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THE KBM DERIVATIVE EXPANSION METHOD IS EQUIVALENT TO THE MULTIPLE-TIME-SCALES METHOD

A. Hassan

School of Mechanical and Materials Engineering, Washington State University, Pullman, Washington 99164-2920, *U.S.A.*

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Several perturbation methods are commonly used to predict the free and forced response of weakly non-linear oscillators. The Krylov-Bogoliubov-Mitropolsky (KBM) and multiple-time-scales (MTS) methods use expansions of dependent variables, ordinary time derivatives, and some system parameters to convert the equations of motion into a set of first order differential equations. Each of these equations represents the slow time modulations of the amplitude and phase of the zeroth order solution(s).

In this paper, a simple correspondence between the expansions of ordinary time derivatives employed in these two methods is used to show that, except for notation, these two methods are identical in the sense that to any order of approximation these two methods will provide identical results when they use the same parameter expansions and identical additional constraints. The KBM method attempts to reduce unneeded algebraic calculations by tailoring the derivative expansions to the simplest applicable form. This, however, requires some experience or a trial and error approach to establish the intermediate expansion variables and the implicit and explicit dependence of perturbation solutions on the different time scales. This relation depends not only on the problem at hand but also on the parameter expansions used in the solution procedure. By using the most general expansion for the time derivatives, the MTS method establishes this dependence as a part of the analysis. In this method, the algebraic details are hidden by using a compact derivative operator type notation. However, these operators do not commute in general.

1. INTRODUCTION

An important aspect of various perturbation methods is their relationship with each other. The main objective of this paper is to develop the equivalence of the Krylov-Bogoliubov-Mitropolsky (KBM) and multiple-time-scales (MTS) methods. For this purpose, attention is focused on single-degree-of-freedom (SDOF), weakly non-linear oscillators which can be expressed as

$$\ddot{x} + x = \epsilon f(x, \dot{x}, \Omega t; \epsilon), \tag{1}$$

where $\epsilon > 0$, is the "small" gauge parameter, over-dots denote derivatives with respect to time t, and Ω is the frequency of external excitation. It is assumed that the function f is analytic in ϵ and has a sufficient number of derivatives with respect to x and $\dot{x}$. Several perturbation methods are available to predict the free and forced response of such oscillators. The KBM [1] and MTS [2] methods use expansions of the dependent variable(s), ordinary time derivative, and possibly some system parameters to approximate the periodic response(s) and the nearby transient solutions for such oscillators. By using

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many examples, Nayfeh [2–4] has shown that the first and second order KBM and MTS results are either identical or can be transformed to identical representation. The relation between higher orders, that is $O(\epsilon^n)$ with $n \geqslant 3$, KBM and MTS approximations is the main topic of this paper.

A very simple correspondence between these two methods is used to prove their equivalence. Indeed it is very surprising that this equivalence has not been reported earlier. Three cases, namely, non-resonance (section 2), resonance with small detuning (section 3), and resonance with large detuning (section 4), are considered for the harmonically excited system. This excitation can be direct or parametric. Approximations for the free response are discussed as a special case of non-resonant response. It is also shown that in general the partial derivative operators used in the MTS method do not commute.

A second objective of this work is to show that the choice of parameters used in the KBM derivative expansions depends on the method used to introduce a detuning parameter in the given equation(s) of the forced system; and that in this regard, incorrect assumptions can lead to inconsistencies which can stall the KBM procedure at higher orders. An extension of the KBM method, which eliminates this problem, is examined. The resulting modified KBM method is shown to be identical to the MTS method with reconstitution. Section 5 is devoted to this discussion. Conclusions are presented in section 6.

2. THE NON-RESONANT CASE

In weakly non-linear oscillators, the resonance phenomenon occurs when the frequency of harmonic excitation is near a rational fraction of the natural frequency of the associated linear system. That is, $\Omega \approx (q/p)\omega$, where p and q are mutually prime integers and ω is the natural frequency of the associated linear system. For convenience, this natural frequency is usually scaled to unity. The $O(\epsilon^n)$ KBM approximation for this case is briefly outlined in the next two paragraphs. These results are then related to the corresponding MTS approximation. The results for the free response are also discussed.

