*Journal of Sound and Vibration* (1997) **200**(4), 491–504

# AN EFFICIENT METHOD FOR THE UNBALANCE RESPONSE ANALYSIS OF ROTOR–BEARING SYSTEMS

S.-W. HONG

*Department of Mechanical and Precision Engineering, Kumoh National University of Technology*, 188 *Sinpyung, Kumi, Kyungbuk, Korea*

AND

J.-H. PARK

*System Engineering Department, Samsung Electronics Co. Ltd.*, 259 *Gongdan, Kumi, Kyungbuk, Korea*

(*Received* 22 *May* 1996, *and in final form* 7 *August* 1996)

Unbalance response analysis is essential in the dynamic analysis of rotor–bearing systems. However, there still remains a problem in the aspect of computational efficiency for unbalance response analysis of large rotor–bearing systems. Gyroscopic terms and local bearing parameters in rotor–bearing systems often make matters worse in unbalance response computation, due to the complicated dynamic properties such as rotational speed dependency and/or anisotropy. In the present paper an efficient method is proposed for unbalance responses of multi-span rotor–bearing systems. An improved substructure synthesis scheme is introduced which makes it possible to compute unbalance responses of the system by coupling unbalance responses of substructures and is easy to manage. The proposed method includes also a scheme to deal with gyroscopic terms and local, coupling or bearing parameters easily. The proposed method causes no errors, even though the computation time is drastically reduced. The proposed method is demonstrated and validated through several test examples.



## 1. INTRODUCTION

The finite element method (FEM) has played an important role in the design or analysis of rotor–bearing systems [1–7]. In particular, unbalance response analysis by FEM is essential in the dynamic analysis of rotor–bearing systems because of its usefulness in vibration diagnosis as well as balancing or identification of parameters involved in rotor bearing systems [7–10]. However, there still remain some difficulties in the computational aspect of unbalance response analysis due to the fact that the classical modal analysis scheme is inconvenient for unbalance response analysis. The presence of local joint elements (e.g., bearings, couplings and seals in rotor–bearing systems) often gives rise to a difficulty in modal anlaysis of rotor–bearing systems, mainly due to the complicated dynamic properties such as rotational speed dependency and/or anisotropy. In addition, gyroscopic terms in rotor–bearing systems often make matters worse in unbalance response computation because they appear as a rotational speed dependent, skew-symmetric matrix in FEM. As a consequence, computation of unbalance responses for rotor–bearing systems relies solely on the direct computation method, which is likely to be inefficient because it

0022–460X/97/090491 + 14 $25.00/0/sv960773 

necessitates repetitive inversion of the full complex dynamic stiffness matrix at every rotational speed of interest.

After Guyan [11] proposed a matrix reduction scheme for dynamic analysis of structural dynamic systems, many kinds of matrix reduction schemes were suggested [12–15]. Among these, the substructure synthesis method has attracted the attention of many investigators. The fundamental idea of the substructure synthesis method is to divide the structure into a few substructures, obtain the dynamic characteristics of the substructures, and then combine the results to obtain the dynamic characteristics of the overall system. For the combining process, various order reduction schemes were suggested. The component mode synthesis method is the most prevelant among various substructure synthesis techniques. In the case of rotating machinery, the component mode synthesis was introduced early and proved useful [14, 15], but the procedure does not seem to be very easy to implement, in addition to which the amount of computational error is uncertain. There still remains a problem associated with quantifying computational errors resulting from substructure synthesis techniques.

In the present paper, an efficient substructure synthesis method for unbalance response analysis is proposed, so as to deal with large multi-span rotor–bearing systems. The proposed method consists of two steps. First, a modal analysis scheme is introduced to obtain unbalance response of a substructural rotor that is a part of the overall rotor–bearing system but does not include any joint parameters. The use of complex co-ordinates in the formulation makes it easy to handle the skew-symmetric property of gyroscopic effects by decomposing the unbalance response formula of a substructural rotor into two modal response formulae that are based on elementary self-adjoint eigenvalue problems. Second, a substructure synthesis scheme is suggested to obtain unbalance responses of the overall rotor–bearing system by incorporating the unbalance responses of the substructures. Here an exact matrix condensation procedure is devised to reduce the order of the system matrix down to the number of co-ordinates connected with local joint elements, which is presumably much smaller than the total number of co-ordinates. Unbalance responses can then be readily obtained by computation of the small, condensed dynamic stiffness matrix with combined uses of the unbalance responses of substructural rotors, already computed in the first step. The proposed method causes no errors, even though the computational time is drastically reduced.

A numerical study is also conducted to validate the efficiency and applicability of the proposed method. In the first numerical example, the proposed method is compared with the direct inversion method. In the second example a realistic re-analysis problem is considered; this is often met in the design stage of a rotor–bearing system. Finally, the proposed method is applied to a rotor–bearing system for a two-spool aircraft engine. The numerical study proves that the proposed method is very efficient and useful for the unbalance response analysis of rotor–bearing systems.

## 2. UNBALANCE RESPONSE OF ROTOR–BEARING SYSTEM

### 2.1. UNBALANCE RESPONSE OF A SUBSTRUCTURAL ROTOR

In Figure 1 is illustrated a multi-span rotor–bearing system that has $n$ substructural rotors connecting each other with couplings and being supported in bearings. The equation of motion for the $i$th rotor except couplings and bearings can be written, in a complex co-ordinate form (see the Appendix) as

$$M_i\ddot{p}_i - \mathrm{j}\Omega G_i\dot{p}_i + K_ip_i = F_i, \tag{1}$$

where $p_i$ and $F_i$ are the complex co-ordinate vector and the corresponding complex force vector of the $i$th rotor, respectively. $M_i$, $G_i$ and $K_i$ are the mass, gyroscopic and stiffness matrices of the $i$th rotor respectively. Equation (A10) in the Appendix leads the unbalance response functions of the $i$th rotor to

$$H_{i,ff} = \{D_{i,ff}\}^{-1} = \{-\Omega^2(M_i - G_i) + K_i\}^{-1} \tag{2a}$$

$$H_{i,bb} = \{D_{i,bb}\}^{-1} = \{-\Omega^2(M_i + G_i) + K_i\}^{-1} \tag{2b}$$

$$H_{i,bf} = H_{i,fb} = 0 \tag{2c}$$

It will prove to be convenient to define the following co-ordinate transform relationship:

$$\underset{m_i \times 1}{p_i^c} = \underset{m_i \times N_i}{T_i} \quad \underset{N_i \times 1}{p_i} \tag{3}$$

where $p_i^c$ is a co-ordinate vector associated with local joint (connecting or supporting) elements, such as bearings, couplings and so forth. $T_i$ is a transform matrix provided for extracting $p_i^c$ from the co-ordinate vector $p_i$ of the $i$th rotor.

