Journal of Sound and Vibration (1997) **200**(3), 315–325

EARTHQUAKE ANALYSIS OF BEAMS RESTING ON ELASTIC FOUNDATIONS BY USING A MODIFIED VLASOV MODEL

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(*Received* 14 *March* 1996, *and in final form* 5 *August* 1996)

The purpose of this paper is to apply the modified Vlasov model to earthquake analysis of beams resting on elastic foundations, and to analyze the effects of the subsoil depth, the beam length and their ratio on its responses. For this purpose, a computer program is coded for the dynamic out-of-plane response of beams resting on elastic foundations. The finite element method is used for spatial integration and the Newmark-β method is used for time integration. The solution technique is an iterative process which is dependent upon the value of the parameter γ. Graphs are presented to evaluate the effects of the beam length and subsoil depth on its responses. Numerical examples show the applicability of the modified Vlasov model to earthquake analysis of beams resting on elastic foundations.

1. INTRODUCTION

The concept of a beam on an elastic foundation has been a convenient tool to obtain solutions for many engineering problems. The behavior of the subsoil or an elastic foundation is hard to model, while it is easy to establish the stiffness characteristics of the beam. Many researchers use the Winkler model for static and dynamic analysis of beams resting on elastic foundations [1], where the reaction forces of the elastic foundation are assumed to be proportional at every point to the deflection of the beam at that point. The Winkler model characterizes the soil as closely spaced linear springs, and neglects the shear between them. When the Winkler model is used, the displacement of the beam is constant when it is subjected to a uniformly distributed load. Therefore, there is no bending moment and no shear force in the beam during this loading condition. Vlasov developed a two-parameter model by introducing an arbitrary parameter, γ, to characterize the distribution of the displacements in the vertical direction in the subsoil [2]. The Vlasov model accounts for the effect of the neglected shear strain energy in the soil and the shear forces that come from the surrounding soil. The Vlasov model requires the estimation of the γ parameter. Jones and Xenophontos [3] established a relationship between the γ parameter and the displacement characteristics, but did not suggest any method for the calculation of its actual value. Recently, an iterative technique has been developed by Vallabhan and Das [4, 5] to solve problems of beams on an elastic foundation by introducing a modified Vlasov model. They calculated the value of γ as a function of the characteristics of the beam and the foundation. Ding [6] solved the problem of vibrations of beams on elastic foundations using the Winkler model. Franciosi and Masi [7] used a two-parameter foundation model to solve the free vibration problem of beams on elastic foundations, and they considered two constant soil parameters which depend on the soil type. Lai *et al*. [8] used the finite element method to solve the differential equation of the

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beam-soil system based on the exact solution and the Winkler foundation model. However, no references have been found for the earthquake analysis of beams resting on elastic foundations by using the modified Vlasov model.

The purpose of this paper is to apply the modified Vlasov model to earthquake analysis of beams resting on elastic foundations, and to analyze the effects of the subsoil depth, the beam length and their ratio on its responses. For this purpose, a computer program is coded for the dynamic out-of-plane response of beams on elastic foundations. The finite element method is used for spatial integration and the Newmark-β method is used for time integration. The solution technique is an iterative process which is dependent upon the value of the parameter, γ.

2. FINITE ELEMENT MODELLING

The governing equation for a beam subjected to an earthquake excitation with no damping is

$$[M]\{\ddot{W}\} + [K]\{W\} = -[M]\{\ddot{u}_g\}, \tag{1}$$

where K is the stiffness and M is the mass matrix of the beam–soil system, W and $\ddot{W}$ are the displacement and acceleration of the beam, respectively, and $\ddot{u}_g$ is the earthquake acceleration. For a beam resting on an elastic foundation, evaluation of the stiffness and mass matrices are given in the following sections.

