alpha_1 x) + (\alpha_1^2/\alpha_2^2)\cos(\alpha_2 x)) + B(\sinh(\alpha_1 x) + (\alpha_2/\alpha_1)\sin(\alpha_2 x)), \tag{4}$$

where the two coefficients A and B are related by

$$A\alpha_1^3\{\cosh(\alpha_1) - \cos(\alpha_2)\} + B\{\alpha_1^3\sinh(\alpha_1) - \alpha_2^3\sin(\alpha_2)\} = 0, \tag{5}$$

Of interest below is the continuity of the full problem in the two following limits: (a) when tension vanishes—in other words, the solutions of BVP (I) should converge toward those of BVP (II) as γ tends to zero; (b) when the bending stiffness of the beam vanishes—in other words, the solutions of BVP (I) should converge toward those of BVP (III) as γ tends to infinity.

The rigid body solution corresponding to a constant mode shape and a zero natural frequency satisfies BVP (I) for all axial loads, and therefore never poses any continuity problem.

4.1. *Lowest non-zero natural frequency for small axial loads*

The continuity of the solutions of equation (2) corresponding to β_i such that $i \geqslant 1$ is an easy matter, since setting $\gamma = 0$ in the characteristic equation (3) effectively yields equation (2), the characteristic equation of the limiting problem. Blevins [5] noted that for some boundary conditions other than free–free, the expression for the natural frequencies $\beta_i (i \geqslant 1)$ with non-zero tensions is quite simple (see, e.g., equations 8–20, 8–21 and 8–22 in reference [5]). For boundary conditions such as free–free, he then proposed equations 8–22 as an approximation, which is allegedly accurate within 1% as long as the magnitude of the axial load does not exceed the critical (buckling) value ($|\gamma| < 1$). In Liu *et al*.'s notation, Blevins' approximation is given below:

$$\beta_i \approx \lambda_i\left\{1 + \gamma\frac{\lambda_1^2}{\lambda_i^2}\right\}^{1/4}, \qquad \text{for } i \geqslant 1, |\gamma| < 1. \tag{6}$$

†The notation of Liu *et al*. [9] is used as much as possible. k^2, defined as TL^2/EI, has been replaced, however, by $\pi^2\gamma$ to avoid possible confusion over the fact that, by definition, k^2 is positive for tensile axial loads, but *negative* for compressive axial loads.

The limiting process of setting $\gamma = 0$ in equation (3), as well as the above formula, fail to account for the lowest non-zero natural frequency β_0, which was only recently discovered by Liu *et al*. [9]. An "inventory" of natural frequencies leaves only the possibility that the mode associated with β_0, a rightful solution of equation (3), "degenerates" into the second (non-constant) rigid body mode satisfying BVP (II) when the tension vanishes. In order to establish this fact, we make the *a priori* assumption that when $|\gamma|$ is small, β is small and of order $\sqrt{\pi}|\gamma|^{1/4}$. The validity of this assumption will be examined *a posteriori*; α_1 and α_2 are then expanded to order β^3:

$$\alpha_1 = \beta + \gamma\pi^2/4\beta + O(\beta^4), \qquad \alpha_2 = \beta - \gamma\pi^2/4\beta + O(\beta^4).$$

These values are now substituted into the characteristic equation (3), where to retain terms of order β^{10}, it is sufficient to develop the transcendental functions in equation (3) as follows:

$$1 - \cosh(\alpha_1)\cos(\alpha_2) = \beta^4/6 - \gamma\pi^2/2 + O(\beta^5), \qquad \sinh(\alpha_1)\sin(\alpha_2) = \beta^2 + O(\beta^3).$$

The characteristic equation then yields

$$\beta^4 = 12\pi^2\gamma, \qquad \text{i.e., } \beta_0 = (12)^{1/4}\sqrt{\pi}\gamma^{1/4}. \tag{7}$$

Equation (7) is the continuity formula for the lowest non-zero natural frequency β_0 of a free–free beam as tension vanishes. Since β^4 is positive, it can be verified that only tensile axial loads ($\gamma > 0$) correspond to the existence of a vibratory mode shape of frequency β_0. Moreover, since $(12)^{1/4} \approx 1{\cdot}861$, our original assumption that β and $\sqrt{\pi}|\gamma|^{1/4}$ be of the same order of magnitude, when both are small, is well satisfied. One can also see that the mode corresponding to β_0 is essentially "tensile", with $\beta_0 \approx 3{\cdot}30\gamma^{1/4}$, since the first solution of the limiting ("cable") boundary value problem BVP (III) is $\mu_0 = \pi\gamma^{1/4}$.