### 2.2. UNBALANCE RESPONSE OF THE OVERALL ROTOR–BEARING SYSTEM

Assume that the superscripts $o$, $c$ and $s$ denote the overall rotor–bearing system, the joint system, and a subsidiary system that is identical to the overall rotor but does not include the joint elements. Then the unbalance response function of the overall rotor–bearing system can be defined by the inverse of dynamic stiffness matrix as given in equation (A9); i.e.,

$$\begin{bmatrix} H_{ff}^o & H_{fb}^o \\ H_{bf}^o & H_{bb}^o \end{bmatrix} = \begin{bmatrix} D_{ff}^o & D_{fb}^o \\ D_{bf}^o & D_{bb}^o \end{bmatrix}^{-1}, \tag{4}$$

where the overall dynamic stiffness matrix can be decomposed as

$$\begin{bmatrix} D_{ff}^o & D_{fb}^o \\ D_{bf}^o & D_{bb}^o \end{bmatrix} = \begin{bmatrix} D_{ff}^s & 0 \\ 0 & D_{bb}^s \end{bmatrix} + \begin{bmatrix} D_{ff}^c & D_{fb}^c \\ D_{bf}^c & D_{bb}^c \end{bmatrix}.$$

Here the first and second terms on the right side represent the dynamic stiffness matrix of the subsidiary system and the joint system matrix that will be derived in the next section. Equation (4) can be rewritten as

$$\begin{bmatrix} H_{ff}^o & H_{fb}^o \\ H_{bf}^o & H_{bb}^o \end{bmatrix} = \left\{ I + \begin{bmatrix} H_{ff}^s & 0 \\ 0 & H_{bb}^s \end{bmatrix} \begin{bmatrix} D_{ff}^c & D_{fb}^c \\ D_{bf}^c & D_{bb}^c \end{bmatrix} \right\}^{-1} \begin{bmatrix} H_{ff}^s & 0 \\ 0 & H_{bb}^s \end{bmatrix}. \tag{5}$$

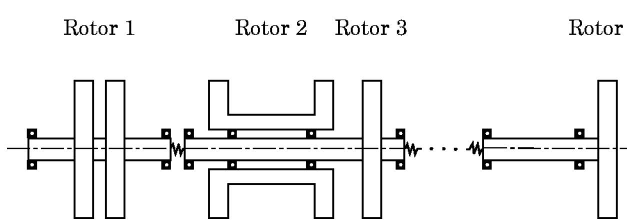


Figure 1. A conceptual system with multi-span rotors.

The dynamic stiffness and unbalance response matrices of the subsidiary system can be represented as

$$D^s_{kk} = \mathrm{diag}\,\{D_{1,kk}, D_{2,kk}, \ldots, D_{n,kk}\}$$
$$H^s_{kk} = \{D^s_{kk}\}^{-1}, \qquad k = f, b. \tag{6}$$

In the case of the direct computation method, equation (4) is often used, but equation (5) is preferable for improving the computational efficiency. In general, there is a need of matrix inversion for a $2N \times 2N$ complex matrix to gain unbalance response.

### 2.3. REPRESENTATION OF JOINT SYSTEM MATRICES

The joint dynamic stiffness matrices, $D^c_{kl}$, $k, l = f, b$, introduced in equations (4) and (5), are apt to be sparse, so that the matrix can be condensed as

$$d^c_{kl} = T_m D^c_{kl} T^T_m, \qquad k, l = f, b. \tag{7}$$

Here $d^c_{kl}$ is a condensed matrix for the joint matrix, the size of which is $m \times m$, $m$ being the degree of freedom of the co-ordinates connected with or supported in the joint elements

$$\left(\sum_{i=1}^{n} m_i\right).$$

The co-ordinate transform matrix is defined using

$$\underset{m\times 1}{p^c} = \underset{m\times N}{T_m} \; \underset{N\times 1}{p}, \tag{8}$$

where $p^c$ is a co-ordinate vector that includes only the co-ordinates related to joint elements. $T_m$ is composed of the transform matrices given in equation (3).

$$T_m = \mathrm{diag}\,\{T_1 \quad T_2 \quad \cdots \quad T_n\}. \tag{9}$$

Consequently, the condensed dynamic stiffness matrices in equation (7) can be easily obtained from $D^c_{kl}$, $k, l = f, b$. Otherwise, the condensed dynamic stiffness matrices can be constructed directly as follows:

$$d^c_{kl} = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} d_{kl,\alpha\beta}\gamma_{\alpha\beta}\gamma^{\mathrm{T}}_{\alpha\beta}, \tag{10}$$

where $d_{kl,\alpha\beta}$ is the dynamic stiffness of a joint element between co-ordinates $\alpha$ and $\beta$ and $\gamma_{\alpha\beta}$ is an $m$-dimensional vector, defined as

$$\gamma_{\alpha\alpha} = \{0 \quad 0 \quad \cdots \quad 0 \quad \underset{\alpha\mathrm{th}}{1} \quad 0 \quad \cdots \quad 0\}^{\mathrm{T}} \qquad \text{for } \alpha = 1, 2, \ldots, m,$$

$$\gamma_{\alpha\beta} = \{0 \quad 0 \quad \cdots \quad 0 \quad \underset{\alpha\mathrm{th}}{1} \quad 0 \quad \cdots \quad \underset{\beta\mathrm{th}}{0} - 1 \quad 0 \quad \cdots \quad 0\}^{\mathrm{T}} \qquad \text{for } \alpha, \beta = 1, 2, \ldots, m, \ \alpha \neq \beta.$$

