APPLICABILITY CONDITIONS OF A NON-LINEAR SUPERPOSITION TECHNIQUE

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The analytical conditions for the applicability of a Non-linear Superposition Technique (NST) are established by using the Normal Form Method (NFM). The superposition technique represents the extension of modal analysis to weakly non-linear vibration systems. In this paper it is shown that the NST leads to a non-uniform perturbation expansion for the solution of dynamical systems when some special non-linear terms are present in the dynamical system. An analytical proof for general dynamical systems and a numerical application to a simple mechanical system is included in the paper.

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1. INTRODUCTION

The Non-linear Superposition Technique [1] (NST) is based upon the concept of invariant manifolds of a dynamical system, called also normal modes in structural dynamics. The first important study on non-linear normal modes is due to Rosenberg [2] in 1966. In the past 30 years there was a lack of interest on normal modes, but recently the activity in this field has grown noticeably. Indeed, many papers have appeared in the literature about normal modes for both weakly and strongly non-linear systems [3, 4], matching techniques for general discrete systems [5], continuous systems [6, 7] and discrete systems with internal resonances [8]. Furthermore, Shaw and Pierre [1] presented a paper focused on a new definition of normal modes in terms of invariant manifolds and developed a method to study the dynamics of weakly non-linear systems. The idea was very appealing, and similar to modal analysis because it consists of uncoupling the system equations through a special projection: in other words, this technique could be considered as an extension of modal analysis to weakly non-linear systems.

In this paper the local analytical solution of non-linear dynamical systems, obtained by the NST, is compared with that obtained by a perturbation method, i.e., the Normal Form Method (NFM) [9–11]. The NST was proposed for structural dynamical applications and good agreements with solutions obtained by the direct numerical integration were achieved for some mechanical applications [1]. In this paper the applicability of the non-linear superposition technique based on the use of the invariant manifolds is investigated: the NFM is used to quantify the limitations of the non-linear modal superposition for general

are determined and the role of special terms, here referred as mixed-coupling terms, is discussed.

The theoretical bases of the non-linear superposition and the normal form methods are presented and compared in sections 2 and 3 respectively; in section 4 the differences between these approaches are illustrated by three simple mechanical examples. Finally in section 5 some conclusions are outlined.

2. NORMAL FORM METHOD

In this section a brief introduction to the normal form method is presented (see reference [11] for details), with emphasis on the evaluation of the portion of the dynamical system (referred to as "non-linear mixed terms" in the following) for which the non-linear superposition does not work.

Consider a general mechanical system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{n}(\dot{\mathbf{x}}, \mathbf{x}) = 0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices respectively, whereas $\mathbf{n}(\dot{\mathbf{x}}, \mathbf{x})$ represents non-linear polynomial terms. Upon introducing the state space vector $\mathbf{z} = [\dot{\mathbf{x}}^\mathrm{T}, \mathbf{x}^\mathrm{T}]^\mathrm{T}$, a $2N$-dimensional set of equations obtained,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathcal{N}(\mathbf{z}), \tag{2}$$

where

$$\mathcal{N}(\mathbf{z}) = \left\{ \begin{matrix} -\mathbf{M}^{-1}\mathbf{n}(\mathbf{z}) \\ 0 \end{matrix} \right\}.$$

The matrix $\mathbf{A}$ can be diagonalized by using the co-ordinate transformation $\mathbf{z} = \mathbf{U}\boldsymbol{\xi}$, where $\mathbf{U}$ is the eigenvector matrix and

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\Lambda}\boldsymbol{\xi} + \hat{\mathbf{f}}(\boldsymbol{\xi}), \tag{3}$$

$\boldsymbol{\Lambda}$ being the diagonal matrix that contains the complex eigenvalues λ_i of the system, and $\hat{\mathbf{f}}(\boldsymbol{\xi}) = \mathbf{U}^{-1}\mathcal{N}(\mathbf{U}\boldsymbol{\xi})$. It must be pointed out that, if the first order dynamical system given by equation (2) is derived from the mechanical system described by equation (1) and there is not supercritical damping then all the eigenvalues (and the corresponding eigenvectors) are complex and conjugate. Now, one can order the eigenvalues and the eigenvectors as follows: $\lambda_i = \overline{\lambda}_{i+N}$, $\mathbf{u}^{(i)} = \overline{\mathbf{u}}^{(i+N)}$, where λ_i is the ith eigenvalue, $\mathbf{u}^{(i)}$ is the ith eigenvector and the superscript($^-$) indicates the complex conjugate; this ordering implies that $\xi_i = \overline{\xi}_{i+N}$ and, if $\hat{f}_i$ is an analytic function, $\hat{f}_i = \overline{\hat{f}}_{i+N}$ corresponds to the statement that the first N equations are the complex conjugates of the second N in this way the initial $2N$-dimensional real system is transformed into a $2N$-dimensional complex system: however, one could consider the N-dimensional complex space given by ξ_i ($i = 1, \ldots, N$), because the remaining part can be trivially obtained through the complex conjugate operation.