2.1. EVALUATION OF THE STIFFNESS MATRIX

The strain energy of a beam–soil system (see Figure 1), upon assuming that the subsoil has a uniform finite depth, can be written as

$$u = \tfrac{1}{2}\int_0^l \left[\frac{d^2w(x)}{dx^2}\right]^T \frac{E_bI_b}{2}\left[\frac{d^2w(x)}{dx^2}\right] dx + \tfrac{1}{2}\int_0^L [w(x)]^T k[w(x)]\, dx + \tfrac{1}{2}\int_0^L \left[\frac{dw(x)}{dx}\right]^T 2t\left[\frac{dw(x)}{dx}\right] dx. \tag{2}$$

Here E_bI_b is the flexural rigidity of the beam, L is the length of the beam, and k and $2t$ are the soil parameters, which can be expressed as

$$k = \int_0^H \frac{E_sb(1-v_s)}{(1+v_s)(1-2v_s)}\left(\frac{d\phi}{dz}\right)^2 dz, \qquad 2t = \int_0^H \frac{E_sb}{2(1+v_s)}\phi^2\, dz, \tag{3, 4}$$

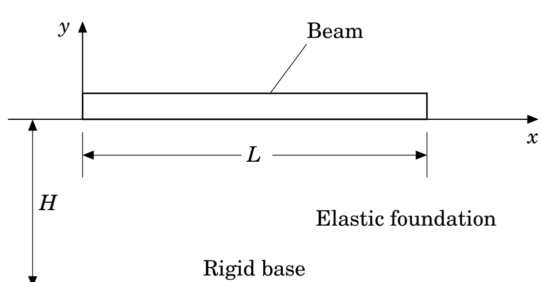

Figure 1. A sample beam on an elastic foundation.

where

$$\phi(z) = \sinh \gamma(1 - z/H)/\sinh \gamma \tag{5}$$

and

$$\left(\frac{\gamma}{H}\right)^2 = \frac{(1-2v_s)}{2(1-v_s)} \frac{\int_0^L (\mathrm{d}w/\mathrm{d}x)^2\,\mathrm{d}x + \frac{1}{2}\sqrt{k/2t}w^2(x)}{\int_0^L w^2(x)\,\mathrm{d}x + \frac{1}{2}\sqrt{2t/k}w^2(x)}. \tag{6}$$

In these expressions, H, b, E_s and v_s are the height, width, modulus of elasticity and Poisson ratio of the subsoil, respectively. Further details can be found in references [4] and [5].

The element stiffness matrices and the equivalent nodal loads can be evaluated by using the cubic displacement function that is standard in finite element beam theory. A cubic field, interpolated from nodal d.o.f. $\{w_e\} = \{w_{i,i}, w_{j,j}\}$, is

$$w(x) = [N_1 \quad N_2 \quad N_3 \quad N_4]\{w_e\}, \tag{7}$$

in which N_1–N_4 are as stated in references [9–11]. By substituting equation (7) into equation (1), the stiffness matrices of the beam and the soil can be evaluated as

$$U = \tfrac{1}{2}\{w_e\}([k_b] + [k_w] + [k_v])\{w_e\} \tag{8}$$

where $[k_b]$ is the conventional uniform-beam stiffness matrix [9], and is given as

$$[k_b] = \frac{E_b I_b}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ & 4l^2 & 6l & 2l^2 \\ & & 12 & 6l \\ & & & 4l^2 \end{bmatrix}, \tag{9}$$

$[k_w]$ is the Winkler foundation stiffness matrix [9],

$$[k_w] = k \begin{bmatrix} \frac{13}{35}l & -\frac{11}{210}l^2 & \frac{9}{70}l & \frac{13}{420}l^2 \\ & \frac{1}{105}l^3 & -\frac{13}{420}l^2 & -\frac{1}{140}l^3 \\ & & \frac{13}{35}l & \frac{11}{210}l^2 \\ & & & \frac{1}{105}l^3 \end{bmatrix}, \tag{10}$$

and $[k_v]$ is the second parameter foundation stiffness matrix [9],

$$[k_v] = 2t \begin{bmatrix} \frac{6}{5}\frac{1}{l} & -\frac{1}{10} & -\frac{6}{5}\frac{1}{l} & -\frac{1}{10} \\ & \frac{2}{15}l & \frac{1}{10} & -\frac{1}{30}l \\ & & \frac{6}{5}\frac{1}{l} & \frac{1}{10} \\ & & & \frac{2}{15}l \end{bmatrix}. \tag{11}$$