## 3. IMPROVEMENT OF UNBALANCE RESPONSE COMPUTATION ALGORITHM

### 3.1. MODAL EXPRESSION OF A SUBSTRUCTURAL ROTOR

Since $M_i$, $G_i$ and $K_i$ in equation (2) are all symmetric, the unbalance response formulae of a substructural rotor can be expressed in a modal expansion form by using a classical modal analysis scheme. Application of the classical modal analysis scheme to $H_{i,ff}$ in equation (2a) can result in

$$H_{i,ff} = \{D_{i,ff}\}^{-1} = \sum_{k=1}^{N} \frac{u_{i,fk} u_{i,fk}^{\mathrm{T}}}{\omega_{i,fk}^2 - \Omega^2}, \tag{11}$$

where the eigenvalues and eigenvectors satisfy the following eigenvalue problem:

$$\{-\omega_{i,fk}^2 (M_i - G_i) + K_i\} u_{i,fk} = 0, \qquad k = 1, 2, \ldots, N. \tag{12}$$

The eigenvectors can be normalized so as to satisfy

$$u_{i,fk}^{\mathrm{T}} (M_i - G_i) u_{i,fl} = \delta_{kl}, \quad u_{i,fk}^{\mathrm{T}} K_i u_{i,fl} = \omega_{i,fk}^2 \delta_{kl}, \qquad k, l = 1, 2, \ldots, N, \tag{13}$$

where $\delta_{kl}$ denotes the Kronecker delta. Similarly to equation (11), $H_{i,bb}$ can be written as

$$H_{i,bb} = \{D_{i,bb}\}^{-1} = \sum_{k=1}^{N} \frac{u_{i,bk} u_{i,bk}^{\mathrm{T}}}{\omega_{i,bk}^2 - \Omega^2}. \tag{14}$$

The eigenvalues and eigenvectors involved in equation (13) satisfy the following eigenvalue problem.

$$\{-\omega_{i,bk}^2 (M_i + G_i) + K_i\} u_{i,bk} = 0, \qquad k = 1, 2, \ldots, N. \tag{15}$$

Here the eigenvectors can be normalized so as to satisfy

$$u_{i,bk}^{\mathrm{T}} (M_i + G_i) u_{i,bl} = \delta_{kl}, \quad u_{i,bk}^{\mathrm{T}} K_i u_{i,bl} = \omega_{i,bk}^2 \delta_{kl}, \qquad k, l = 1, 2, \ldots, N. \tag{16}$$

As a result, the unbalance response functions of a substructural rotor can be deduced from equations (11) and (14), which are based upon elementary, self-adjoint eigenvalue problems. It is noteworthy that the use of complex co-ordinates makes it possible to replace the non-self-adjoint system due to the skew-symmetric nature of the gyroscopic matrix by two equivalent self-adjoint systems.

### 3.2. SUBSTRUCTURAL SYNTHESIS FOR UNBALANCE RESPONSE FORMULAE

Equation (5) gives

$$\begin{bmatrix} H_{ff}^s & 0 \\ 0 & H_{bb}^s \end{bmatrix} - \begin{bmatrix} H_{ff}^o & H_{fb}^o \\ H_{bf}^o & H_{bb}^o \end{bmatrix} = \begin{bmatrix} H_{ff}^s & 0 \\ 0 & H_{bb}^s \end{bmatrix} \begin{bmatrix} D_{ff}^c & D_{fb}^c \\ D_{bf}^c & D_{bb}^c \end{bmatrix} \begin{bmatrix} H_{ff}^o & H_{fb}^o \\ H_{bf}^o & H_{bb}^o \end{bmatrix}. \tag{17}$$

Since the joint matrix is sparse, substitution of equation (7) into equation (17) may yield

$$\begin{bmatrix} H_{ff}^s & 0 \\ 0 & H_{bb}^s \end{bmatrix} - \begin{bmatrix} H_{ff}^o & H_{fb}^o \\ H_{bf}^o & H_{bb}^o \end{bmatrix} = \begin{bmatrix} H_{ffmN}^{s\mathrm{T}} & 0 \\ 0 & H_{bbmN}^{s\mathrm{T}} \end{bmatrix} \begin{bmatrix} d_{ff}^c & d_{fb}^c \\ d_{bf}^c & d_{bb}^c \end{bmatrix} \begin{bmatrix} H_{ffmN}^o & H_{fbmN}^o \\ H_{bfmN}^o & H_{bbmN}^o \end{bmatrix}. \tag{18}$$

Upon premultiplying equation (18) by diag $\{T_m, T_m\}$,

$$\begin{bmatrix} H_{ffmN}^o & H_{fbmN}^o \\ H_{bfmN}^o & H_{bbmN}^o \end{bmatrix} = \left\{ I + \begin{bmatrix} H_{ffmm}^s & 0 \\ 0 & H_{bbmm}^s \end{bmatrix} \begin{bmatrix} d_{ff}^c & d_{fb}^c \\ d_{bf}^c & d_{bb}^c \end{bmatrix} \right\}^{-1} \begin{bmatrix} H_{ffmN}^s & 0 \\ 0 & H_{bbmN}^s \end{bmatrix}, \tag{19}$$

where

$$H_{;;mN}^k = T_m H_{;;}^k, \qquad k = o, s,$$

$$H_{;;mm}^s = T_m H_{;;}^s T_m^{\mathrm{T}} = \operatorname{diag}\{T_1 H_{1;;} T_1^{\mathrm{T}} \quad T_2 H_{2;;} T_2^{\mathrm{T}} \quad \cdots \quad T_n H_{n;;} T_n^{\mathrm{T}}\}.$$

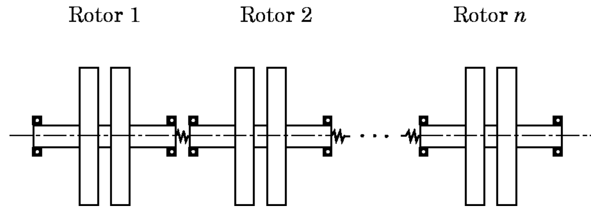


Figure 2. Numerical model 1: a system with $n$ rotors connected by an identical coupling.