Note that the generic component of $\hat{\mathbf{f}}$ can be rewritten in a power series;

$$\hat{f}_i = \sum_{j,k=1}^{2N} \hat{f}_{i,jk}\xi_j\xi_k + \sum_{j,k,r=1}^{2N} \hat{f}_{i,jkr}\xi_j\xi_k\xi_r + \cdots, \qquad i = 1, \ldots, 2N, \tag{4}$$

where, e.g., $\hat{f}_{i,jk} = \frac{1}{2}\partial^2\hat{f}_i/\partial\xi_j\,\partial\xi_k|_{\xi=0}$, because $\hat{\mathbf{f}}$ is an analytic function of $\boldsymbol{\xi}$.

Now it is convenient to decompose the $\hat{f}_i$ into two parts,

$$\hat{f}_i(\boldsymbol{\xi}) = \sum_{k=1}^{2N} \hat{f}'_{ik}(\xi_k, \overline{\xi}_k) + \hat{f}''_i(\boldsymbol{\xi}), \tag{5}$$

where we refer to $\Sigma_{k=1}^{2N} \hat{f}'_{ik}$ as *unmixed-coupling terms* and $\hat{f}''_i$ as *mixed-coupling terms*. Specifically, we denote by unmixed-coupling terms the unmixed polynomials of degree m (e.g., for $m = 3$, x_1^3, x_2^3, etc.) and by mixed-coupling terms the mixed polynomials of degree m (e.g., for $m = 3$, $x_1 x_2^2$, $x_1^2 x_2$, etc.). Furthermore, the unmixed terms may be expressed as a sum (see equation (5)) in k of a generic $\hat{f}'_{ik}$, where each term is the portion of $\hat{f}_i$ that contains only the two space state variables ξ_k and $\overline{\xi}_k$. It is straightforwardly observed that $\hat{f}''_i = \hat{f}_i - \Sigma_{k=1}^{2N} \hat{f}'_{ik}$. The notation *mixed* implies that the term contains more than one variable, for any power combination of the simple factor of the monomial. As will be shown later, the presence of the *mixed-coupling terms* does not allow non-linear modal superposition.

Next, in order to obtain a perturbative solution for the ordinary differential equations by means of the NFM, it is necessary to introduce an ordering real parameter ε, as

$$\boldsymbol{\xi} = \varepsilon^{1/(m-1)}\boldsymbol{\zeta},$$

where m represents the lower degree of the series expansion for $\hat{\mathbf{f}}$. Then equation (3) becomes

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\Lambda}\boldsymbol{\zeta} + \varepsilon^{-1/(m-1)}\hat{\mathbf{f}}(\varepsilon^{1/(m-1)}\boldsymbol{\zeta}, \varepsilon^{1/(m-1)}\overline{\boldsymbol{\zeta}}) = \boldsymbol{\Lambda}\boldsymbol{\zeta} + \varepsilon\mathbf{f}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}) + \mathcal{O}(\varepsilon^2), \tag{6}$$

$$\dot{\overline{\boldsymbol{\zeta}}} = \overline{\boldsymbol{\Lambda}}\overline{\boldsymbol{\zeta}} + \varepsilon^{-1/(m-1)}\overline{\hat{\mathbf{f}}}(\varepsilon^{1/(m-1)}\varepsilon, \varepsilon^{1/(m-1)}\overline{\boldsymbol{\zeta}}) = \overline{\boldsymbol{\Lambda}}\overline{\boldsymbol{\zeta}} + \varepsilon\overline{\mathbf{f}}(\boldsymbol{\zeta}, \overline{\boldsymbol{\zeta}}) + \mathcal{O}(\varepsilon^2). \tag{7}$$

For example, if $\hat{f} = x^3$, it follows that $m = 3$, $\xi = \sqrt{\varepsilon}\zeta$ and $(\sqrt{\varepsilon})^{-1} (\sqrt{\varepsilon})^3\zeta^3 = \varepsilon\zeta^3$.

One can now apply the NFM to the previous system. The method is based on a non-linear transformation of co-ordinates (near identity transformation) that leads to a simpler form of the equations, i.e., the so called normal forms. The *near identity transformation* (also known as the *Lie transformation* [11, 12]) is

$$\zeta_i = \eta_i + \varepsilon h_i(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}), \qquad i = 1, \ldots, N, \tag{8}$$

where h_i is an unknown non-linear (e.g., polynomial) function of $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$ and the complex conjugate set of equations is not explicitly considered. The same kind of decomposition given by equation (5) can be obtained for the above transformation law: $h_i = \Sigma_k h'_{ik} + h''_i$. Consequently, by using equations (5), (6) and (8) and searching for the simplest form of the transformed problem, (i.e., satisfying the resonance condition [11]) the normal form equations are obtained;

$$\dot{\eta}_i = \lambda_i\eta_i + \varepsilon g_i(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) \qquad \text{or} \qquad \dot{\eta}_i = \lambda_i\eta_i + \varepsilon[g'_{ii}(\eta_i, \overline{\eta}_i) + g''_i(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}})], \tag{9}$$

where $g_i = g'_{ii} + g''_i$ represent the so-called secular terms (the Einstein notation for repeated indices is not used). Note that the linear portion of the normal forms corresponds to that of the original problem. Furthermore, combining equations (6), (8) and (9), one obtains

the so-called ''homology'' relations [11]

$$g'_{ii} + g''_i + \sum_{k=1}^{2N}\left[\frac{\partial h'_{ik}}{\partial \eta_k}\lambda_k\eta_k + \frac{\partial h'_{ik}}{\partial \bar{\eta}_k}\bar{\lambda}_k\bar{\eta}_k - \lambda_i h'_{ik}\right] + \sum_{j=1}^{2N}\left[\frac{\partial h''_i}{\partial \eta_j}\lambda_j\eta_j + \frac{\partial h''_i}{\partial \bar{\eta}_j}\bar{\lambda}_j\bar{\eta}_j\right]$$

