Journal of Sound and Vibration (1997) **200**(1), 31–40

AN IDENTIFICATION TECHNIQUE FOR NON-LINEAR DYNAMICAL SYSTEMS UNDER STOCHASTIC EXCITATIONS

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(*Received* 21 *December* 1995, *and in final form* 22 *July* 1996)

An identification technique is devised for SDOF dynamical mechanical systems under random excitations. The system is assumed to be governed by a non-linear equation of motion in general form, in which the restoring force and the dissipative terms are given by arbitrary power functions. Algebraic equations are obtained for the expectations of some suitable excitation and response quantities. It is shown that these equations are valid for any stationary random excitations if the system attains the steady state. Based on these equations, an identification technique has been devised and verified experimentally for white noise and coloured (pink) noise random excitations.

1. INTRODUCTION

Mechanical systems, such as machine elements, are usually subjected to complex dynamical excitations. Such excitations are often idealized as stationary stochastic processes, both time continuous and of impulsive type [1–3]. As concerns the structural element, its service life expectancy, which depends on the degree of wear or deterioration, is the most essential. The wear, or deterioration, is determined by the change of element properties, which are due to the dynamical loadings. If the behaviour of the mechanical system is linear, then the identification techniques based on the experimental modal analysis may be used to evaluate directly the modal parameters [4–6]. These parameters define the dynamical properties of the system under investigation and their changes determine the deterioration of the system.

In general, it appears that an *a priori* assumed linear model of the structural element behaviour is an oversimplification if, for example, the energy losses taking place in a spring-damping machine element and processes connected with material aging are to be determined precisely. In these cases suitable tests can be performed on especially designed simple dynamical systems, in which the spring-damping element plays the dominant role in the overall system behaviour (e.g., systems having one or one and a half degrees of freedom). Then, in the cases in which the dissipation function is essentially non-linear, it is much easier to determine the change of its form. On the other hand, the identification of the physical system (especially of the non-linear one) should be performed for a wider class of excitations than purely harmonic or single-impulse excitations.

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In references [7–10] the procedure is presented for the determination of the damping characteristics of the mechanical system governed by the equation

$$m\ddot{x} + F_d(\dot{x}) + F_s(x) = p(t). \qquad (1)$$

In the above-mentioned procedure no *a priori* assumed specific form of the dissipative function $F_d(\dot{x})$ is required. However, the excitation must be either harmonic, or periodic of such a form that the induced response is harmonic [9, 10]. The hypothesis of parallel action of the purely dissipative element $F_d(\dot{x})$ and purely elastic element $F_s(x)$, which is in fact expressed by equation (1), may seem questionable. It can, however, be verified experimentally [9, 11].

Let us assume that as a result of the above-mentioned procedure the equation governing the system behaviour has been established in the form

$$m\ddot{x} + h \operatorname{sgn} \dot{x} + \sum_{\nu=1}^{n} c_\nu \dot{x}^\nu + \sum_{\mu=1}^{q} k_\mu x^\mu = p(t) \qquad (2)$$

(in which n and q are arbitrary large numbers) and in what follows only the changes of parameters m, h, c_ν and k_μ of the system, which are due to the arbitrary dynamic excitation $p(t)$, are of interest. If the excitation can be idealized as an arbitrary deterministic periodic function, with period T, then the parameters appearing in equation (2) can be evaluated with the help of methods given in references [12–14]. These methods are essentially based on the following identification equations: energy balance equation,

$$\alpha_x^p = h\alpha_x^{S(v)} + \sum_{\nu=1}^{n} c_\nu \alpha_x^{v^\nu}; \qquad (3)$$

power balance equation,

$$\alpha_v^p = m\alpha_v^a + \sum_{\mu=1}^{q} k_\mu \alpha_v^{x^\mu}. \qquad (4)$$

Here α_y^z denotes the area within the closed loop of the relationship $z(y)$, where z and y denote suitable signals, e.g., excitation $p(t)$, displacement $x(t)$, velocity $v(t)$, acceleration $a(t)$, sign of velocity $S(v)$, etc. For example, α_p^x denotes the area within the well known dynamic hysteresis loop and α_v^x the area within the phase portrait $v(x)$.