Thus, substituting equation (19) into equation (18) may lead to the final equation for unbalance response function as follows:

$$H^o_{ff} = H^s_{ff} - \{H^s_{ffmN}\}^{\mathrm{T}} P H^s_{ffmN}, \tag{20}$$

$$H^o_{bf} = -\{H^s_{bbmN}\}^{\mathrm{T}} Q H^s_{ffmN}, \tag{21}$$

where

$$\begin{bmatrix} P & R \\ Q & S \end{bmatrix} = \begin{bmatrix} d^c_{ff} & d^c_{fb} \\ d^c_{bf} & d^c_{bb} \end{bmatrix} \left\{ I + \begin{bmatrix} H^s_{ffmm} & 0 \\ 0 & H^s_{bbmm} \end{bmatrix} \begin{bmatrix} d^c_{ff} & d^c_{fb} \\ d^c_{bf} & d^c_{bb} \end{bmatrix} \right\}^{-1}. \tag{22}$$

Here $P$, $Q$, $R$ and $S$ are all $m \times m$ partitioned matrices. The backward components $H^o_{fb}$ and $H^o_{bb}$ are omitted hereinafter because they do not affect the unbalance responses. Equations (20) and (21) can be simplified as

$$H^o_{ff}(\alpha, \beta) = H_{\alpha,ff}\delta_{\alpha\beta} - \{T_\alpha H_{\alpha,ff}\}^{\mathrm{T}} P(\alpha, \beta) T_\beta H_{\beta,ff}, \tag{23}$$

$$H^o_{bf}(\alpha, \beta) = -\{T_\alpha H_{\alpha,bb}\}^{\mathrm{T}} Q(\alpha, \beta) T_\beta H_{\beta,ff}, \tag{24}$$

The notation $(\alpha, \beta)$ indicates a partitioned matrix of size, $m_\alpha \times m_\beta$, to represent the response of the $\alpha$th rotor subjected to the input of the $\beta$th rotor. Equations (23) and (24)

TABLE 1

*Specifications of a substructural rotor in numerical model* 1

| | | | |
|---|---|---|---|
| Shaft | Length | | 1·2 m |
| | Diameter | | 8·0 cm |
| | Young's modulus | | $2{\cdot}0 \times 10^{11}$ N/m |
| | Density | | 8000 kg/m$^3$ |
| | Number of finite elements | | 12 (equal length) |
| Disk (two identical) | Mass | | 20 kg |
| | Polar moment of inertia | | 0·163 kg m$^2$ |
| | Diametral moment of inertia | | 0·085 kg m$^2$ |
| | Location (distance from left) | | Nodes 6 and 8 (0·5, 0·7 m) |
| Bearings (two identical) | Location (distance from left) | | Nodes 1 and 13 (0, 1·2 m) |
| | Stiffness | $k_{yy}$ | 20 MN/m |
| | | $k_{zz}$ | 25 MN/m |
| | | $k_{yz}$ | −15 MN/m |
| | | $k_{zy}$ | 10 MN/m |
| | Damping | $c_{yy}$ | 60 000 Ns/m |
| | | $c_{zz}$ | 80 000 Ns/m |
| | | $c_{yz}$ | −40 000 Ns/m |
| | | $c_{zy}$ | −40 000 Ns/m |

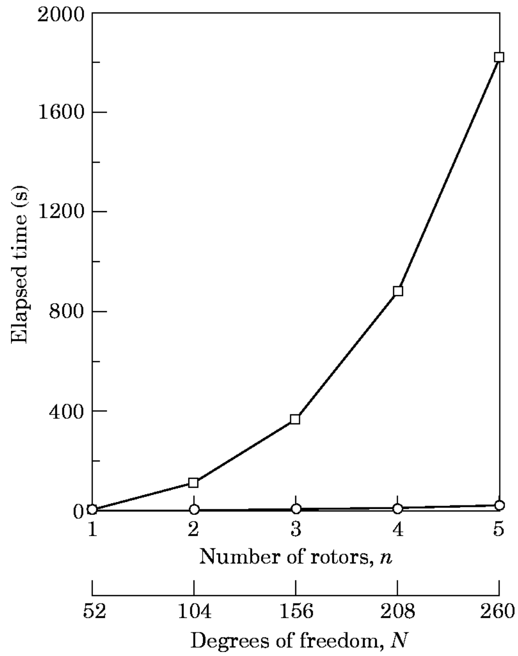


Figure 3. A comparison of the proposed and direct computation methods on time elapsed for one point unbalance response computation. —□—, Direct method; —○—, proposed method.

are surely simple to solve, because the matrix required to be inverted becomes $2m \times 2m$, and is presumably small.

Generally, the complex unbalance vector can be represented by

$$W^{\mathrm{T}} = \{W_1^{\mathrm{T}} \quad W_2^{\mathrm{T}} \quad \cdots \quad W_n^{\mathrm{T}}\}, \tag{25}$$

where $W_i$, $i = 1, 2, \ldots, n$, is the unbalance vector at the $i$th rotor. Equation (A9) yields the unbalance responses as

$$p_f = H_{ff}^{o} W \Omega^2, \qquad p_b = H_{bf}^{o} W \Omega^2. \tag{26}$$

## 4. NUMERICAL EXAMPLES AND DISCUSSION

### 4.1. NUMERICAL EXAMPLE 1

The present example deals with an artificial system with $n$-span rotors to investigate the computational efficiency of the proposed method in comparison with the direct computation method. Both the proposed method and the direct computation method are implemented on a PC with Matlab [16]. The model, in which every substructural rotor is assumed to be identical and serially connected to the neighboring rotor with a coupling, is shown in Figure 2. Two bearings support each rotor. The detailed specifications of the

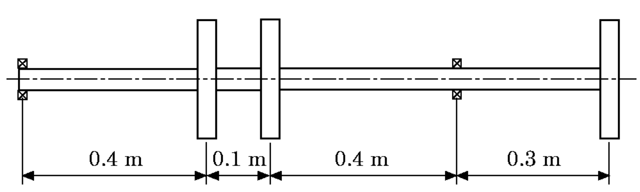


Figure 4. Numerical model 2: a laboratory test model (dimensions in m).

TABLE 2

*Specifications of the rotor in numerical model 2*

| | | |
|---|---|---|
| Shaft | Length | 1·2 m |
| | Diameter | 8·0 cm |
| | Young's modulus | $2{\cdot}0 \times 10^{11}$ N/m |
| | Density | 8000 kg/m$^3$ |
| | Number of finite elements | 12 (equal length) |
| Disk (three identical) | Mass | 20 kg |
| | Polar moment of inertia | 0·163 kg m$^2$ |
| | Diametral moment of inertia | 0·085 kg m$^2$ |
| | Location (distance from left) | Nodes 5, 6 and 13 (0·4, 0·5, 1·2 m) |
| Bearing 1 | Location (distance from left) | Node 1 (0 m) |
| | Load | 29·42 kgf |
| | $L/D$ | 0·5 |
| | $C/R$ | 2/1000 |
| | Viscosity | 9·37 mPas |
| Bearing 2 | Location (distance from left) | Node 10 (0·9 m) |
| | Load | 78·84 kgf |
| | $L/D$ | 0·5 |
| | $C/R$ | 2/1000 |
| | Viscosity | 9·37 mPas |

substructural rotor is shown in Table 1. Every coupling is identical and is modelled as a translation stiffness. The number of nodal points for a single rotor is taken to be 13, so that the direct method requires inverse of a $26n \times 26n$ complex dynamic stiffness matrix for unbalance response analysis, while the proposed method deals with a $3n \times 3n$ matrix equation. In the direct method, the Gauss elimination method was adopted for inversion of a matrix. Of course, both methods yield the same unbalance response.