$$- \lambda_i h''_i - \sum_{k=1}^{2N}f'_{ik} - f''_i = 0: \tag{10}$$

i.e., a relationship that correlates the terms of the old problem f'_{ik} and f''_i with those of the new one g'_{ii} and g''_i through the unknown non-linear transformation h'_{ik} and h''_i. Choosing g'_{ii} and g''_i as resonant terms of the original problem (i.e., the resonant part of f'_{ik} and f''_i), one determines the non-linear transformation terms h'_{ik} and h''_i [11, 12]. Note that, if the *mixed-coupling terms* are not present, the previous equation (10) can be reduced to

$$\delta_{ik}g'_{ii} + \frac{\partial h'_{ik}}{\partial \eta_k}\lambda_k\eta_k + \frac{\partial h'_{ik}}{\partial \bar{\eta}_k}\bar{\lambda}_k\bar{\eta}_k - \lambda_i h'_{ik} - f'_{ik} = 0, \qquad i = 1, \ldots, N, \quad k = 1, \ldots, N, \tag{11}$$

each term in the first summation of equation (10) being independent of the other. In the next section the above equation is compared with the one obtained by the NST approach.

3. NON-LINEAR SUPERPOSITION METHOD

The non-linear superposition method is based on the use of invariant manifolds of dynamical systems and a special projection on it: i.e., a non-linear transformation that uncouples the equations. A method for performing the uncoupling was developed in reference [1] and it is briefly outlined in the following. If bifurcations and internal resonances are avoided, the mth manifold is two-dimensional and tangent to the mth eigenspace (see references [1, 14]). In the following, the NST is presented with consideration of the complex state space variables, in order to compare this approach with the NFM outlined in the previous section (in Appendix A a real domain approach to NST is also presented).

Consider a particular motion that belongs to a two-dimensional invariant manifold, so that the state vector ζ depends on two variables as
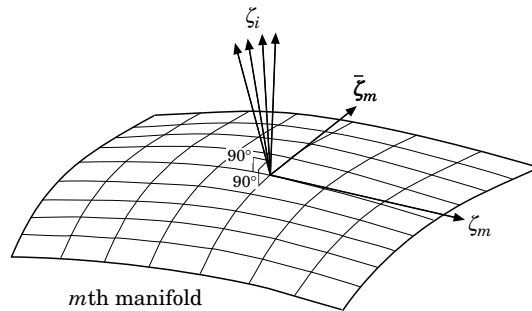
$$\zeta = \mathscr{L}(u, v), \tag{12}$$

Figure 1. The invariant manifold.

where u and v are the material co-ordinates of the manifold. It is worth pointing out that an invariant manifold is a regular surface tangent at the fixed point to the corresponding eigenspace. If one would describe the mth manifold by using the co-ordinate variables (ζ_i), then the manifold can be parametrized in a unique way as (See Figure 1)

$$u \equiv \zeta_m, \qquad v \equiv \overline{\zeta}_m, \tag{13}$$

where the index indicates that the mth manifold is considered (the motivation for the use of the conjugate co-ordinate has been described in the previous section).

Using equations (12 and 13) one can write the motion equations as follows:

$$\dot{\zeta}_i(u, v) = \frac{\partial \mathscr{Z}_i(u, v)}{\partial u} \dot{u} + \frac{\partial \mathscr{Z}_i(u, v)}{\partial v} \dot{v} = \lambda_i \mathscr{Z}_i(u, v) + \varepsilon f_i \quad \mathscr{Z}(u, v)). \tag{14}$$

Considering the parametrization of u and v and the equation of motion for the mth co-ordinate, one obtains the manifold equations:

$$\frac{\partial \mathscr{Z}_i}{\partial \zeta_m} (\lambda_m \zeta_m + \varepsilon f_m) + \frac{\partial \mathscr{Z}_i}{\partial \overline{\zeta}_m} (\overline{\lambda}_m \overline{\zeta}_m + \varepsilon \overline{f}_m) = \lambda_i \mathscr{Z}_i + f_i, \qquad i, m = 1, \ldots, N. \tag{15}$$

Considering also that the manifold is tangent to the corresponding eigenspace [14], it can be described through a vector function $(\mathscr{Z})$ that does not contains linear terms (see Appendix B). Therefore one can write

$$\zeta_i^{(m)} = \varepsilon H_i^{(m)}(\zeta_m^{(m)}, \overline{\zeta}_m^{(m)}), \qquad i = 1, \ldots, N, \qquad i \neq m, \tag{16}$$

where the parametric variables (u, v) have been replaced by $(\zeta_m^{(m)}, \overline{\zeta}_m^{(m)})$. The symbol $H_i^{(m)}$ represents a polynomial function of degree $\geqslant 2$ and the superscript (m) indicates the mth manifold. The dynamics on the manifold is described by (see equation 6)

$$\dot{\zeta}_m^{(m)} = \lambda_m \zeta_m^{(m)} + \varepsilon f_{mm}'(\zeta_m^{(m)}, \overline{\zeta}_m^{(m)}) + \mathcal{O}(\varepsilon^2), \tag{17}$$