Equations (3) and (4) are satisfied for any dynamic excitations which induce the periodic response of the system. Equation (3) results from multiplying both sides of equation (2) by an infinitesimal displacement $\mathrm{d}x = \dot{x}\,\mathrm{d}t$ and subsequently integrating over the whole period T. It can be noted that if the periodicity relationships $p(t) = p(t + T)$ and $x(t) = x(t + T)$ hold, then the energy balance equation is obtained in the form (3) [12]. Similarly, multiplying both sides of equation (2) by an infinitesimal velocity $\mathrm{d}v = \ddot{x}\,\mathrm{d}t$ and integrating yields equation (4).

In the present paper it is shown that similar equations can be obtained for dynamic excitations in form of ergodic, stationary stochastic processes. The equations are next used for identification of damping in a certain non-linear dynamical system subjected to a white noise and one type of coloured noise excitation. The specific coloured noise with a spectral density inversely proportional to the frequency was used. Such a noise is termed as a pink noise (cf., [15]), by analogy with pink light, because the major part of its power spectrum is located in the low frequency region.

2. ENERGY BALANCE EQUATION FOR RANDOM EXCITATIONS

Assume that the excitation $p(t)$ in equation (2) is a stationary stochastic process and, in particular, that its one-dimensional probability distribution is time-invariant. In addition, assume that the process is ergodic. The dynamical system is assumed to be asymptotically stable: i.e., it attains steady state when $t \to \infty$. As is well known, the steady state response to a stationary excitation is also a stationary process, and in particular the steady state response statistical moments (the moments of the process itself and of its first derivative) are constant.

Multiplying both sides of equation (2) by the elementary displacement $\mathrm{d}x = \dot{x}\,\mathrm{d}t$ and performing time averaging over the interval T, one obtains

$$m\frac{1}{T}\int_0^T \ddot{x}\dot{x}\,\mathrm{d}t + \frac{1}{T}h\int_0^T (\operatorname{sgn}\dot{x})\dot{x}\,\mathrm{d}t + \frac{1}{T}\int_0^T \sum_{v=1}^{n} c_v\dot{x}^v\dot{x}\,\mathrm{d}t + \frac{1}{T}\int_0^T \sum_{\mu=1}^{q} k_\mu x^\mu\dot{x}\,\mathrm{d}t = \frac{1}{T}\int_0^T p(t)\dot{x}\,\mathrm{d}t. \tag{5}$$

Assuming that the stationary response process is also ergodic, one can replace the time averaging by ensemble averaging: i.e., by the expectation. Then it can be noticed that the first integral and the fourth group of integrals on the left side of equation (5) are equal to zero. For example, in the case of the first integral one has

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T \ddot{x}\dot{x}\,\mathrm{d}t\right] = \mathrm{E}[\ddot{x}\dot{x}] = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[\dot{x}^2]. \tag{6}$$

However, since the velocity $v(t) = \dot{x}(t)$ is a stationary process, it follows that

$$\mathrm{E}[\dot{x}^2] = \text{constant} \Rightarrow (\mathrm{d}/\mathrm{d}t)\mathrm{E}[\dot{x}^2] = 0. \tag{7}$$

Likewise, in the case of any of the integrals of the fourth group the following relationshp is valid,

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T x^\mu\dot{x}\,\mathrm{d}t\right] = \mathrm{E}[x^\mu\dot{x}] = \frac{1}{\mu+1}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[x^{\mu+1}], \tag{8}$$

and as the process $x(t)$ is stationary it follows that

$$\mathrm{E}[x^{\mu+1}] = \text{constant} \Rightarrow (\mathrm{d}/\mathrm{d}t)\mathrm{E}[x^{\mu+1}] = 0. \tag{9}$$