In Figure 3 is shown a plot of the number of rotors ($n$) versus time elapsed in the computation of one point unbalance response for two methods, the proposed and the direct methods. From Figure 3 it is evident that the proposed method is far more efficient than the direct method and can save more time as the number of degrees of freedom becomes large. Although the proposed method necessitates solutions of eigenvalue problems, no significant computational burden is caused because all the eigenvalue problems are self-adjoint and required to be solved only once. The present example confirms that the proposed method can significantly reduce the computation time without resulting in any errors.

TABLE 3

*Specification of fluid film bearings used in numerical model 2*

| Bearing type | Parameters |
|---|---|
| Two axial groove | Oil groove angle = 10° |
| Four tilting pad | Tilting pad angle = 80°<br>Preload factor = 0<br>LBP type |
| Five tilting pad | Tilting pad angle = 60°<br>Preload factor = 0<br>LBP type |

### 4.2. NUMERICAL EXAMPLE 2

In the present example a typical re-analysis problem for a rotor–bearing model is considered, as shown in Figure 4. The detail specifications of the rotor are given in Table 2.

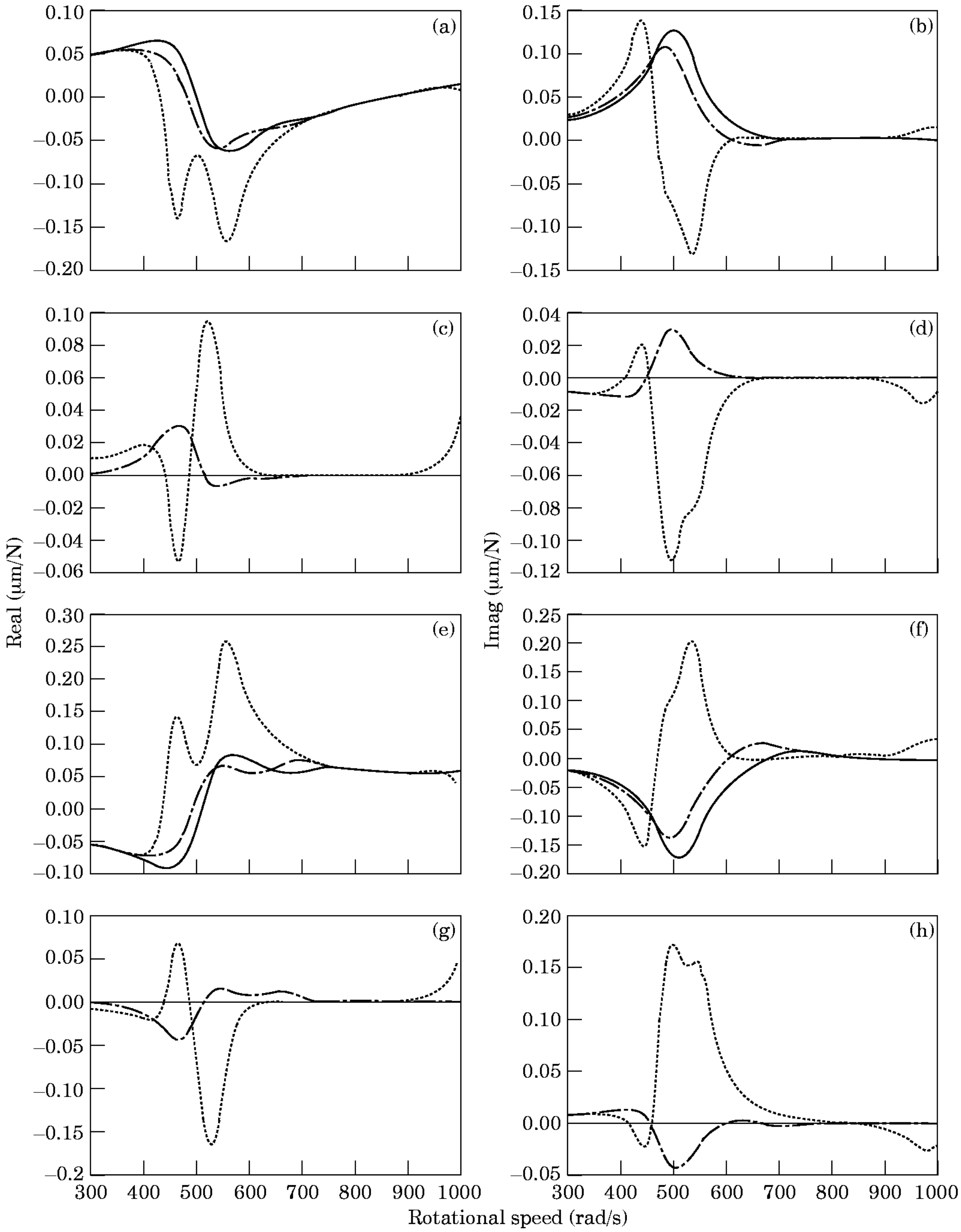


Figure 5. Unbalance response functions at nodes 5 and 13 (disks 1 and 3). (a) Forward real ($h_{ff5,5}$); (b) forward imaginary ($h_{ff5,5}$); (c) backward real ($h_{bf5,5}$); (d) backward imaginary ($h_{bf5,5}$); (e) forward real ($h_{ff13,5}$); (f) forward imaginary ($h_{ff13,5}$); (g) backward real ($h_{bf13,5}$); (h) backward imaginary ($h_{bf13,5}$).· · · · , Two-axial groove; ——, four tilting pad; ———, five tilting pad.