where

$$\varepsilon f_{mm}'(\zeta_m^{(m)}, \overline{\zeta}_m^{(m)}) + \mathcal{O}(\varepsilon^2) = \varepsilon f_m(\varepsilon H_1^{(m)}, \ldots, \zeta_m^{(m)}, \ldots, \zeta_{N+m}^{(m)}, \ldots, \varepsilon H_{2N}^{(m)})$$

and the generating equations for the manifolds (see equation (15)) are

$$\frac{\partial H_i^{(m)}}{\partial \zeta_m^{(m)}} \lambda_m \zeta_m^{(m)} + \frac{\partial H_i^{(m)}}{\partial \overline{\zeta}_m^{(m)}} \overline{\lambda}_m \overline{\zeta}_m^{(m)} - \lambda_i H_i^{(m)} - f_{im}' = 0, \tag{18}$$

which clearly implies (see the homology equation (11))

$$H_i^{(m)} \equiv h_{im}', \qquad i \neq m, \tag{19}$$

if no mixed-coupling terms are present.

In reference [1] equation (17) was numerically integrated and the result was compared with the solution obtained by the direct integration of the initial equations. Alternatively, equation (17) can be studied through a perturbation technique. When using a first order perturbation procedure, the solution will be within the order of approximation considered to obtain the manifolds (see equation (18)), i.e., the error is bounded at the second order and the present analysis is still valid. Indeed, our task is to show that the NST may lead to a non-uniform first order expansion; then, using the NFM to analyze equation (17), one obtains

$$\zeta_m^{(m)} = \eta_m^{(m)} + \varepsilon p_m(\eta_m^{(m)}, \bar{\eta}_m^{(m)}), \qquad \dot{\eta}_m^{(m)} = \lambda_m \eta_m^{(m)} + \varepsilon q_m(\eta_m^{(m)}, \bar{\eta}_m^{(m)}), \tag{20, 21}$$

where p_m is given by the relationship

$$q_m + \left\{ \frac{\partial p_m}{\partial \eta_m^{(m)}} \lambda_m \eta_m^{(m)} + \frac{\partial p_m}{\partial \bar{\eta}_m^{(m)}} \bar{\lambda}_m \bar{\eta}_m^{(m)} \right\} - \lambda_m p_m - f''_{mm} = 0. \tag{22}$$

Comparing equation (11), when $i, k = m$, with equation (22), one obtains

$$p_m \equiv h'_{mm}, \qquad q_m \equiv g'_{mm} \tag{23}$$

In reference [1] it was assumed that the general motion is obtained as the superposition of motions evolving on manifolds; such superposition is given by

$$\zeta_i = \zeta_i^{(i)} + \sum_{m=1, m \neq i}^{N} \varepsilon H_i^{(m)}(\zeta_m^{(m)}, \bar{\zeta}_m^{(m)}), \tag{24}$$

where $\zeta_m^{(m)}$ is obtained from equation (17).

Use of equations (19), (20), (21), (23) and (24) shows that the method investigated does not consider the contributions h''_i of the non-linear transformation given by equation (24) and g''_i of the normal form equations: i.e., an $\mathcal{O}(\varepsilon)$ contribution leads to an $\mathcal{O}(\varepsilon)$ error if the f''_i are present. This fact shows the fundamental role of mixed-coupling terms f''_i on the applicability of the NST. It is worth pointing out that the lost contributions are of the same order as the corrections.

4. SOME APPLICATIONS

In this section three cases concerning two-dimensional systems are considered. The first is an example for which the NST cannot be applied at all. In the second example, a physical system is analyzed and numerical simulations are performed to show quantitatively and qualitatively the error arising in the application of the NST. Finally, the third case is an example in which the NST gives the same result as the NFM.

4.1. A CASE OF TOTAL FAILURE OF THE NST

Consider a two-dimensional Hamiltonian dynamical system with the potential energy. $\mathcal{E} = \frac{1}{2}\omega_1^2 x_1^2 + \frac{1}{2}\omega_2^2 x_2^2 + \varepsilon \frac{1}{2} K_3 x_1^2 x_2^2)$. Consequently, the restoring forces are $f_1 = (k_1 + \varepsilon k_3 x_2^2)x_1$, and $f_2 = (\omega_2^2 + \varepsilon k_3 x_1^2)x_2$, and the governing equations are

$$\ddot{x}_1 + \omega_1^2 x_1 + \varepsilon k_3 x_1 x_2^2 = 0, \qquad \ddot{x}_2 + \omega_2^2 x_2 + \varepsilon k_3 x_2 x_1^2 = 0. \tag{25}$$