We may now turn our attention to the continuity of the mode shape $y_0(x)$. With equation (7) being satisfied, the expansion of α_1 and α_2 to order β^4 becomes

$$\alpha_1 = \beta + \beta^3/48 + O(\beta^4), \qquad \alpha_2 = \beta - \beta^3/48 + O(\beta^4).$$

Taking the arbitrary constant A equal to 1/2 in equation (4), while keeping only terms up to order βx in the brackets containing functions of x, we easily obtain

$$y(x) = 1 + 2B\beta x + O(\beta^2 x^2).$$

In the above, the second term is only necessary in as much as B is not of order β or smaller. From equation (5), the expression for B is

$$B = -\frac{\cosh(\alpha_1) - \cos(\alpha_2)}{2\{\sinh(\alpha_1) - (\alpha_2^3/\alpha_1^3)\sin(\alpha_2)\}}.$$

The leading term for the numerator is simply β^2. In expanding the denominator to order β^3, it is important to develop α_1^3 and α_2^3 to order β^5, so that the ratio α_2^3/α_1^3 is correctly $(1 - \beta^2/8)$; on the other hand, we have

$$\sinh(\alpha_1) = \beta + 9\beta^3/48 + O(\beta^4), \qquad \sin(\alpha_2) = \beta - 9\beta^3/48 + O(\beta^4),$$

Thus, the leading term of B is easily determined to be $-1/\beta$, and the mode shape $y_0(x)$ is given by

$$y_0(x) = 1 - 2x + O(\beta^2 x^2). \tag{8}$$

As discussed earlier, the second (non-constant) rigid body mode shape obtained when solving BVP (II) could theoretically be any straight line, and thus was not systematically orthogonal to the constant rigid body mode shape. With the normalization conditions $y(0) = 1$, it was seen that if one selected the particular straight line orthogonal to $y(x) = 1$, one obtained $y_0(x) = 1 - 2x$. With the onset of a small axial tensile force, not only β_0 becomes differentiated from 0, but the nearly linear mode shape given by equation (8) is the straight line orthogonal to all other mode shapes with distinct natural frequencies, as mathematical theory demands (see, e.g., Timoshenko *et al*. [2] and Liu *et al*. [9]).

4.2. *The case of large tensions*

For these cases or, equivalently, for vanishing bending stiffness, the behavior of the vibrating structural member is expected to approach that of a cable. Consequently, we make the *a priori* assumption that when γ is large, β is large and of order $\gamma^{1/4}\sqrt{\pi}$.

It is clear that with γ and β both large, α_1 is a very large number and the hyperbolic sine and cosine in equation (3) will be about equal to $\exp(\alpha_1)/2$. On the other hand, β^6 is much smaller that $(\pi^2\gamma)^3$, a term of order β^{12} under our assumption. It follows that the characteristic equation can only be satisfied if we have

$$\sin(\alpha_2) = 0.$$

The non-zero solutions of the above equation are immediately available as

$$\alpha_2 = n\pi, \quad n = 1, 2, \ldots;$$

recall that the characteristic equation is defined for non-zero natural frequencies β, which also implies that α_2 must be non-zero.

Using the definition of α_2, the above equation straightforwardly yields

$$\beta_i = \gamma^{1/4}\pi\sqrt{i+1}\left\{1 + \frac{(i+1)^2}{\gamma}\right\}^{1/4}, \qquad i = 0, 1, 2, \ldots.$$

Note that n, starting from 1, has been redefined as i, starting from 0, to be consistent with Liu *et al*.'s definition of the lowest non-zero frequency as β_0.