If $\{q\}$ denotes $-[M]\{\ddot{u}_g\}$, equivalent nodal loads, $\{F\}$, for a beam can be found as

$$\{F\} = \int_0^l [N]^{\mathrm{T}}\{q\}\,\mathrm{d}x. \tag{12}$$

2.2. EVALUATION OF THE MASS MATRIX

The dynamics of elastic structures are based on Hamilton's variational principle, with the kinetic energy

$$\Pi_k = \tfrac{1}{2}\int_\Omega v^{\mathrm{T}}\mu\dot{v}\,\mathrm{d}\Omega, \tag{13}$$

where μ is the mass density matrix and v represents the vector of generalized displacement components relevant to inertial forces. The dot denotes the partial derivative with respect to the time variable t. The consistent mass matrix is obtained by substituting $v = N_I w_e$ into equation (13):

$$M = \int_\Omega N_I^{\mathrm{T}}\,\mu N_I\,\mathrm{d}\Omega. \tag{14}$$

The mass matrix for the beam–soil system, M, needs to be analyzed. The matrix μ in equation (13) is a square symmetric matrix of the form

$$\mu = \begin{bmatrix} \rho_b h + \frac{1}{3}\rho_s H & 0 \\ 0 & \frac{1}{12}\rho_b h^3 \end{bmatrix}, \tag{15}$$

where ρ_b is the beam mass density, h is the depth of the beam cross-section, and ρ_s is the mass density of the soil. Further details can be found in reference [12]. In view of equation (7), the following expression can be written for each finite piece:

$$[N_I] = \begin{bmatrix} N \\ -\mathrm{d}N/\mathrm{d}x \end{bmatrix}. \tag{16}$$

The consistent mass matrix of the beam and the soil can be evaluated after substituting equation (16) into equation (14), and integrating it from zero to l.

It should be noted that, in this study, the Newmark-β method is used for the time integration of equation (1) by using the average acceleration method [13].

3. THE SOLUTION TECHNIQUE

The solution technique is an iterative process which is dependent upon the γ parameter [12]. When the time and the velocity is equal to zero at the beginning, γ is initially set equal to one. Then, the values of the two soil parameters, k and $2t$, are calculated. These values are used to construct the coefficient matrix of the beam–soil system and to calculate the displacements at discrete points along the beam. Next, the value of γ is calculated by using the displacements found in the previous step. A comparison between this calculated value of γ and the initially assumed γ or the previously calculated γ value is then made. If the difference between the two successive γ values is within a prescribed tolerance, the analysis is completed for the related time increment. Otherwise, another iteration is performed and the process is repeated until convergence is obtained. The final value of γ at this time increment is taken to be the initial value of the γ parameter for the next time increment.

4. NUMERICAL EXAMPLES

4.1. DATA FOR NUMERICAL EXAMPLES

In the light of the results given in reference [4] for beams resting on elastic foundations, the depths, H, of the soil stratum used are 5 m, 10 m and 15 m. The ratios H/L used are 0·25, 0·50, 0·77 and 1·0 for each subsoil depth considered. The cross-section of the beams considered in this study is 30 cm in width and 50 cm in depth. In the calculation of the mass matrix, the mass densities of the beam and the subsoil are taken to be 2500 kg/m^3 and 1700 kg/m^3, respectively.

To obtain the response of each beam, the first 15 s of the vertical component of the 13 March 1992 Erzincan earthquake in Turkey is used, because the peak value of the record occurs in this range [14]. Also, after the fifteenth second, it is seen that the responses of the beam remain almost constant, since no damping is assumed in this study.