In the non-resonant case, the KBM method uses the following perturbation expansions:

$$x = x_0(a, \psi) + \epsilon x_1(a, \psi, \Omega t) + \epsilon^2 x_2(a, \psi, \Omega t) + \cdots, \tag{2}$$

$$\mathrm{d}a/\mathrm{d}t = \epsilon A_1(a) + \epsilon^2 A_2(a) + \cdots, \qquad \text{and} \qquad \mathrm{d}\psi/\mathrm{d}t = 1 + \epsilon B_1(a) + \epsilon^2 B_2(a) + \cdots, \tag{3, 4}$$

where $\psi = t + \theta$, x_i are periodic functions with a period 2π in both the arguments ψ and Ωt, and the symbols a and θ represent the amplitude and phase of the zeroth order solution x_0, which is usually taken as $x_0 = a\cos(t + \theta)$. The above expansions are used to obtain the following:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial x_0}{\partial \psi} + \epsilon\left[\left(\frac{\partial}{\partial \psi} + \frac{\partial}{\partial t}\right)x_1 + \left(A_1\frac{\partial}{\partial a} + B_1\frac{\partial}{\partial \psi}\right)x_0\right]$$

$$+ \epsilon^2\left[\left(\frac{\partial}{\partial \psi} + \frac{\partial}{\partial t}\right)x_2 + \left(A_1\frac{\partial}{\partial a} + B_1\frac{\partial}{\partial \psi}\right)x_1 + \left(A_2\frac{\partial}{\partial a} + B_2\frac{\partial}{\partial \psi}\right)x_0\right] + \cdots, \tag{5}$$

and

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\partial^2 x_0}{\partial\psi^2} + \epsilon\left\{\left(\frac{\partial^2}{\partial\psi^2} + 2\,\frac{\partial^2}{\partial\psi\,\partial t} + \frac{\partial^2}{\partial t^2}\right)x_1 + 2\left[A_1\left(\frac{\partial^2}{\partial a\,\partial\psi} + \frac{\partial^2}{\partial a\,\partial t}\right) + B_1\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\psi\,\partial t}\right)\right]x_0\right\}$$

$$+ \epsilon^2\left\{\left(\frac{\partial^2}{\partial\psi^2} + 2\,\frac{\partial^2}{\partial\psi\,\partial t} + \frac{\partial^2}{\partial t^2}\right)x_2 + 2\left[A_1\left(\frac{\partial^2}{\partial a\,\partial\psi} + \frac{\partial^2}{\partial a\,\partial t}\right) + B_1\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\psi\,\partial t}\right)\right]x_1\right.$$

$$+ 2\left[A_2\left(\frac{\partial^2}{\partial a\,\partial\psi} + \frac{\partial^2}{\partial a\,\partial t}\right) + B_2\left(\frac{\partial^2}{\partial\psi^2}\,\frac{\partial^2}{\partial\psi\,\partial t}\right)\right]x_0$$

$$\left.+ \left(A_1^2\,\frac{\partial^2}{\partial a^2} + B_1^2\,\frac{\partial^2}{\partial\psi^2} + 2A_1B_1\,\frac{\partial^2}{\partial a\,\partial\psi} + A_1\,\frac{\mathrm{d}A_1}{\mathrm{d}a}\,\frac{\partial}{\partial a} + A_1\,\frac{\mathrm{d}B_1}{\mathrm{d}a}\,\frac{\partial}{\partial\psi}\right)x_0\right\} + \cdots. \tag{6}$$

The function f is expanded in a Taylor series about $\epsilon = 0$,

$$f(x, \dot{x}, \Omega t; \epsilon) = f_1(x, \dot{x}, \Omega t) + \epsilon f_2(x, \dot{x}, \Omega t) + \cdots,$$

and equations (2) and (5) are used to expand $f_i(x, \dot{x}, \Omega t)$, $i = 1, 2, \ldots$, about $(x_0, \partial x_0/\partial\psi)$. These expansions are substituted in the governing equation (1) and the coefficients of ϵ^i, $i = 0, 1, 2, \ldots$, are equated to zero to obtain a hierarchy of linear oscillators, which are solved in sequence. The relationship between this KBM approach and the MTS method with reconstitution is developed below.