The expression for β_i can be formally recast in a manner somewhat similar to equation (6):

$$\beta_i = \mu_i\left\{1 + \frac{1}{\gamma}\frac{\mu_i^4}{\mu_0^4}\right\}^{1/4}, \qquad i = 0, 1, 2, \ldots. \tag{9}$$

As γ tends to infinity, the above formula clearly shows continuity between boundary value problems (I) and (III).

Strictly speaking, the above derivation relied on the *a priori* assumption that β was of order $\gamma^{1/4}\sqrt{\pi}$. The result obtained subsequently shows that our assumption is acceptable as long as $\sqrt{\pi(i+1)}\{1 + (i+1)^2/\gamma\}^{1/4}$ is of order one, or, say, less than 10. In other words, the following approximate condition should hold:

$$(i+1)\sqrt{1 + (i+1)^2/\gamma} \ll 100/\pi.$$

This condition is by no means very restrictive, especially for the lower natural frequencies (low indices i). It confirms that for a given value of γ, a beam will first have a cable-like behavior in the lower frequency mode shapes. Moreover, if the above condition is not satisfied, the analysis could be repeated with a different *a priori* assumption on the order of magnitude of β, and the results should not be fundamentally different.

We will now examine the continuity of the mode shapes. Retaining only the leading terms in equation (5) leads to $B = -A$. It follows that the mode shapes described by equation (4) are given approximately by

$$y_i(x) = (\alpha_1^2/\alpha_2^2)\cos(\alpha_2 x).$$

Imposing the normalization conditions $y_i(0) = 1$, we have

$$y_i(x) = \cos(\alpha_2 x), \qquad \text{i.e., } y_i(x) = \cos\{(i+1)\pi x\}, \quad i = 0, 1, 2, \ldots .$$

These are precisely the mode shapes obtained when solving BVP (III). Once again, the

TABLE 1

β_0 and approximating expressions

γ	β_0 Exact (numerical)	γ	μ_0	γ	β_0 equation (7)	γ	β_0 equation (9)
0	0	0	0	0	0	0	3·1415927
0·1	1·8529856	0·1	1·766647402	0·1	1·855112497	0·1	3·21734809
0·2	2·2011743	0·2	2·10090966	0·2	2·206112981	0·2	3·288101313
0·3	2·4334568	0·3	2·325038736	0·3	2·441465348	0·3	3·354561182
0·4	2·6123106	0·4	2·498416716	0·4	2·623525253	0·4	3·417290412
0·5	2·7595459	0·5	2·64175404	0·5	2·774040211	0·5	3·47674384
0·6	2·8855982	0·6	2·764952607	0·6	2·903407963	0·6	3·533294804
0·7	2·9963464	0·7	2·873587257	0·7	3·017482507	0·7	3·587253842
0·8	3·0954585	0·8	2·971134935	0·8	3·119914897	0·8	3·638882274
0·9	3·1853907	0·9	3·05992306	0·9	3·213149099	0·9	3·688402259
1	3·267875	1	3·1415927	1	3·298908357	1	3·736004391
5	4·80412	5	4·697776815	5	4·933018591	5	4·916858292
10	5·6693	10	5·586629613	10	5·866380806	10	5·721343863
50	8·388	50	8·353959783	50	8·772285389	50	8·395419914
100	9·95618	100	9·934588413	100	10·4320642	100	9·959332286
500	14·8627	500	14·85567467	500	15·59957449	500	14·86309695
1000	17·6707	1000	17·66647402	1000	18·55112497	1000	17·67088898

TABLE 2

Relative errors for β_0 approximations

γ	$(\beta_0 - \mu_0)/\beta_0$	γ	$\{\beta_0(7) - \beta_0\}/\beta_0$	γ	$\{\beta_0(9) - \beta_0\}/\beta_0$
0·1	0·0465941	0·1	0·001147822	0·1	0·736304961
0·2	0·0455505	0·2	0·002243671	0·2	0·49379418
0·3	0·0445531	0·3	0·003291038	0·3	0·37851687
0·4	0·0435989	0·4	0·004293005	0·4	0·308148589
0·5	0·0426852	0·5	0·005252426	0·5	0·259897087
0·6	0·0418096	0·6	0·006171952	0·6	0·224458352
0·7	0·0409696	0·7	0·007053973	0·7	0·197209338
0·8	0·0401632	0·8	0·007900732	0·8	0·175555176
0·9	0·0393885	0·9	0·008714284	0·9	0·157912045
1	0·0386436	1	0·009496494	1	0·143251927
5	0·0221358	5	0·026830843	5	0·023467002
10	0·0145821	10	0·034762811	10	0·009179945
50	0·0040582	50	0·045813709	50	0·000884587
100	0·0021687	100	0·04779787	100	0·000316616
500	0·0004727	500	0·049578777	500	2·67076E − 05
1000	0·0002392	1000	0·049824001	1000	1·06948E − 05