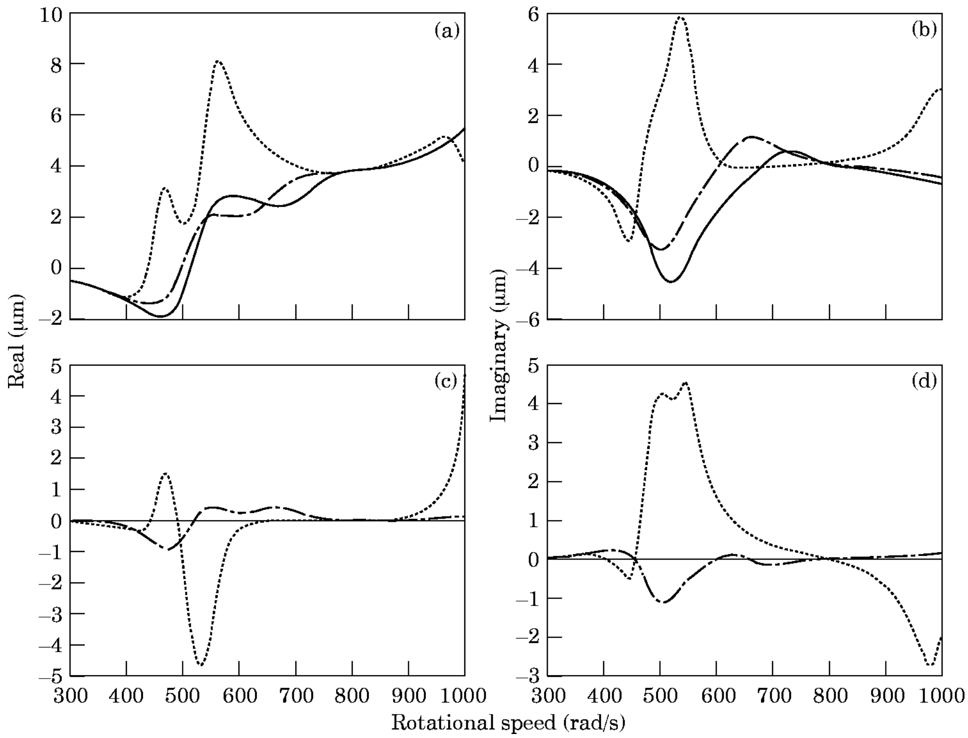


Figure 6. Unbalance responses at disk 3 (node 13) when an unbalance of 10 g-cm is attached to disk 1 (node 5). (a) Forward real; (b) forward imaginary; (c) backward real; (d) backward imaginary. · · · · , Two-axial groove; ——, four tilting pad; — · — · —, five tilting pad.

The number of nodal points in this example is 13, so that two $52 \times 52$ complex dynamic stiffness matrices need to be solved for unbalance response analysis in the direct method. However, the number of bearings is just two, and the proposed method requires inversion of two $4 \times 4$ matrix equations. Three types of bearings are considered in this example:

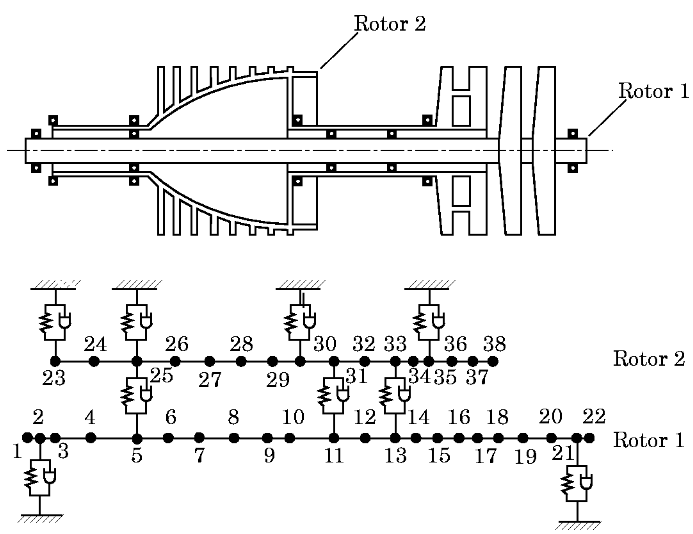


Figure 7. Numerical model 3: a two-spool aircraft engine [15].

four-pad and five-pad tilting pad bearings, and a two-axial-groove bearing. The bearing characteristics are given in Table 3. The stiffness and damping coefficients for the journal bearings are obtained by linear interpolation of the data in the reference [17]. Since the present example treats the identical rotor except bearings, the proposed method does not demand any more computation of modal responses of the rotor, after just solving the first case.

Typical unbalance response functions at disks 1 and 3 are illustrated in Figure 5. The case of a two-axial groove bearing causes larger responses than the other cases. No backward responses are observed in the case of the four-pad tilting pad bearing. In Figure 6 is shown an unbalance response plot when an unbalance of 10 g-cm is assigned to disk 1. This plot can easily be obtained by multiplying the unbalance vector by the unbalance response functions. It can be found from this example that the proposed method will be useful in the design of a rotor–bearing system that often requires repetitive computation of responses with changing joint locations or properties.

### 4.3. NUMERICAL EXAMPLE 3

In this example an aircraft engine rotor system, shown in Figure 7, is considered. The system has two substructural rotors, one of which (rotor 2) is supposed to be well balanced and to rotate at a constant speed of 15 000 rpm (1570 rad/s), while the other (rotor 1) contains some unbalance and the rotational speed varies from 100 to 2000 rad/s. The unbalance response functions of substructural rotor 2 are computed by the direct method, apart from the modal expressions, equations (10) and (13), because rotor 2 rotates at a constant speed. In Figure 8 are shown the unbalance responses when two unbalances of