The energy balance equation for stationary, ergodic random excitations is obtained in the form

$$h\mathrm{E}[(\operatorname{sgn} v)v] + \sum_{v=1}^{n} c_v\mathrm{E}[v^{v+1}] = \mathrm{E}[pv]. \tag{10}$$

This equation can be used to evaluate the constants h and c_v which determine the damping function of the actual system, if the expected (mean) values of the signals $((\operatorname{sgn} v)v,\ v^{v+1},\ pv)$ are measured.

3. POWER BALANCE EQUATION FOR RANDOM EXCITATIONS

Under the same assumptions about the excitation process $p(t)$ and the dynamical system as in section 2, the power balance equation is obtained.

Multiplying both sides of equation (2) by the elementary velocity $\mathrm{d}v = \ddot{x}\,\mathrm{d}t$ and performing the time averaging over the interval T, one obtains

$$m\frac{1}{T}\int_0^T \dddot{x}\ddot{x}\,\mathrm{d}t + \frac{1}{T}h\int_0^T (\operatorname{sgn}\dot{x})\ddot{x}\,\mathrm{d}t + \frac{1}{T}\int_0^T \sum_{v=1}^{n} c_v \dot{x}^v \ddot{x}\,\mathrm{d}t + \frac{1}{T}\int_0^T \sum_{\mu=1}^{q} k_\mu x^\mu \ddot{x}\,\mathrm{d}t = \frac{1}{T}\int_0^T p(t)\ddot{x}\,\mathrm{d}t. \tag{11}$$

As before, the time averaging is replaced by the expectation. Here, however, the second integral and the third group of integrals on the left side of equation (11) are equal to zero. For example, in the case of the second integral, which describes the dry friction, one has

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T (\operatorname{sgn} v)\dot{v}\,\mathrm{d}t\right] = \mathrm{E}[S(v)\dot{v}] = (\mathrm{d}/\mathrm{d}t)\mathrm{E}[|v|]. \tag{12}$$

As the process $v(t)$ is stationary (its probability density function is time invariant) it follows that

$$\mathrm{E}[|v|] = \text{constant} \Rightarrow (\mathrm{d}/\mathrm{d}t)\mathrm{E}[|v|] = 0. \tag{13}$$

In the case of the third group of integrals in equation (19), the following relationship is valid:

$$\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T v^v \dot{v}\,\mathrm{d}t\right] = \frac{1}{v+1}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[v^{v+1}] = 0. \tag{14}$$

Upon using equations (12)–(14), equation (11) is obtained as

$$m\mathrm{E}[a^2] + \sum_{\mu=1}^{q} k_\mu \mathrm{E}[x^\mu a] = \mathrm{E}[pa], \tag{15}$$

where a denotes the acceleration $a(t)$ of the mass m.

Notice that equations (10) and (15) are analogous to equations (3) and (4), respectively. These equations are, in principle, satisfied by any random excitations (of time-continuous or impulsive type) if only these excitations induce steady state response. Such equations can be used to evaluate the unknown parameters m, h, c_v and k_μ of the system under investigation if the expectations which appear in these equations are known (e.g., from experiments) and provided that for these specific values of the parameters equations (10) and (15) are linearly independent.

4. EXAMPLE APPLICATION

In order to verify the above described technique, an experiment has been performed for the non-linear dynamical system shown in Figure 1. This system has been designed as an analogue machine. The form of the non-linear damping function is specified by the three coefficients c_1, c_2 and c_3 (see Figure 2).