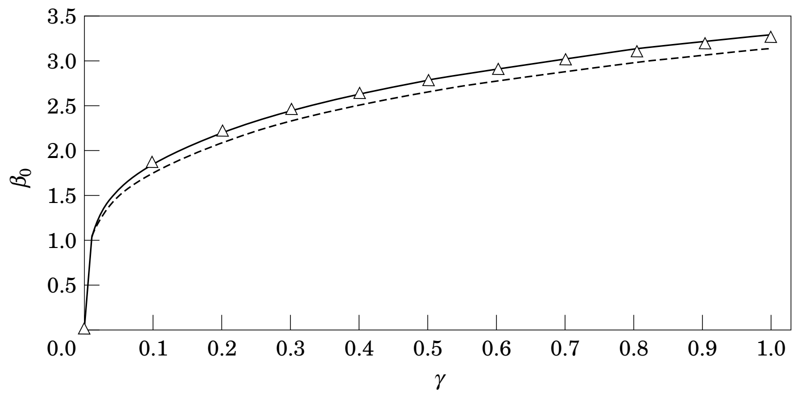

Figure 1. Small-tension approximations of the lowest natural freuqncy. △, Exact (numerical, Liu *et al*. [9]); ——, equation (7); – – –, μ_0.

continuity of BVP (I) for large values of γ removes any doubt about the definition of free–free boundary conditions for a cable: the argument proposed earlier is confirmed that, of the four boundary conditions necessary to solve for BVP (I), only the two zero-shear end conditions (1b) should be retained as the bending stiffness of the beam vanishes, and the order of the beam motion differential equation drops from the fourth to the second order (in x). In other words, free–free cable mode shapes have zero end slopes.

5. APPROXIMATE FORMULAS FOR β_0

The existence of β_0 was only recently established by Liu *et al*. [9] through an improved numerical solution of equation (3). While examining the theoretical question of the continuity of BVP (I), as the axial load tends to either zero or infinity, a number of very simple approximate expressions for the fundamental frequency β_0 were derived in the case of tensile loads (β_0 does not exist for compressive axial loads).

Equations (7) and (9) provide an easy way to evaluate β_0, in the respective limits of small and large tensions, while avoiding the numerical difficulties associated with the hyperbolic functions in equation (3). Moreover, the fundamental frequency for BVP (III), μ_0, was observed to deviate little from β_0, even for small tensions: it is trivial to show that the maximum relative error committed when replacing β_0 by μ_0 is $(1 - \sqrt{\pi}/12^{1/4}) \approx 4{\cdot}77\%$ when γ tends to zero. Reciprocally, the small-tension approximation given by equation (7), written $\beta_0(7)$, remains relatively close to β_0 for large tensions: it can be shown that the maximum relative error committed when replacing β_0 by $\beta_0(7)$ is $(12^{1/4}/\sqrt{\pi} - 1) \approx 5{\cdot}01\%$ when γ tends to infinity; since β_0 becomes large, however, the absolute error also grows in this limit.

The accuracy of the various approximations of β_0 available through equations (7), (9) and μ_0 in Tables 1 and 2 is quantified. The best small-γ approximations are graphically illustrated in Figure 1.

6. CONCLUSIONS

The present study has verified that the modal boundary value problem for a vibrating free–free straight beam subjected to a uniform axial load is continuous in two well-known limits: first, when the axial load vanishes, and second, when the axial load becomes very large (vanishing bending stiffness). In both cases, the proof relied on a careful derivation of the limits of the characteristic equation, and resulted in very simple and accurate approximating formulas for the lowest non-zero natural frequency.