For the sake of accuracy in the results, rather than starting with a set consisting of a finite element mesh size and time increment, the mesh size and the time increment required to produce the desired accuracy are determined. This analysis is undertaken separately for the mesh size and time increment. To find out the required finite element mesh size, the time increment is fixed, and the convergence of the maximum displacement is checked for different mesh sizes. To obtain the required time increment, the finite element mesh size is fixed, and the convergence of the maximum displacement is checked for different time increments. In this way, it is concluded that the results have an acceptable error when using equally spaced 20 elements for a 10 m beam if a 0·005 s time increment is used. The element length is kept constant for different lengths of the beam.

4.2. RESULTS

In this study, the time histories of the displacement of the beams at different points are presented for different H and L values. The time histories of the mid-span of the beam for $L = 10$ m and 20 m when $H = 5$ m, for $L = 10$ m and 20 m when $H = 10$ m, and for $L = 20$ m when $H = 15$ m are given in Figures 2, 3 and 4, respectively. As seen from Figures 2(a) and 2(b), the mid-span displacement of the 10 m and 20 m beams for $H = 5$ m reaches its absolute maximum values of 10·19 mm at 11·435 s, and of 18·56 mm at 10·125 s, respectively. As expected in a static sense, the maximum displacement of the 20 m beam is larger than that of the 10 m beam, since the 20 m beam is more flexible than the 10 m beam. As seen from Figures 3(a) and 3(b), the mid-span displacement of the 10 m and 20 m beams for $H = 10$ m reaches its absolute maximum values of 20·46 mm at 4·155 s, and of 14·92 mm at 1·85 s, respectively. These figures also show that the time histories of beams

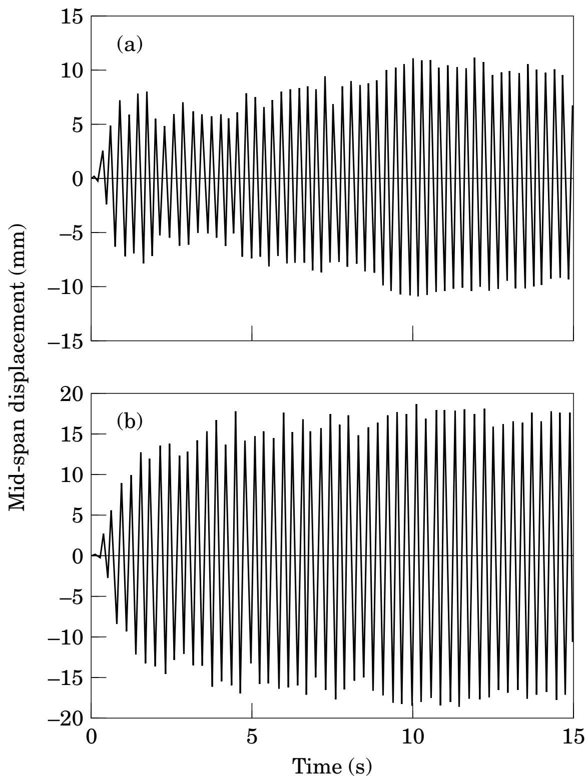

Figure 2. The time history of the mid-span displacement of the beam: (a) for $H = 5$ m and $L = 10$ m; (b) for $H = 5$ m and $L = 20$ m.

with different lengths are quite different even with the same subsoil depth, because the dynamic characteristics of the system affect the responses. In Figure 4 it is shown that the time histories of the 20 m beam for $H = 15$ m are considerably different from previous time histories, and its absolute maximum value occurs at 3·565 s, being 20·02 mm.

These three figures indicate that the time histories of the mid-span displacements of the beams differ from each other depending on the dynamic characteristics of the system.

All these figures show that the periods of the mid-span displacements are becoming larger with increasing subsoil depth for a fixed beam length, and with increasing beam length for a fixed subsoil depth. This is expected, because the system considered is becoming more flexible when the subsoil depth and/or the beam length increase.