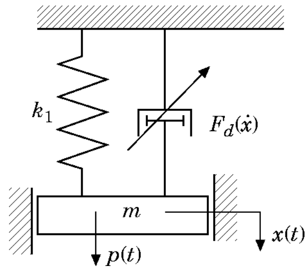

Figure 1. The dynamical system. $m\ddot{x} + F_d(\dot{x}) + F_s(x) = p(t)$; $F_d(\dot{x}) = c_1\dot{x} + c_3\dot{x}^3 + c_5\dot{x}^5$, $F_5(x) = kx$; $m = 8$ kg, $k_1 = 9800$ kg/s^2, $c_1 = 260$ kg/s, $c_3 = -340$ kgs/m^2, $c_5 = 320$ kg s^3/m^4.

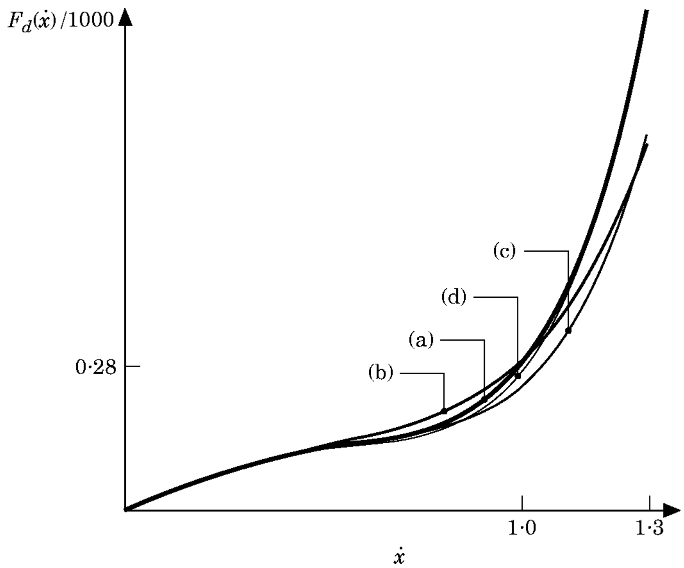

Figure 2. The damping function of the system tested (a), and the damping functions obtained in the cases (b), (c) and (d). (a) Fixed damping function $F_d(\dot{x}) = 260\dot{x} - 340\dot{x}^3 + 320\dot{x}^5$; (b) for RN only, $\hat{F}_d(\dot{x}) = 265,4\dot{x} - 451\dot{x}^3 + 425,8\dot{x}^5$; (c) for PN only, $\hat{F}_d(\dot{x}) = 258,1\dot{x} - 309,8\dot{x}^3 + 251,2\dot{x}^5$; (d) for RN and PN, $\hat{F}_d(\dot{x}) = 261,28\dot{x} - 376\dot{x}^3 + 340,16\dot{x}^5$.

The purpose of the experiment was to determine the damping function $F_d(\dot{x})$ of the system under random excitations. Two types of random excitations have been used: random noise excitation (RN); pink noise excitation (PN). In the experiment, as a generator of random noises a two-channel analyzer HP35665 was used, which was connected with the tested system as shown in Figure 3.

In the case of the tested system equation (10) simplifies to

$$c_1\mathrm{E}[v^2] + c_3\mathrm{E}[v^4] + c_5\mathrm{E}[v^6] = \mathrm{E}[pv]. \tag{16}$$

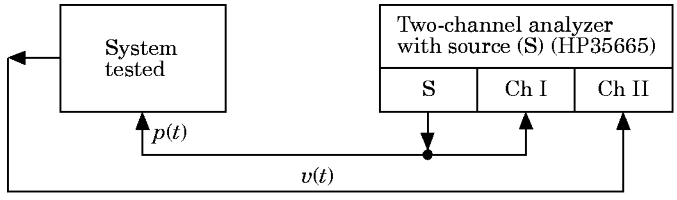

Figure 3. The connection of two-channel dynamic signal analyzer in the procedure.