Such a system might represent a spring-mass system with varying stiffness, obtained, for example, by a control system. Here it is shown that the NST completely fails the analysis because it does not furnish any correction to the linear model. Indeed, in terms of complex co-ordinates the system given by equation (25) becomes

$$\dot{\zeta}_1 = j\omega_1\zeta_1 - \varepsilon \frac{k_3}{2j\omega_1}(\zeta_1 + \bar{\zeta}_1)(\zeta_2 + \bar{\zeta}_2)^2, \qquad \dot{\zeta}_2 = j\omega_2\zeta_2 - \varepsilon \frac{k_3}{2j\omega_2}(\zeta_2 + \bar{\zeta}_2)(\zeta_1 + \bar{\zeta}_1)^2 \tag{26}$$

where $x_i = \zeta_i + \bar{\zeta}_i$ and $\dot{x}_i = j\omega_i(\zeta_i - \bar{\zeta}_i)$. The first non-linear mode of the system is given by

$$\zeta_1^{(2)} = \varepsilon H_2^{(1)}(\zeta_1^{(1)}, \bar{\zeta}_1^{(1)}) \tag{27}$$

where

$$\dot{\zeta}_1^{(1)} = j\omega_1\omega\zeta_1^{(1)}, + \mathcal{O}(\varepsilon^2),$$
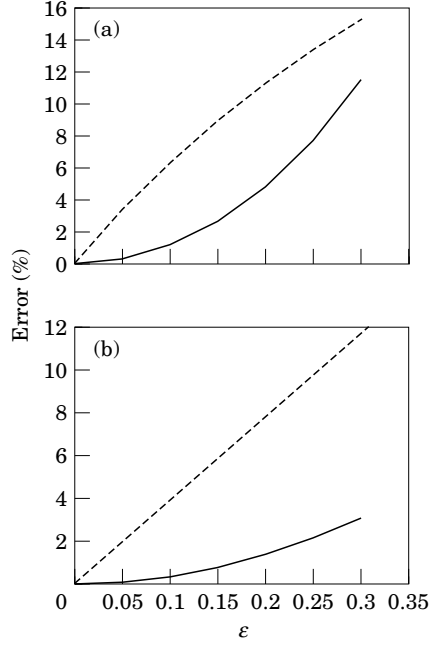
Figure 2. Behaviour of the mean error. (a) First degree of freedom; (b) second degree of freedom. ——, Normal form; - - - -, non-linear superposition.

whereas $H_2^{(1)}$ is given by

$$\frac{\partial H_2^{(1)}}{\partial \zeta_1^{(1)}} j\omega_1 \zeta_1^{(1)} + \frac{\partial H_2^{(1)}}{\partial \overline{\zeta}_1^{(1)}} (-j\omega_1)\overline{\zeta}_1^{(1)} - j\omega_2 H_2^{(1)} + \mathcal{O}(\varepsilon^2) = 0$$

i.e., $\Rightarrow H_2^{(1)} \equiv 0$. Therefore, in this case the non-linear effect is completely lost by the procedure. On the other hand, its effect is not negligible because, applying the NFM to equation (26), one obtains

$$\dot{\eta}_1 = j\omega_1\eta_1 + \varepsilon \frac{jk_3}{2\omega_1} \eta_1\eta_2\bar{\eta}_2, \qquad \dot{\eta}_2 = j\omega_2\eta_2 + \varepsilon \frac{jk_3}{2\omega_2} \eta_2\eta_1\bar{\eta}_1, \qquad (28, 29)$$

Using the polar form $\eta_i = a_i e^{j(\omega_i t + \theta_i(t))}$, one finds the non-linear frequencies

$$\tilde{\omega}_1 = \omega_1 + \varepsilon (k_3/2\omega_1)a_2^2, \qquad \tilde{\omega}_2 = \omega_2 + \varepsilon (k_3 2\omega_2)a_1^2,$$

and, successively, non-trivial non-linear transformations for h_i. Therefore there is a non-linear $\mathcal{O}(1)$ effect on the frequencies and higher order contributions given by the transformations h_i.

4.2. A PHYSICAL EXAMPLE OF PARTIAL FAILURE OF THE NST

Consider a two-dimensional Hamiltonian dynamical system with potential energy $\mathscr{E} = \frac{1}{2}\omega_1^2 x_1^2 + \frac{1}{2}\omega_2^2 x_2^2 + \varepsilon(\frac{1}{4}k_{11}x_1^4 + \frac{1}{4}k_{22}x_2^4 - \frac{1}{2}k_{12}x_1^2 x_2^2)$. The resulting dynamical system is

$$\ddot{x}_1 + \omega_1^2 x_1 + \varepsilon(k_{11}x_1^3 - k_{12}x_1 x_2^2) = 0, \qquad \ddot{x}_2 + \omega_2^2 x_2 + \varepsilon(k_{22}x_2^3 - k_{12}x_2 x_1^2) = 0. \quad (30)$$

Such a system might represent a mass connected with two orthogonal springs. Here the terms $k_{11}x_1^3$ and $k_{22}x_2^3$ are unmixed terms, while $k_{12}x_1 x_2^2$ and $k_{12}x_2 x_1^2$ are the mixed-coupling terms. In Figure 2 is shown the behaviour of the mean error of the NST and NFM versus ε; the continuous line referes to the NST solution while the dotted line

refers to the NFM solution. It can be seen that the first method leads to an error behaviour that is not a quadratic function of ε; i.e., an $\mathcal{O}(\varepsilon^2)$ error. In fact, for $\varepsilon \to 0$ the NST error is not tangent to the horizontal axis. In Figure 3 is shown in the time history of the two degrees of freedom; here the separation of the NST from the numerical reconstruction is evident. Finally, in Figure 4 the previous signal is shown in the frequency domain: note that part of the high frequency behaviour is lost when using the NST.