It has been suggested that the KBM "method can be viewed as a multiple scales method with a and θ being the scales" [3, p. 173]. Here, however, these are taken as secondary variables which depend on multiple time scales. That is, $a = a(t_0, t_1, t_2, \ldots)$, $\theta = \theta(t_0, t_1, t_2, \ldots)$, where $t_i = \epsilon^i t$ for $i = 0, 1, 2, \ldots$. Next, following the MTS method, define

$$D_i(\,) = \partial(\,)/\partial t_i, \qquad D_i a = A_i, \qquad D_i\psi = B_i, \qquad \text{for } i = 1, 2, \ldots, \tag{7}$$

where $A_i = A_i(a)$, $B_i = B_i(a)$ for $i \geqslant 1$. With this notation and the assumptions that $D_0 a = 0$ and $D_0\psi = 1$ (which are also used in the MTS method), one can write

$$D_0 u = (\partial/\partial\psi + \partial/\partial t)u, \qquad \text{and} \qquad D_i u = (A_i\,\partial/\partial a + B_i\,\partial/\partial\psi)u, \qquad i \geqslant 1,$$

where $u(a, \psi, t)$ is an arbitrary, differentiable function. It is also clear that for $i \geqslant 1$,

$$D_0 D_i(\,) = D_i D_0(\,) = \left[A_i\left(\frac{\partial^2}{\partial a\,\partial\psi} + \frac{\partial^2}{\partial a\,\partial t}\right) + B_i\left(\frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\psi\,\partial t}\right)\right](\,),$$

and for $i \neq j$, $D_i D_j \neq D_j D_i$, unless i or j is zero. That is, the operators D_i, D_j, $i \neq j$, do not commute unless i or j is zero. Using this notation, the expansions (5) and (6) of the KBM method can be, respectively, written as

$$\mathrm{d}x/\mathrm{d}t = D_0x_0 + \epsilon(D_0x_1 + D_1x_0) + \epsilon^2(D_0x_2 + D_1x_1 + D_2x_0) + \cdots,$$

$$\mathrm{d}^2x/\mathrm{d}t^2 = D_0^2x_0 + \epsilon[D_0^2x_1 + (D_0D_1 + D_1D_0)x_0]$$

$$+ \epsilon^2[D_0^2x_2 + (D_0D_1 + D_1D_0)x_1 + (D_0D_2 + D_1^2 + D_2D_0)x_0] + \cdots,$$

which are precisely the expansions used in the MTS method. It is also clear that the ordinary time derivative expands according to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\,) = \left(\sum_{i \geqslant 0} \epsilon^i D_i\right)(\,),$$

and the derivative expansions (3) and (4), respectively, become

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\sum_{i \geqslant 0} \epsilon^i D_i\right) a, \qquad \text{and} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \left(\sum_{i \geqslant 0} \epsilon^i D_i\right) \theta,$$

which are the two components of the reconstitution equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \left(\sum_{i \geqslant 0} \epsilon^i D_i\right) A, \tag{8}$$

used in the MTS method with reconstitution, where $A = a\,\mathrm{e}^{\mathrm{i}\theta}/2$, $i = \sqrt{-1}$. This reconstitution preserves the order ($O(\epsilon^n)$, $n \geqslant 0$) of all terms. It is, therefore, clear that all equations of the KBM method, including the hierarchical set of linear oscillators, are directly translated into the corresponding equations of the MTS method with reconstitution and *vice versa*. Hence, the $O(\epsilon^n)$, $n \geqslant 1$, KBM and MTS reconstitution approximations for the non-resonance situation of equation (1) must be identical. The approximate solutions for the unforced response can be established by setting Ω and the coefficient of the harmonic excitation to zero in the above development. Therefore, the above equivalence of the two methods is also valid for the case of free response of equation (1). The KBM expansions and the corresponding MTS results can also be described in terms of t, a and θ by substituting $\psi = t + \theta$ in equations (2) and (4). Then the operator D_0 can be written in its conventional form $D_0(\,) = \partial(\,)/\partial t_0$. Equivalence of these two methods for the resonance situation is described next.