The purpose of this study is to present the time histories of the displacement at different points of the beam for different beam lengths and subsoil depths, but only the maximum displacement for different subsoil depths and beam lengths are presented, since presenting all of the time histories would take up excessive space. This simplification to presentation only of the maximum responses is supported by the fact that the maximum values of these quantities are the most important ones for design. The results are presented in graphical, rather than in tabular form, in Figures 5 and 6. These graphs show the maximum displacements for different H and L values, and for different H/L ratios, respectively. In these figures, the bottom part shows the downward displacements, and the top part shows the upward displacements. As also seen from these figures, the maximum dynamic displacements do not vary in a smooth simple way as in the case of static displacements:

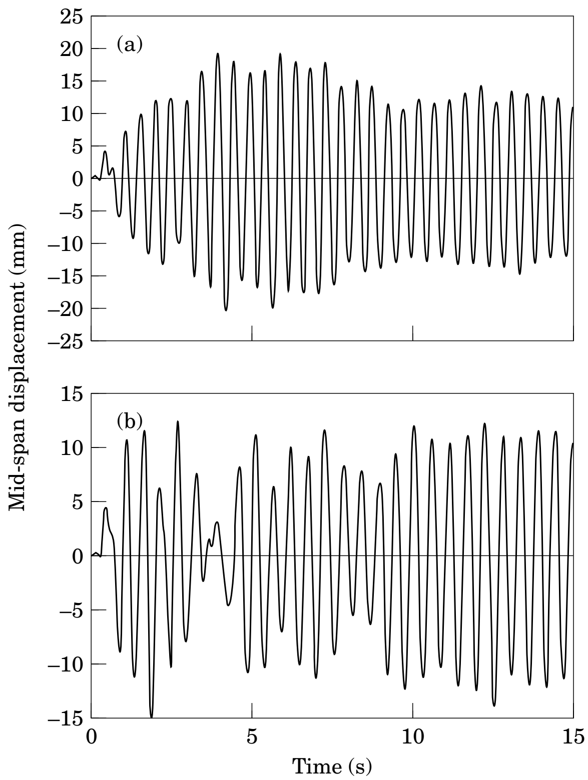

Figure 3. The time history of the mid-span displacement of the beam: (a) for $H = 10$ m and $L = 10$ m; (b) for $H = 10$ m and $L = 20$ m.

instead they are irregular. This behavior is due to the sensitivity of the response to small changes in the dynamic characteristics of the beams resting on elastic foundation.

Several general trends illustrated in Figures 5 and 6 are instructive, despite the somewhat irregular patterns in the curves. These trends seen from these figures are as follows.

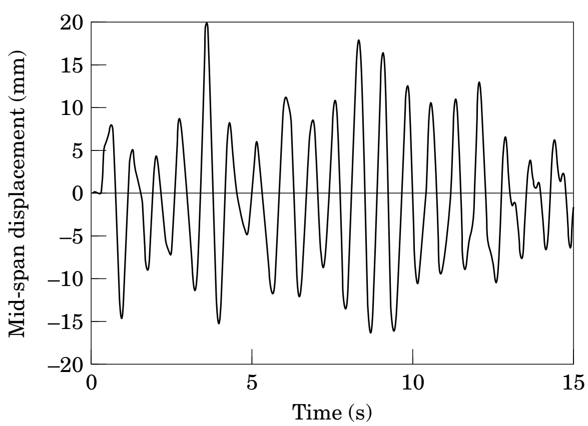

Figure 4. The time history of the mid-span displacement of the beam for $H = 15$ m and $L = 20$ m.

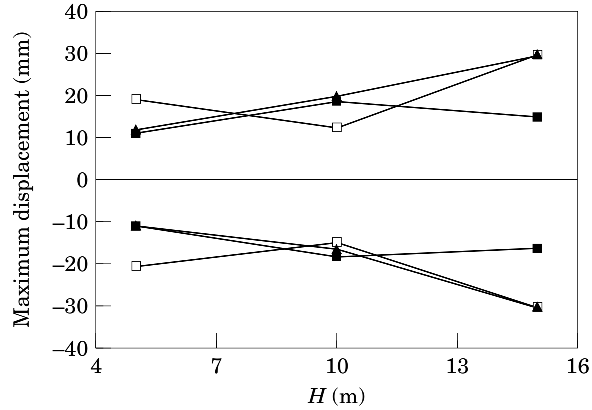

Figure 5. The maximum displacement of the beam for different beam lengths and subsoil depths. —■—, $L = 10$ m; —□—, $L = 20$ m; —▲—, $L = 30$ m.