4.3. A CASE OF APPLICABILITY OF THE NST

Consider a two-dimensional dynamical system governed by the equations

$$\ddot{x}_1 + \omega_1^2 x_1 + \varepsilon(kx_1^3 + cx_2^3) = 0 \qquad \ddot{x}_2 + \omega_2^2 x_2 + \varepsilon(kx_2^3 + cx_1^3) = 0.$$

Note that no elastic potential energy can correspond to the above system; however, it could be realized through a control system. The first order form is

$$\dot{\zeta}_1 = j\omega_1\zeta_1 + \varepsilon(j/2\omega_1)\left[k(\zeta_1 + \overline{\zeta}_1)^3 + c(\zeta_2 + \overline{\zeta}_2)^3\right], \tag{31}$$

$$\dot{\zeta}_2 = j\omega_2\zeta_2 + \varepsilon(j/2\omega_2)\left[k(\zeta_2 + \overline{\zeta}_2)^3 + c(\zeta_1 + \overline{\zeta}_1)^3\right], \tag{32}$$

where $x_i = \zeta_i + \overline{\zeta}_i$ and $\dot{x}_i = j\omega_i(\zeta_i - \overline{\zeta}_i)$.
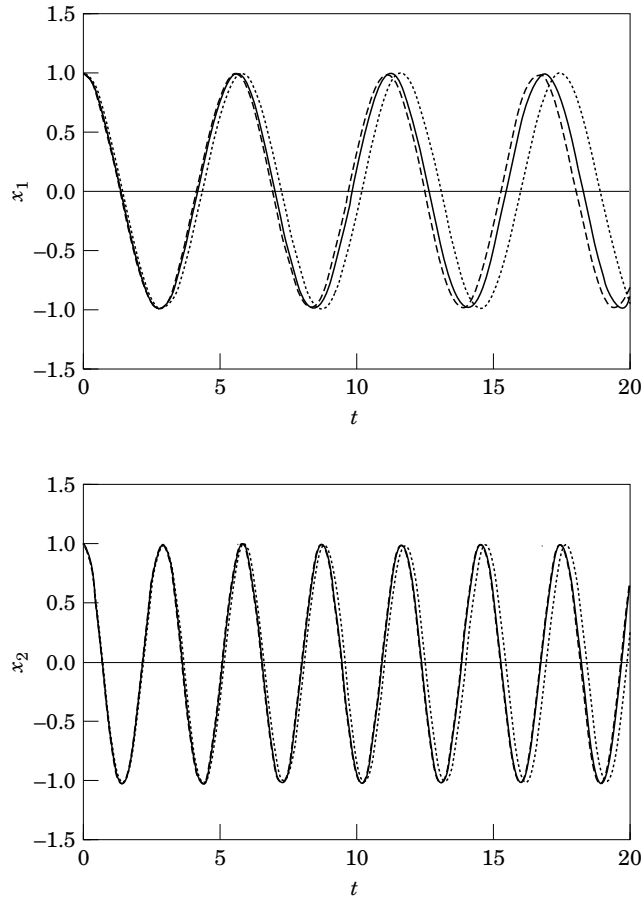

Figure 3. A comparison between the NFM (- - - -), NST ($\cdots$) and numerical methods (——).
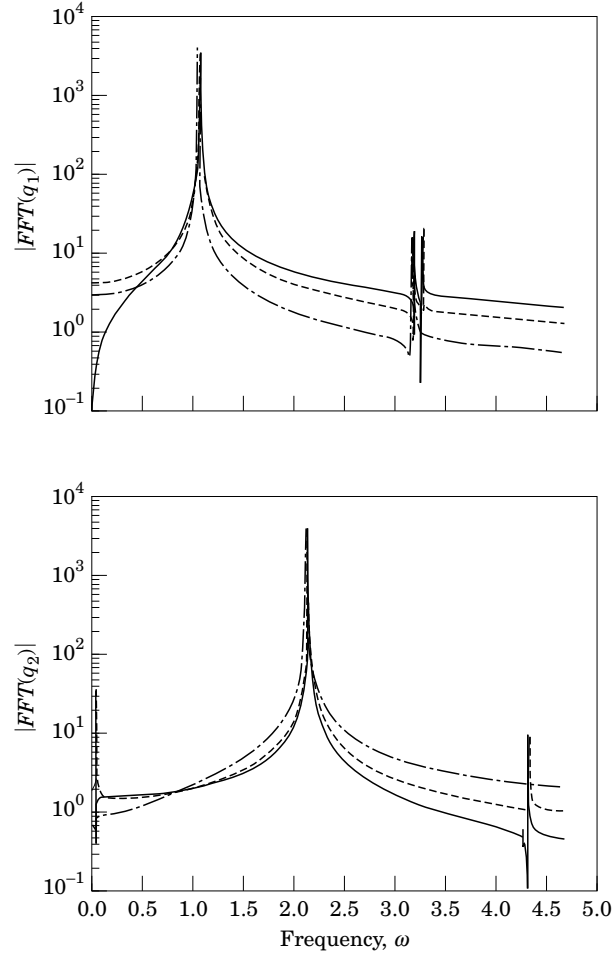
Figure 4. A comparison between the NFM (- - - -), NST ($-\cdot-\cdot-$) and numerical methods (——).