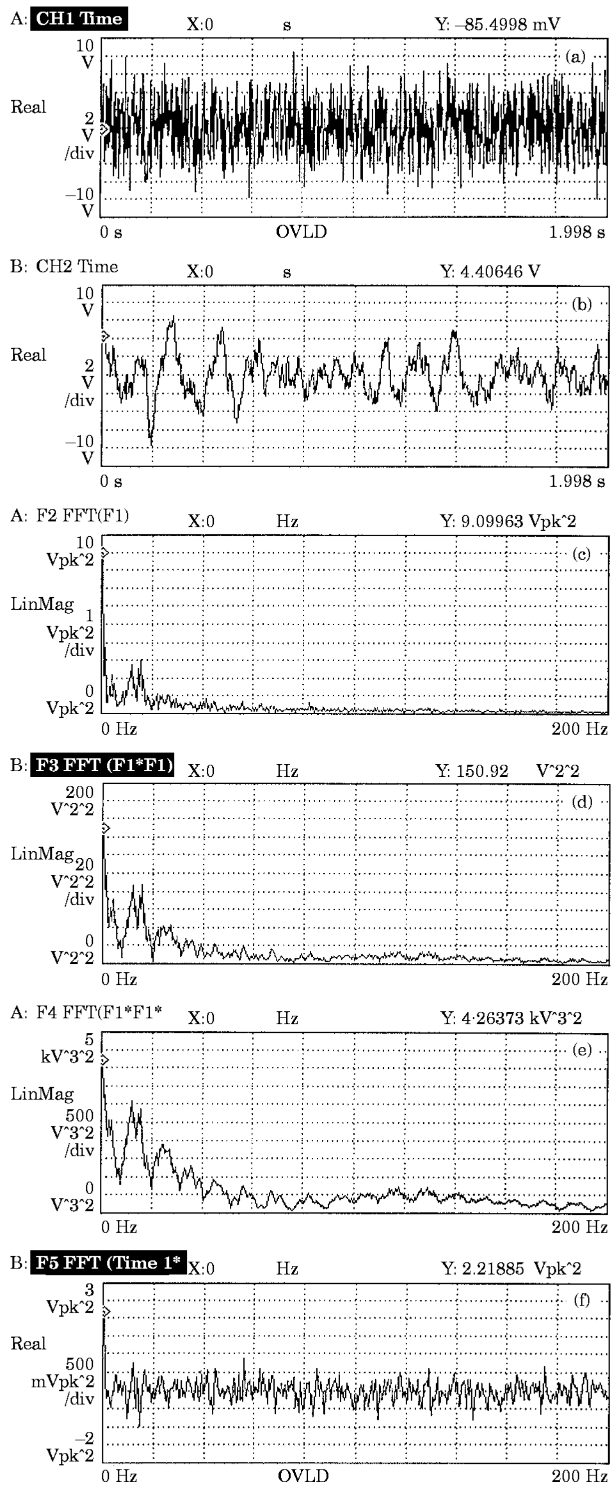

Figure 4. Examples of the RN excitation (a), the response (b) and functions F2, F3, . . . , F5, (c)–(f) (according to row 4 in Table 1).