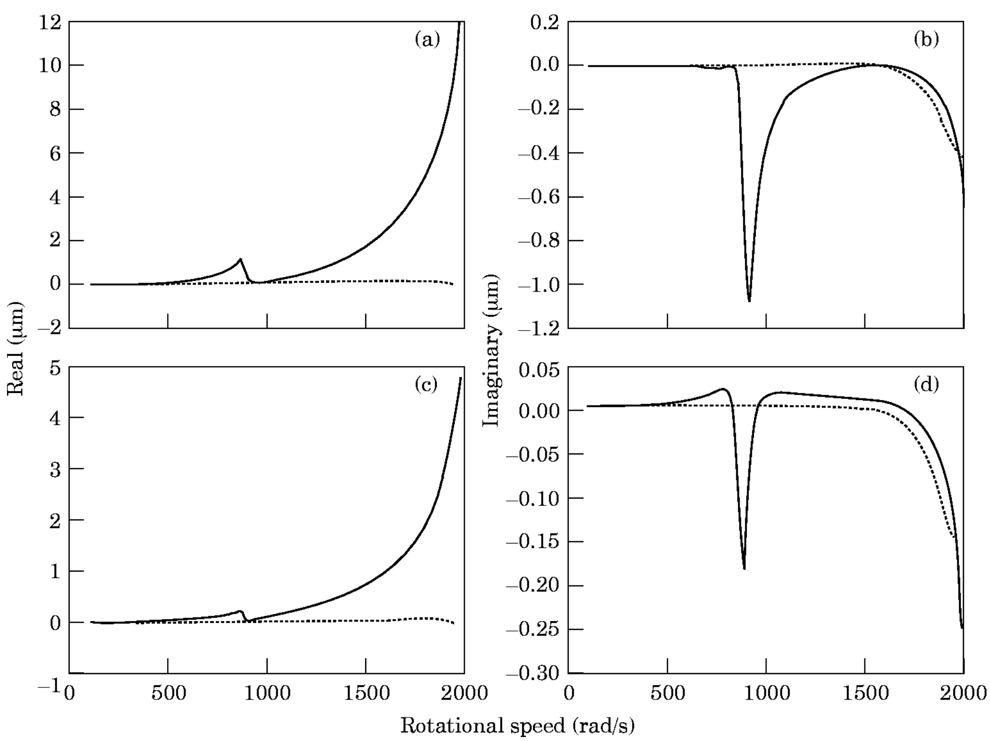


Figure 8. Unbalance responses at nodes 5 and 25 of numerical model 3. (a) Real of forward and backward at node 5; (b) imaginary of forward and backward at node 5; (c) real of forward and backward at node 25; (d) imaginary of forward and backward at node 25. ——, Forward; · · · ·, backward.

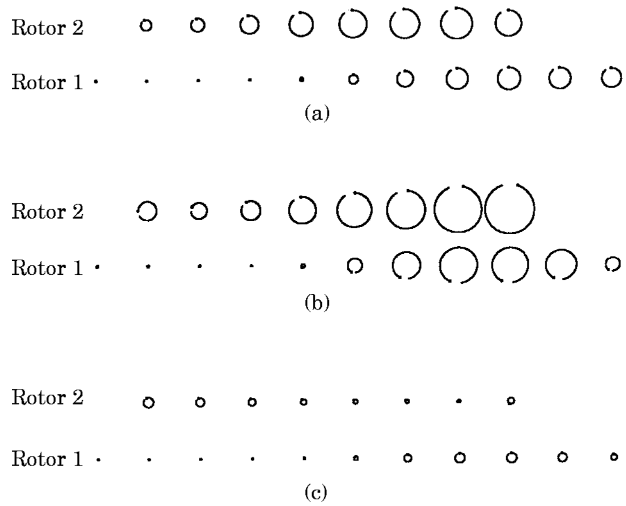


Figure 9. Orbital plots along the shaft axes. (a) 1300 rad/s; (b) 1000 rad/s; (c) 700 rad/s.

0·0036 kg cm each are assumed to be present at nodes 9 and 20 of a rotor, phased 180° apart from each other. The orbital plots taken along the shaft axis, at three different rotational speeds near a critical speed are, are illustrated in Figure 9. The center of the orbit is the nodal point in the model.

## 5. CONCLUDING REMARKS

In the present paper an improved substructure synthesis method is proposed for the unbalance response analysis of large, multi-span rotor–bearing systems. The proposed method consists of two steps. First, the unbalance responses for substructural rotors are obtained with an elementary undamped modal analysis of substructural rotors. Second, an improved substructure synthesis is carried out to compute the unbalance response by coupling the unbalance responses of the substructures. In the coupling procedure, an exact matrix condensation is introduced which makes the system matrix small. The use of complex co-ordinates in the formulation can decompose the skew-symmetric properties into forward and backward properties, which is of use in dealing with unbalance response analysis. A test example is illustrated and compared with the direct method to verify the proposed method. The applicability of the proposed method is shown through two practical examples.

The proposed method is very effective in consideration of the facts that most rotor–bearing systems include only a few bearings and/or couplings and that a complicated rotor system is mostly composed of multi-span rotors. The proposed method is of great use for preliminary design, which often requires repetitive computation of responses with changing bearing locations or properties. It is worthwhile mentioning that the proposed method can easily be extended to general harmonic response analysis, but is confined only to steady state responses.

## REFERENCES

1. R. L. Ruhl 1970 *Ph.D. Dissertation*, *Cornell University*. Dynamics of distributed parameter rotor systems: transfer matrix and finite elements methods.

2. R. GASCH 1972 *Journal of Sound and Vibration* **89**, 53–73. Vibraton of large turbo rotors in fluid film bearings on an elastic foundation.
3. H. D. NELSON and J. M. MCVAUGH 1976 *Transactions of the American Society of Mechanical Engineers*, *Journal of Engineering for Industry* **98**, 593–600. The dynamics of rotor bearing systems using finite elements.
4. H. D. NELSON 1980 *Transactions of the American Society of Mechanical Engineers*, *Journal of Mechanical Design* **102**, 793–803. A finite rotating shaft element using Timoshenko beam theory.
5. K. E. ROUCH and J. S. KAO 1979 *Journal of Sound and Vibration* **66**, 119–140. A tapered beam finite element for rotor dynamic analysis.
6. E. HASHISH and T. S. SANKER 1984 *Transactions of the American Society of Mechanical Engineers*, *Journal of Vibration*, *Acoustics*, *Stress and Reliability in Design* **106**, 80–89. Finite element and modal analyses of rotor bearing systems under stochastic loading conditions.
7. F. F. EHRICH 1992 *Handbook of Rotordynamics*. New York: McGraw-Hill.
8. C. W. LEE and S. W. HONG 1990 *International Journal of Analytical and Experimental Modal Analysis* **5**, 51–65. Asynchronous harmonic response analysis of rotor bearing systems.
9. N. KLOMPAS 1983 *Transactions of the American Society of Mechanical Engineers*, *Journal of Engineering for Power* **105**, 184–191. Unbalance response analysis of a complete turbomachine.
10. S. W. HONG and C. W. LEE 1992 *Proceedings of the Fifth International Conference on Vibrations in Rotating Machinery*, *Bath*, *U.K.* Identification of bearing dynamic coefficients by using measured and computed unbalance response in flexible rotor bearing systems.
11. R. J. GUYAN 1965 *American Institute of Aeronautics and Astronautics Journal* **3**, 380. Reduction of stiffness and mass matrices.
12. L. MEIROVITCH 1980 *Computational Methods in Structural Dynamics*. Alphen aan den Rijn: Sijthoff & Noordhoff.
13. U. J. FAN and S. T. NOAH 1988 *American Institute of Aeronautics and Astronautics Journal of Propulsion* **5**, 602–609. Vibration analysis of rotor systems using reduced subsystem models.
14. D. A. GLASGOW and H. D. NELSON 1980 *Transactions of the American Society of Mechanical Engineers*, *Journal of Mechanical Design* **102**, 353–359. Stability analysis of rotor bearing systems using component mode synthesis.
15. W. WANG and J. KIRKHOPE 1994 *Journal of Sound and Vibration* **173**, 537–555. Component mode synthesis for multi-shaft rotors with flexible inter-shaft bearings.
16. MATLAB 1992 *User's Guide*. The Math-Works, Inc.
17. T. SOMEYA (editor) 1988 *Journal Bearing Databook*. Berlin: Springer-Verlag.