Now applying the NST, one finds the first normal mode to be

$$\dot{\zeta}_1^{(1)} = j\omega_1\zeta_1 + \varepsilon \frac{j}{2\omega_1}[k(\zeta_1^{(1)} + \overline{\zeta}_1^{(1)})^3 + c(\varepsilon H_2^{(1)} + \underline{\varepsilon}\overline{H}_2^{(1)})^3] = j\omega_1\zeta_1 + \varepsilon \frac{j}{2\omega_1}[k(\zeta_1^{(1)} + \overline{\zeta}_1^{(1)})^3], \quad (33)$$

where

$$H_1^{(1)} = \Lambda_1^{(1)}\zeta_1^{(1)^3} + \Lambda_2^{(1)}\zeta_1^{(1)^2}\overline{\zeta}_1^{(1)} + \Lambda_3^{(1)}\zeta_1^{(1)}\overline{\zeta}_1^{(1)^2} + \Lambda_4^{(1)}\overline{\zeta}_1^{(1)^3},$$

$$\Lambda_1^{(1)} = \frac{c}{2\omega_2(3\omega_1 - \omega_2)}, \qquad \Lambda_2^{(1)} = \frac{3c}{2\omega_2(\omega_1 - \omega_2)},$$

$$\Lambda_3^{(1)} = -\frac{3c}{2\omega_2(\omega_1 + \omega_2)}, \qquad \Lambda_4^{(1)} = -\frac{c}{2\omega_2(3\omega_1 + \omega_2)}.$$

Applying the NFM to equation (33) yields

$$\zeta_1^{(1)} = \xi_1^{(1)} + \varepsilon p_1^{(1)},$$

$$p_1^{(1)} = \Gamma_1^{(1)}\xi_1^{(1)3} + \Gamma_2^{(1)}\xi_1^{(1)2}\overline{\xi}_1^{(1)} + \Gamma_3^{(1)}\xi_1^{(1)}\overline{\xi}_1^{(1)2} + \Gamma_4^{(1)}\overline{\xi}_1^{(1)3},$$

$$\dot{\xi}_1^{(1)} = j\omega_1\xi_1^{(1)} + j\varepsilon(3k/2\omega_1)\,\xi_1^{(1)2}\overline{\xi}_1^{(1)}, \tag{34}$$

$$\Gamma_1^{(1)} = (k/4\omega_1^2), \Gamma_2^{(1)} = 0, \qquad \Gamma_3^{(1)} = -(3k/4\omega_1^2), \qquad \Gamma_4^{(1)} = -(k/8\omega_1^2).$$

The first mode is given by

$$\zeta_1^{(1)} = \xi_1^{(1)} + \varepsilon p_1^{(1)}, \qquad \zeta_2^{(1)} = \varepsilon H_2^{(1)}. \tag{35}$$

After a suitable change of indices in the above expressions, the second mode can be obtained and the final modal superposition is given by

$$\zeta_1 = \zeta_1^{(1)} + \zeta_1^{(2)}, \qquad \zeta_2 = \zeta_2^{(1)} + \zeta_2^{(2)}. \tag{36}$$

Consider the normal form for equation (31):

$$\dot{\xi}_1 = j\omega_1\xi_1 + j\varepsilon(3k/2\omega_1)\,\xi_1^2\overline{\xi}_1, \quad \dot{\xi}_2 = j\omega_2\xi_2 + j\varepsilon(3k/2\omega_2)\xi_2^2\overline{\xi}_2 \tag{37, 38}$$

Note that equation (37) is identical to the first modal equation (see equation (34)) and it can be shown that equation (38) is equal to the second modal equation. Furthermore, the near identity transformation is given by

$$\zeta_1 = \xi_1 + \varepsilon h_1, \qquad \zeta_2 = \xi_2 + \varepsilon h_2, \tag{39}$$

where

$$h_1 = \alpha_1\xi_1^3 + \alpha_2\xi_1^2\overline{\xi}_1 + \alpha_3\xi_1\overline{\xi}_1^2 + \alpha_4\overline{\xi}_1^3 + \alpha_5\xi_2^3 + \alpha_6\xi_2^2\overline{\xi}_2 + \alpha_7\xi_2\overline{\xi}_2^2 + \alpha_8\overline{\xi}_2^3, \tag{40}$$

$$h_2 = \beta_1\xi_1^3 + \beta_2\xi_1^2\overline{\xi}_1 + \beta_3\xi_1\overline{\xi}_1^2 + \beta_4\overline{\xi}_1^3 + \beta_5\xi_2^3 + \beta_6\xi_2^2\overline{\xi}_2 + \beta_7\xi_2\overline{\xi}_2^2 + \beta_8\overline{\xi}_2^3. \tag{41}$$

The above coefficients can be obtained by using the homology equation (11); comparing the NFM solution with the NST solution one has

$$\alpha_1 = \Gamma_1^{(1)}, \qquad \alpha_2 = \Gamma_2^{(1)}, \qquad \alpha_3 = \Gamma_3^{(1)}, \qquad \alpha_4 = \Gamma_4^{(1)}, \tag{42}$$

$$\beta_1 = \Lambda_1^{(1)}, \qquad \beta_2 = \Lambda_2^{(1)}, \qquad \beta_3 = \Lambda_3^{(1)}, \qquad \beta_4 = \Lambda_4^{(1)}, \tag{43}$$

Similar relationships hold for the other constants α_i, β_i, $\Gamma_i^{(j)}$, and $\Lambda_i^{(j)}$. Therefore, in this case NST and NFM yield the same result.