In the large axial load limit, the BVP for a cable is "recovered" if one defines a free–free cable as having zero slopes, i.e., vanishing shear, at the ends.

In the small axial load limit, it was confirmed that only tensions give rise to the lowest non-zero frequency mode numerically discovered by Liu *et al*. [9]. As tension vanishes, this mode shape tends to the non-constant rigid body solution of limiting BVP (II), $y(x) = 1 - 2x$, which is no longer vibratory, but remains orthogonal to all other modes. This orthogonality property is preserved in the limiting process; by contrast, a solution of BVP (II) alone yields an indeterminate slope for the non-constant rigid body solution.

This work is complete in the context of vibratory modes. On the other hand, the lack of a natural frequency for small compressive loads, which would correspond to a mode converging to $y(x) = 1 - 2x$ as the axial compression vanishes, leaves the overall continuity analysis somewhat incomplete. Work is currently under way to investigate time-exponential (non-vibratory) modes in the presence of axial loads, and it is expected that results from this on-going work will shed some light on the question.

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APPENDIX A: PROOF OF THE NON-EXISTENCE OF TIME-EXPONENTIAL ($C < 0$) MODE SHAPES FOR A FREE–FREE STRAIGHT BEAM WITH NO AXIAL LOAD

The BVP for the time-exponential mode shapes of a free–free straight beam without axial load is written below:

$$\frac{\partial^4 y}{\partial x^4} - Cy = 0 \quad (C < 0), \qquad \frac{\partial^2 y}{\partial x^2}(0) = \frac{\partial^3 y}{\partial x^3}(0), \qquad \frac{\partial^2 y}{\partial x^2}(1) = \frac{\partial^3 y}{\partial x^3}(1) = 0. \quad \text{(A1–A3)}$$

This BVP is formally the same as BVP (II), except for the sign of the constant C, which corresponds here to time-exponential modes (these modes are not vibratory if they exist). The different sign for C results in a different set of four fundamental functions satisfying equation (A1); instead of being a linear combination of cosh, sinh, cos and sin, the general solution of equation (A1) may be written

$$y(x) = A\,\mathrm{e}^{ax}\cos(ax) + B\,\mathrm{e}^{ax}\sin(ax) + C\,\mathrm{e}^{-ax}\cos(ax) + D\,\mathrm{e}^{-ax}\sin(ax),$$

where A, B, C and D are constants, and $a = (-C)^{1/4}/\sqrt{2}$. Since $C < 0$, only values of a such that $a > 0$ are admissible.

Substituting the above general solution into the boundary conditions (A2) at $x = 0$ yields

$$D = B, \qquad C = A - 2B. \tag{A4, 5}$$

Repeating the procedure at $x = 1$, we further obtain

$$-A\,\mathrm{e}^{a}\sin(a) + B\,\mathrm{e}^{a}\cos(a) + C\,\mathrm{e}^{-a}\sin(a) - D\,\mathrm{e}^{-a}\cos(a) = 0, \tag{A6}$$

$$-A\,\mathrm{e}^{a}\{\sin(a) + \cos(a)\} + B\,\mathrm{e}^{a}\{\cos(a) - \sin(a)\} + C\,\mathrm{e}^{-a}\{\cos(a) - \sin(a)\} + D\,\mathrm{e}^{-a}\{\sin(a) + \cos(a)\} = 0. \tag{A7}$$

For a non-trivial solution to exist, for which A, B, C and D are not all zero, the determinant made up with the coefficients of these constants in equation (A4) through (A7), must vanish. The simple form of equation (A4) and (A5) lends itself to substitution, and after some elementary algebra, the zero-determinant requirement reduces to

$$\sinh^2 a - \sin^2 a = 0. \tag{A8}$$

The above result represents the characteristic equation of the BVP formulated in Equation (A1)–(A3); it is the equivalent to equation (2) when C is negative. It can be seen, however, that no strictly positive value of a satisfies equation (A8). Therefore no mode shape $y(x)$ exists for a free–free straight beam when the corresponding time dependence is exponential ($C < 0$) rather than trigonometric ($C > 0$). A similar proof can easily be established for a straight beam subjected to different boundary conditions.