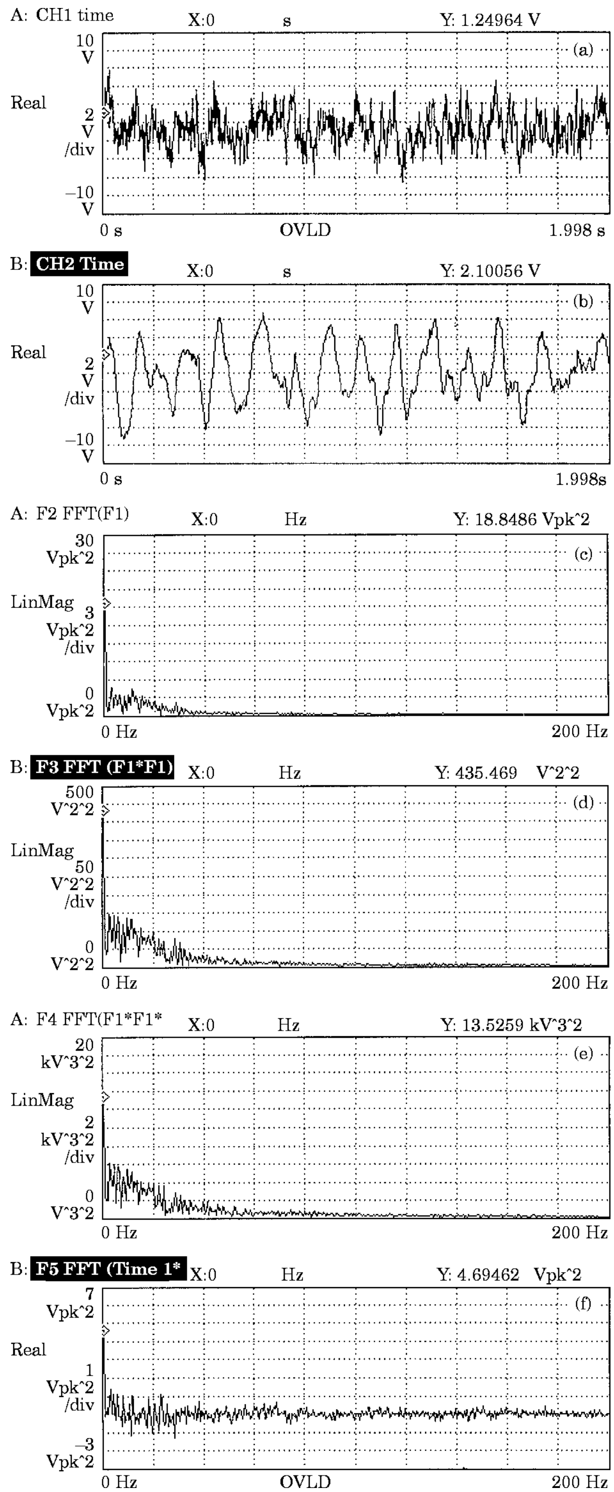

Figure 5. As Figure 4, but for PN excitation (according to row 17 in Table 2).

TABLE 1

Mean values obtained from the experiment with the help of F2, F3, F4 *and* F5 (*random noise*)

No.	F2 = E[v^2]	F3 = E[v^4]	F4 = E[v^6]	F5 = E[pv]
1	0·18966	0·051314	0·021031	0·045425
2	0·12500	0·022541	0·0060864	0·03150
3	0·13159	0·027070	0·0089229	0·032954
4	0·090996	0·015092	0·0042637	0·022198
5	0·02695	0·0011436	0·00008032	0·008040
6	0·16211	0·039727	0·015581	0·03832
⋮	⋮	⋮	⋮	⋮
32	0·10614	0·0159143	0·0034727	0·02755

TABLE 2

Mean values obtained from the experiment with the help of F2, F3, F4, *and* F5, (*pink noise*)

No.	F2 = E[v^2]	F3 = E[v^4]	F4 = E[v^6]	F5 = E[pv]
1	0·131721	0·026851	0·00927514	0·0338197
2	0·16147	0·0435021	0·0177616	0·039988
3	0·20299	0·069316	0·0406493	0·049971
⋮	⋮	⋮	⋮	⋮
17	0·188486	0·043547	0·013526	0·046946
⋮	⋮	⋮	⋮	⋮
32	0·251428	0·0753756	0·0309312	0·062872

In order to measure the mean values E[v^2], E[v^4], E[v^6] and E[pv] which appear in equation (16) the system analyzer was used, which allowed one to create the following functions of the signals $v(t)$ and $p(t)$: F1 = TIME2 ∗ TIME2 ⇒ time history of $v^2(t)$; F2 = FFT(F1) ⇒ Fourier transform of $v^2(t)$; F3 = FFT (F1 ∗ F1) ⇒ Fourier transform of $v^4(t)$; F4 = FFT (F1 ∗ F1 ∗ F1) ⇒ Fourier transform of $v^6(t)$; F5 = FFT(TIME1 ∗ TIME2) ⇒ Fourier transform of $p(t)v(t)$.