3. THE RESONANCE CONDITION WITH SMALL DETUNING

As mentioned earlier, a resonance situation arises in the forced response of equation (1) when $\Omega \approx q/p$, where p and q are mutually prime integers. In this case, the excitation frequency is related to the natural frequency, here unity, of the associated linear system. When the difference between these two values, called detuning, is small, it is assumed to be of $O(\epsilon)$. This relation can be expressed in several different ways [5–7], such as $p\Omega/q = 1 + \epsilon\sigma$ or $1 = (p\Omega/q)^2 + \epsilon\sigma$. The particular equation used in the analysis actually defines the detuning parameter σ. Different values of p and q are used to classify the response into sub-, super-, or ultra-superharmonic resonance or various parametric resonances. For example, the situation of direct excitation with $p = q$ is called primary resonance. As shown in section 5, use of expansion $\Omega = 1 + \epsilon\sigma$ for this case can lead to a situation where the resulting set of linear systems and periodicity conditions may not meet the assumptions used in the KBM expansions. Thus, the KBM method can stall at a higher order where this contradiction first appears. The expansion $1 = (p\Omega/q)^2 + \epsilon\sigma$ is used in the following discussion. However, the results are valid for any such expansion which does not lead to the above mentioned problem.

In this case, equation (1) can be written as

$$\ddot{x} + ((p/q)\Omega)^2 x = \epsilon F(x, \dot{x}, \Omega t; \epsilon),$$

where $F = f(x, \dot{x}, \Omega t; \epsilon) - \sigma x$. Now the KBM method uses expansion (2) for the dependent variable, with $\psi = (p\Omega/q)t + \theta$, and the functions a and θ are assumed to satisfy the following conditions:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sum_{i \geqslant 1} \epsilon^i A_i(a, \theta), \qquad \text{and} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \sum_{i \geqslant 1} \epsilon^i B_i(a, \theta). \tag{9, 10}$$

These conditions are more general versions of equations (3) and (4), respectively. It is easy to check that by using definition (8) with $A_i = A_i(a, \theta)$, $B_i = B_i(a, \theta)$ for $i \geqslant 1$ and the assumptions $D_0 a = 0$ and $D_0\theta = 0$ (which are also used in the MTS method), all equations of the KBM method for this case can be directly translated into the corresponding equations of the MTS method with reconstitution and *vice versa*. In particular, equations (9) and (10) become the two components of the reconstitution equation (8) of the MTS method. Therefore, for small detuning, the $O(\epsilon^n)$ KBM and MTS reconstitution approximations for the sub-, super-, and ultra-superharmonic responses or other parametric resonances in equation (1) will be identical when the two approaches use the same method of defining a detuning parameter.

Again, the operators D_i, D_j, $i \neq j$, do not commute unless i (or j) is zero. The situation for large detuning, that is when $O(1 - p\Omega/q) = O(1)$, is described in the next section.

4. THE RESONANCE CONDITION WITH LARGE DETUNING

When the detuning, that is the absolute value of $(1 - p\Omega/q)$, is not small, the KBM method uses the expansion (2) for the dependent variable with the conditions (9) and

$$\mathrm{d}\theta/\mathrm{d}t = (1 - (p/q)\Omega) + \epsilon B_1(a, \theta) + \epsilon^2 B_2(a, \theta) + \cdots.$$

Again, all equations of the KBM method for this case can be directly translated into the corresponding equations of the MTS reconstitution method and *vice versa* by using the definition (8) with $A_i = A_i(a, \theta)$, $B_i = B_i(a, \theta)$ for $i \geqslant 1$, and the assumptions that $D_0 a = 0$ and $D_0\theta = 1 - (p\Omega/q)$. So that now

$$D_0 u = [(1 - (p/q)\Omega)\, \partial/\partial\theta + \partial/\partial t_0]u,$$

where $u = (a, \theta, t_j)$ is an arbitrary, differentiable function. In this case, the operators D_i, D_j, $i \neq j$, do not commute for all $i, j \geqslant 0$. In particular, D_0 does not commute with D_i for all $i \geqslant 1$, which is different from the situation $D_0 D_i = D_i D_0$ for $i \geqslant 1$ observed in Sections 2 and 3.