## APPENDIX: DERIVATION OF UNBALANCE RESPONSE IN COMPLEX CO-ORDINATE SYSTEMS FOR ROTOR–BEARING SYSTEMS

The finite elements equation of motion for a typical rotor–bearing system can be written as

$$M^r\ddot{q}(t) + \{C^c(\Omega) + \Omega G^r\}\dot{q}(t) + \{K^r + K^c(\Omega)\}q(t) = f(t), \tag{A1}$$

where the superscripts $r$ and $c$ denote the rotor/shaft and connecting/supporting systems, respectively. The global co-ordinate vector $q(t)$ and the corresponding force vector $f(t)$ can be written as follows:

$$q^{\mathrm{T}} = \{y^{\mathrm{T}} \quad z^{\mathrm{T}}\}, \qquad f^{\mathrm{T}} = \{f_y^{\mathrm{T}} \quad f_z^{\mathrm{T}}\}. \tag{A2}$$

Here $y$ and $z$ represent the $y$-directional and $z$-directional nodal co-ordinate vectors, respectively, and $f_y$ and $f_z$ are the force vectors corresponding to $y$ and $z$. The rotational speed ($\Omega$) dependent stiffness and damping matrices of the connecting/supporting system, $K^c(\Omega)$ and $C^c(\Omega)$, respectively, are generally sparse and non-symmetric. The mass (or inertia) matrix $M^r$ and the stiffness matrix $K^r$ are symmetric, while the gyroscopic matrix is skew-symmetric. The system matrices are of the order $2N \times 2N$, $N$ being the dimension of the $y$ or $z$ co-ordinate vector ($N = 2 \times$ number of nodal points). The system matrices are represented, in a partitioned form, by

$$M^r = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \qquad G^r = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix},$$

$$C^c = \begin{bmatrix} C_{yy} & C_{yz} \\ C_{zy} & C_{zz} \end{bmatrix}, \qquad K^c = \begin{bmatrix} K_{yy} & K_{yz} \\ K_{zy} & K_{zz} \end{bmatrix}, \qquad K^r = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}, \tag{A3}$$

where the partitioned system matrices $M$, $G$ and $K$ are symmetric.

Introducing the complex displacement and force vectors defined by

$$p = y + \mathrm{j}z, \qquad F = f_y + \mathrm{j}f_z, \tag{A4}$$

equation (A1) can be rewritten as

$$M\ddot{p} - \mathrm{j}\Omega G\dot{p} + C_f\dot{p} + C_b\dot{\bar{p}} + Kp + K_f p + K_b\bar{p} = F, \tag{A5}$$

where the subscripts $f$ and $b$ denote forward and backward, respectively, and the overbar represents the complex conjugate. Here the connecting/supporting system matrices are represented by

$$2C_f = C_{yy} + C_{zz} - \mathrm{j}(C_{yz} - C_{zy}), \qquad 2C_b = C_{yy} - C_{zz} + \mathrm{j}(C_{yz} + C_{zy}),$$
$$2K_f = K_{yy} + K_{zz} - \mathrm{j}(K_{yz} - K_{zy}), \qquad 2K_b = K_{yy} - K_{zz} + \mathrm{j}(K_{yz} + K_{zy}). \tag{A6}$$

Provided that the external force is due only to unbalance force, the external force vector can be represented by

$$F = W\Omega^2\,\mathrm{e}^{\mathrm{j}\Omega t}, \tag{A7}$$

where $W$ is a complex unbalance distribution vector.

In general, the unbalance response is composed of two synchronous vibrations, forward and backward: i.e.,

$$p = p_f\,\mathrm{e}^{\mathrm{j}\Omega t} + \bar{p}_b\,\mathrm{e}^{-\mathrm{j}\Omega t}, \tag{A8}$$

where $p_f$ and $p_b$ are the forward and backward whirl response vectors, respectively.

Then the unbalance response vectors can be obtained from

$$\begin{bmatrix} p_f \\ p_b \end{bmatrix} = \begin{bmatrix} D_{ff} & D_{fb} \\ D_{bf} & D_{bb} \end{bmatrix}^{-1} \begin{bmatrix} W\Omega^2 \\ 0 \end{bmatrix} = \begin{bmatrix} H_{ff} & H_{fb} \\ H_{bf} & H_{bb} \end{bmatrix} \begin{bmatrix} W\Omega^2 \\ 0 \end{bmatrix} = \begin{bmatrix} H_{ff} \\ H_{bf} \end{bmatrix} W\Omega^2, \tag{A9}$$

where

$$D_{ff} = -\Omega^2 M + \Omega^2 G + K + K_f + \mathrm{j}\Omega C_f,$$
$$D_{bb} = -\Omega^2 M - \Omega^2 G + K + \bar{K}_f + \mathrm{j}\Omega\bar{C}_f,$$
$$D_{fb} = K_b + \mathrm{j}\Omega C_b, \qquad D_{bf} = \bar{K}_b + \mathrm{j}\Omega\bar{C}_b. \tag{A10}$$

$H_{ff}$, $H_{bf}$, $H_{fb}$ and $H_{bb}$ in equation (A9) are the unbalance response functions, which are similar to the frequency response functions.