(i) The maximum displacement (see Figure 5) generally increases with an increasing value of H for any values of the beam length, L. The decreases as exceptions to this trend would occur if the periods of the system happened to be changing along the left or increasing sides of the spectrum peak.

(ii) The maximum displacements at first decrease with increasing H/L ratios for any value of subsoil depth, H. The exceptions to this trend in Figure 6 are caused by dramatic changes in the dynamic amplification factor as peaks and valleys are traversed due to changes in the period of the beam on a foundation, as in the case of a plate [15].

(iii) The maximum displacement generally increases with an increasing value of H for any values of H/L, but tends to level off as the H/L ratio becomes large, especially for $H = 5$ m. This behavior is understandable in that a beam on an elastic foundation with a larger subsoil depth becomes more flexible and thus less resistant to the load in a static sense.

The maximum displacements obtained by using the Winkler model are presented in Figure 7 for different values of H and H/L ratios. As seen from this figure, in contrast to the results of the modified Vlasov model, the curves vary in a smooth way, and the

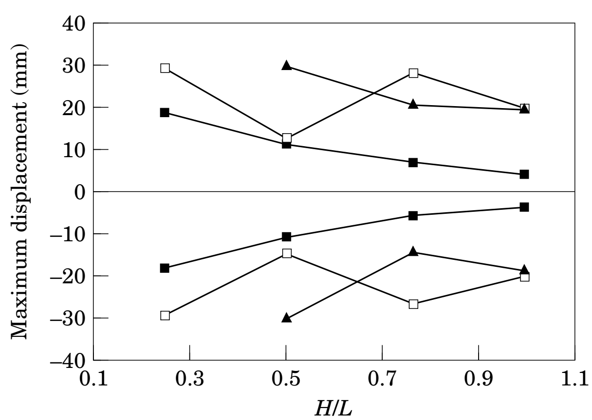

Figure 6. The maximum displacement of the beam for different H/L ratios and subsoil depths. —■—, $H = 5$ m; —□—, $H = 10$ m; —▲—, $H = 15$ m.

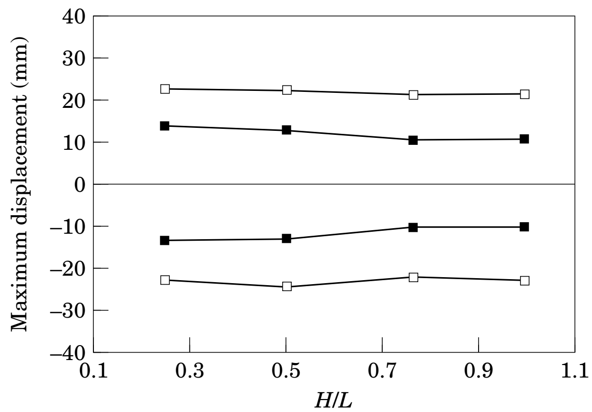

Figure 7. The maximum displacement of the beam for different H/L ratios and subsoil depths for the Winkler model. Key as Figure 6.

maximum displacements are almost constant for a fixed subsoil depth when the H/L ratio increases.

Although the results of the Winkler model are presented here, it is thought that it may not be appropriate to compare the results of both methods since the stiffness parameter, k, which has a constant value in the Winkler model at all time increments, takes on different values in the modified Vlasov model at each time increment. It should be noted that the value of the stiffness parameter determined by the modified Vlasov model for the first time increment is used to obtain the results of the Winkler model.

The deflected shapes of the 20 m beam for the time at which the maximum displacement occurs are given in Figure 8.