It is clear that for a SDOF, weakly non-linear oscillator, the $O(\epsilon^n)$ KBM and MTS reconstitution approximations will be identical when the same definition of a detuning parameter is used in these methods. One drawback of the above KBM method is described next.

5. PRIMARY RESONANCE IN THE DUFFING OSCILLATOR

As mentioned earlier, success of the classical KBM derivative expansions (9) and (10) depends on the method used to introduce a detuning parameter in the equation(s) of motion. These expansions use the amplitude and phase of the zeroth order solution as secondary variables which are assumed to be implicit functions of the slow time scales t_i,

$i \geqslant 1$. That is the functions A_i, B_i, and the solutions x_i are assumed to implicitly depend on these time scales through a and θ. Also, $\mathrm{d}a/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$ are assumed to be zero in the steady state situation. However, these assumptions are somewhat restrictive and some definitions of the detuning parameter can lead to situations where A_i, B_i, and x_i, $i \geqslant 1$, implicitly (through a and θ) and explicitly depend on the slow time scales t_i, $i \geqslant 1$. In addition, the steady state is not characterized by $\mathrm{d}\theta/\mathrm{d}t = 0$. Instead, the time derivative of some other phase variable, which depends on θ and the slow time scales, must be equated to zero to obtain the periodic steady state results. For example, this situation arises when the primary resonance in the Duffing oscillator,

$$\ddot{x} + \epsilon\delta\dot{x} + x + \epsilon x^3 = \epsilon p \cos \Omega t, \tag{11}$$

is analyzed with the detuning, σ, defined by $\Omega = 1 + \epsilon\sigma$. In this case, the following zeroth and first order systems can be established by using equations (9) and (10) in equation (11):

$$\epsilon^0: \quad \partial^2 x_0/\partial t^2 + x_0 = 0, \tag{12}$$

$$\epsilon^1: \quad \frac{\partial^2 x_1}{\partial t^2} + 2A_1 \frac{\partial^2 x_0}{\partial a\, \partial t} + 2B_1 \frac{\partial^2 x_0}{\partial \theta\, \partial t} + \delta \frac{\partial x_0}{\partial t} + x_1 + x_0^3 = \frac{p}{2} \mathrm{e}^{\mathrm{i}\Omega t} + cc, \tag{13}$$

where cc stands for the complex conjugate of the preceding terms. The anticipated solution $x_0 = a \cos(t + \theta)$ of equation (12) is substituted in equation (13) and the secular terms are removed by using $\Omega = 1 + \epsilon\sigma$ to provide the following $O(\epsilon)$ periodicity conditions:

$$A_1 = (-\delta a/2) + (p/2) \sin \gamma, \qquad \text{and} \qquad B_1 = (3a^2/8) - (p/2) \cos \gamma,$$

where $\gamma = \epsilon\sigma t - \theta$. Thus A_1 and B_1 explicitly depend on the slow time scale $t_1 = \epsilon t$, which violates the assumptions $A_i = A_i(a, \theta)$, $B_i = B_i(a, \theta)$ used in expansions (9) and (10). Although the above $O(\epsilon)$ approximation is correct, this KBM process cannot be carried further without some modification. One solution is to define a detuning parameter which does not lead to this problem. However, this situation can also be easily resolved by allowing the functions A_i, B_i, and x_i to explicitly depend on the slow time scales t_i, $i \geqslant 1$. The KBM expansions used in reference [4] allow A_i, B_i to depend on t_1. The explicit dependence of x_i on t_i is not mentioned in reference [4]. This, however, does not affect the first and second order results presented in reference [4] because in the analysis presented there, the dependence of x_i on t_1 arises first in the third order approximation. Thus for the situations in which a detuning parameter is defined at $O(\epsilon)$, the KBM expansions can be generalized to

$$x = x_0(a, \theta) + \epsilon x_1(a, \theta, t_i, i \geqslant 1) + \cdots, \tag{14}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sum_{i \geqslant 1} \epsilon^i A_i(a, \theta, t_j, j \geqslant 1), \qquad \text{and} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \sum_{i \geqslant 1} \epsilon^i B_i(a, \theta, t_j, j \geqslant 1). \tag{15, 16}$$