Some examples of the results of the analysis are shown in Figures 4 and 5 and the corresponding data set is given in Table 1. The set covers all data which were used in the identification procedure. For example, the data of row 4 (see Table 1) have been obtained from results shown in Figure 4 as follows: (1) from F2 (Figure 4(c)) one has Y: 9·0996 V pk^2 and hence E[v^2] = 0·090996(m/s^2) (according to column F2 = E[v^2] in Table 1); (2) from F3 (Figure 4(d)) one has Y: 150·92 kV^{22} and hence E[v^4] = 0·015092(m/s)^4 (according to column F3 = E[v^4] in Table 1); (3) from F4 (Figure 4(e)) one has Y: 4·2637 kV^{32} and hence E[v^6] = 0·00426378(m/s)^6 (according to column F4 = E[v^6] in Table 1); (4) from F5 (Figure 4(f)) one has Y: 2·2198 mV pk^2 and hence F5 = E[pv] = 0·022198(Nm/800 s)† (according to column E[pv] in Table 1).

The parameters c_1, c_3 and c_5 of the damping function have been estimated based on equation (16) and on data given in Tables 1 and 2, for the random noise only (Table 1), for the pink noise only (Table 2) and for both excitations together. The results obtained on applying the least sum of squared errors condition are shown in Table 3.

In computations a modified equation in the form

$$c_1 + c_3 y_1 + c_5 y_2 = z \tag{17}$$

† The number 800 appearing in the dimension of the E[pv] results from the measurement of the rescaled excitation $p(t)$ of the analogue model.

TABLE 3
Numerical results of experiments

	Given values	Results obtained for random noise (RN)	Results obtained for pink noise (PN)	Results obtained for both noises
c_1	260	265·4	258·1	261·28
c_3	−340	−451	−309·8	−376·0
c_5	320	425·8	251·6	340·16

was used, which was obtained from equation (16) ($y_1 = \mathrm{E}[v^4]/\mathrm{E}[v^2]$, $y_2 = \mathrm{E}[v^6]/\mathrm{E}[v^2]$ and $z = \mathrm{E}[pv]/\mathrm{E}[v^2]$. The level of random signals has been assumed in such a way as to obtain uniform distribution of the velocity amplitude over the interval [0, 1 m/s]. The estimates of the damping function obtained for each of the cases considered are compared in Figure 2.

5. CONCLUSIONS

The comparison of experimental results (Table 3) reveals that values of the estimators $\hat{c}_1$ of coefficient c_1 are in each of the three cases considered very close to the given value $c_1 = 260$. Larger differences occur in the case of coefficients c_3 and c_5. Nevertheless, the functions $\hat{F}_d(\dot{x})$ estimated from the estimators of the latter coefficients are (in the considered interval of velocity variation) very close (in a qualitative sense) to the given function (cf., Figure 2). The results obtained should be considered satisfactory as the tests have been carried out in large part with the help of analogue technique.

Another problem, which is not dealt with in the present paper, is how the change of the observation time and the resulting frequency range affect the results. In the present case the frequency range was assumed *a priori* as 0–200 Hz, because the system natural frequency was 5–8 Hz (cf., Figures 4 and 5).

Similar investigations for systems with discontinuous characteristics (e.g., with dry friction) have also been carried out by the authors and the results are to be published.

It should be noted that the technique presented herein for a SDOF system can be extended to chain-like MDOF systems (cf., [9]).

ACKNOWLEDGMENT

The present research was supported by the Committee for Scientific Research (K.B.N.) through grant no. PB 0964/S1/94/06.

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