5. CONCLUSION

In the present paper the Normal Form Method (NFM) has been used to investigate the mathematical conditions of the applicability of a Non-linear Superposition Technique (NST). It was shown that the blind use of the NST may lead to an asymptotic expansion of the solution which is not uniformly valid. This is due to the presence of some special terms of the dynamical system, called the mixed-coupling terms. These terms are clearly identified in this paper. Indeed, the possibility of application of the NST is connected with the nature of the non-linearities. The range of applicability seems to be lower whenever the dimension of the problem is increased: in fact, in this case it is more likely that there exist the so-called mixed-coupling terms, which are responsible for the failure of the NST. This issue affects the general applicability of NST to N-d.o.f. systems although such applicability to N-d.o.f. systems would be the main objective of this methodology.

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APPPENDIX A: THE REAL DOMAIN APPROACH FOR THE NST

Consider the mechanical system given by equation (1) and its state space representation

$$\dot{\mathbf{x}} = \mathbf{y}, \qquad \dot{\mathbf{y}} = \mathbf{f}(\mathbf{x}, \mathbf{y}), \tag{A1}$$

where in $\mathbf{f}$ both linear and non-linear contributions are included. Then, consider a particular motion belonging to a two-dimensional invariant manifold

$$\mathbf{x} = \mathbf{X}(u, v), \qquad \mathbf{y} = \mathbf{Y}(u, v). \tag{A2}$$

Impose the conditions that $u = x_1$ and $v = y_1$. Upon using equations (A1) and (A2), the generating functions for the manifolds are found to be

$$\mathbf{X}_{,u} v + \mathbf{X}_{,v} f_1 = \mathbf{Y}, \qquad \mathbf{Y}_{,u} v + \mathbf{Y}_{,v} f_1 = \mathbf{f}, \tag{A3}$$

where $f_1 = f_1(X_1(u, v), \ldots, Y_1(u, v), \ldots)$ and e.g., $\mathbf{X}_{,u} = \partial \mathbf{X}/\partial u$. By expanding $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{f}$ as power series in u and v (e.g., $\mathbf{X}^{(m)} = \mathbf{X}_0^{(m)} + \mathbf{X}_1^{(m)} + \ldots$) one obtains a cascade of equations that allows one to obtain manifolds. It must be pointed out that at the first order this procedure gives linear invariant manifolds: i.e., the linear normal modes. On the other hand, the equations to be solved at the first order are non-linear (see reference [1] for details), whereas the successive orders are governed by linear equations. The global dynamics is then rebuilt by a simple superposition, upon considering the state vector for a manifold:

$$\mathbf{z}^{(m)} = [\mathbf{x}_1^{(m)\mathrm{T}}, \cdots, \mathbf{x}_N^{(m)\mathrm{T}} \cdots, \mathbf{y}_1^{(m)\mathrm{T}}, \mathbf{y}_N^{(m)\mathrm{T}}]^{\mathrm{T}}$$

The superposition is then performed by writing

$$\mathbf{z} = \sum_{m=1}^{N} (\mathbf{Z}_0^{(m)} + \mathbf{Z}_1^{(m)} + \dots).$$

The non-linear transformation can be inverted in a perturbation form, obtaining

$$\begin{Bmatrix} u^{(1)} \\ u^{(2)} \\ . \\ v^{(1)} \\ v^{(2)} \\ . \end{Bmatrix} = \mathbf{S}_0(\mathbf{z}) + \mathbf{S}_1(\mathbf{z}) + \dots.$$

Then the initial conditions can be projected into the ''modal basis'' and the dynamics can be studied easily by a set of ''uncoupled'' equations.

APPENDIX B: TANGENCY OF THE INVARIANT MANIFOLDS VERSUS EIGENSPACE

The tangency of the invariant manifolds to the corresponding eigenspaces is briefly shown in the present appendix. If one separates the linear and the non-linear part of $\mathcal{K}$, by considering a series expansion, one obtains

$$\mathcal{K} = \mathbf{A} \begin{Bmatrix} u \\ v \end{Bmatrix} + \mathbf{H},$$

where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ . & . \\ . & . \\ . & . \\ a_{2N} & b_{2N} \end{bmatrix}$$

and $\mathbf{H}$ contains the higher orders. If one analyzes the system in a perturbation form, at the order $\mathcal{O}(1)$ one obtains

$$\mathcal{O}(1): \quad \begin{Bmatrix} \zeta_m a_i \lambda_m + \overline{\zeta}_m b_i \overline{\lambda}_m = \zeta_m a_i \lambda_i + \overline{\zeta}_m b_i \lambda_i \\ \overline{\zeta}_m \bar{a}_i \overline{\lambda}_m + \zeta_m b_i \lambda_m = \overline{\zeta}_m \bar{a}_i \overline{\lambda}_i + \zeta_m \overline{b}_i \overline{\lambda}_i \end{Bmatrix}$$

which implies

$$a_1(\lambda_m - \lambda_i) = 0 \Rightarrow a_i = 0 \qquad \text{if } \lambda_i \neq \lambda_m,$$

$$b_i(\overline{\lambda}_m - \lambda_i) = 0 \Rightarrow b_i = 0 \qquad \text{if } \lambda_i \neq \overline{\lambda}_m. \tag{B1}$$

It follows that, once one assumes $i \neq m$ (that is enough to satisfy equation (B1) if no internal resonances are present), the linear part of $\mathcal{K}$ can be eliminated.