The results obtained by these expansions can be easily translated into the MTS reconstitution results by defining

$$D_i a = A_i, \qquad D_i \theta = B_i, \qquad \text{and}$$

$$D_i u = (A_i\, \partial/\partial a + B_i\, \partial/\partial\theta + \partial/\partial t_i)u, \qquad \text{for} \qquad \mathrm{i} \geqslant 1, \tag{17}$$

together with the assumptions that $D_0 a = D_0 \gamma = 0$, where $u = u(a, \theta, t_j, j \geqslant 0)$ is an arbitrary, differentiable function and γ is defined in terms of θ and t_i, $i \geqslant 1$. The above

KBM method can also be applied to the systems with slowly varying parameters, multi-degree-of-freedom systems, systems with multi-frequency direct and/or parametric excitations, etc. It is clear that the KBM results for these systems will be identical to the MTS reconstitution results provided that the same method(s) of introducing detuning parameter(s) are used in both methods.

6. DISCUSSION AND CONCLUSIONS

The general KBM derivative expansion has been shown to be equivalent to the MTS method with reconstitution. These results for the SDOF oscillator can be easily extended to the coupled systems, multiple direct and/or parametric excitations, etc. The MTS method provides compact notation through the operators D_i. Consequently, the most general expansions (14)–(17) can be applied to the problem under investigation, and when needed this analysis reduces to the simpler cases, such as equations (2)–(4). In general, however, the operators D_i, D_j, $i \neq j$ do not commute. In addition, the definition of D_0 may depend on the problem being analyzed even though most problems are treated by assuming $D_0 a = D_0 \theta = 0$. On the other hand, the KBM method uses expanded forms of these operators, which leads to lengthy equations/representations. Consequently, the KBM approach attempts to reduce the algebra by tailoring the derivative expansions to the simplest form that can be used for the problem under consideration. Specifically, the dependence of A_i, B_i, and x_i on a, θ, and t_j is fixed in advance to the simplest form that may work. As a result, the expansion of operators D_i reduces to their simplest form with the consequent reduction in algebraic details that must be carried through. This, however, requires some *a priori* knowledge about the problem being analyzed. For the situation of free response, for example, the KBM method assumes that A_i, B_i depend only on the amplitude a. In contrast, the MTS method establishes this dependence as a part of the analysis. Although a and θ do not explicitly depend on the slow time-scales t_j, $j \geqslant 1$, this explicit dependence is allowed for their derivatives, e.g., equations (15) and (16), and the higher order solutions x_i. As a result, the steady state solution may require that $\mathrm{d}a/\mathrm{d}t = 0$ and $\mathrm{d}\gamma/\mathrm{d}t = 0$, where γ explicitly depends on θ and the slow time scales. This situation is handled as a part of the MTS analysis. However, due to the early use of the expanded forms of D_i, $i \geqslant 0$, use of the KBM method for such situations requires some experience or trial and error applications of this method so that a secondary variable, like γ above, is suitably defined for the problem being considered.

Both methods allow the use of a homogeneous solution at each order of approximation. However, these functions must be determined by imposing additional constraints. This is specially true for the free vibration problem where a perturbation approximation for the response frequency is also determined as a part of the analysis. It is clear that the two methods will provide identical results when they use the same parameter expansions and identical additional constraints.

In summary, except for notation, the KBM and MTS derivative expansion methods are identical. By hiding details, the short notation of the MTS method provides some advantages in presentation and the actual algebraic manipulations.

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