A beam resting on an elastic foundation with a larger subsoil depth and beam length becomes more flexible and thus less resistant to the load, so that this system will have a larger displacement in a static sense, but the 20 m beam for $H = 10$ m has a smaller maximum displacement than for $H = 5$ m (see Figure 8), because the maximum displacement of the dynamic system changes depending on the dynamic characteristics of the system.

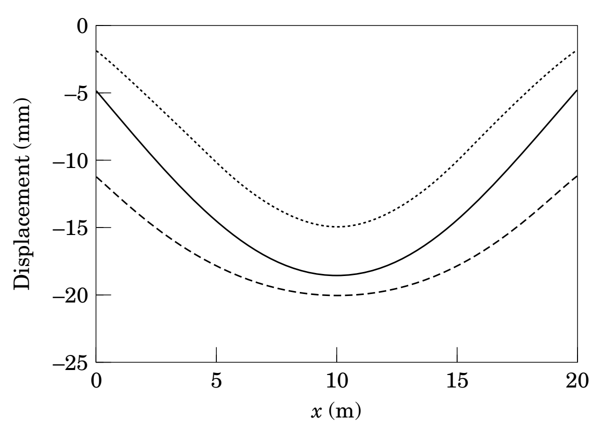

Figure 8. The deflected shape of the beam for different subsoil depths ($L = 20$ m). ——, $H = 5$ m; · · · ·, $H = 10$ m; - - - -, $H = 15$ m.

TABLE 1

The maximum and minimum γ values for different subsoil depths and beam lengths

H (m)	L (m)	γ_{min}	γ_{max}
5	5	0·78	3·79
	6·5	0·68	1·79
	10	0·58	2·14
	20	0·36	1·99
10	10	0·85	3·90
	13	0·72	4·26
	20	0·58	2·54
	40	0·36	1·97
15	15	0·83	5·26
	19·5	0·74	3·28
	30	0·58	3·73

In conclusion, it may be said that the maximum displacements of a beam resting on an elastic foundation sometimes tend to be sensitive to small changes in the dynamic characteristics such as the period as the beam length and/or subsoil depth are changed. Despite their irregularities, the curves show behavior trends that can be readily understood, and that can be utilized in choosing the beam length, depending on the subsoil depth if known.

The maximum and minimum values of the γ parameter obtained at each run for different subsoil depths and beam lengths are presented in Table 1. As seen from this table, the values of γ generally increase with increasing subsoil depth for any values of the beam length, and the minimum γ value generally decreases as the beam length increases for any values of the subsoil depth. This result agrees with the conclusion reached in reference [4]. As the value of γ increases, the ϕ function of equation (5) represents a rapidly dissipating displacement, which is typical for large values of H. When the value of γ approaches zero, the function ϕ yields a linear variation of displacements from top to bottom [4].

It should be noted that most of the values of γ obtained at each time increment are close to the minimum γ for small values of the subsoil depth. When the depth of the subsoil becomes larger, the values of γ obtained at each time increment fluctuate between γ_{min} and γ_{max}.

If a beam resting on an elastic foundation is subjected to the vertical component of an earthquake excitation, it is somewhat difficult to talk about the effects of the beam length and subsoil depth on the response, because both the frequency content of the earthquake and the exact natural frequency of the particular beam can make a difference to its response. The curves presented herein can help the designer to anticipate the effects of the beam length and the subsoil depth on the earthquake response of a beam resting on an elastic foundation.

5. CONCLUSIONS

The modified Vlasov model has been applied effectively to the earthquake analysis of beams resting on elastic foundations. Two soil parameters are calculated iteratively in terms of the other parameter, γ, which controls the decay of the stress distribution within the foundation.

In addition, the following conclusions can be drawn from the results obtained in this study.

The maximum displacement generally increases as the subsoil depth increases.

The maximum displacement generally decreases as the H/L ratio increases for any values of the subsoil depth.

In general, the subsoil depth has a stronger influence on the maximum displacement than the beam length.

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