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RAYLEIGH'S OPTIMIZATION CONCEPT AND THE USE OF SINUSOIDAL CO-ORDINATE FUNCTIONS

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1. Introduction

Lord Rayleigh suggested, over a century ago, the inclusion of an undetermined exponential parameter in the co-ordinate functions when employing his now famous method [1]. Since his method yields upper bounds for the eigenvalues, one is able to optimize them by minimizing the characteristic values with respect to the undetermined parameter.* In the case of a field problem, one minimizes the functional with respect to the exponential parameter as shown by Bert when solving a heat conduction problem [3].

Professor C. W. Bert (University of Oklahoma) and R. Schmidt (University of Detroit) have contributed significantly to the development of the method by solving numerous important applied mechanics problems. Other research groups from Argentina (Institute of Applied Mechanics; Universidad Nacional del Sur; Facultad Regional Bahía Blanca, UTN, and Universidad Nacional de Mar del Plata) have also reported some research performed on the subject matter, based on Schmidt and Bert's work.

During the past ten years, the approach suggested by Rayleigh (essentially a non-linear optimization procedure) has been applied to a variety of problems; column buckling, beam vibration, plate buckling and vibration, elastic torsion, etc.

It is important to point out that, apparently unaware of Lord Rayleigh's suggestion, well known authors such as Stodola [4], Pauling and Bright-Wilson [5] and Timoshenko and Goodier [6] also made use of Rayleigh's optimization concept.

Exponential co-ordinate functions containing an undetermined exponential parameter have been employed in references [5–7].

A thorough discussion on application of other deflection functions with undetermined parameters, namely functions with real exponential and trigonometric terms, is due to Schmidt in the context of buckling problems [8].

The present note reports some numerical experiments performed on the determination of the fundamental frequency vibration of a rectangular plate with three simply supported edges while the fourth is free; see Figure 1. The optimization parameter is contained in the argument of the sinusoidal terms of a truncated Fourier series. An interesting and rather novel feature of the approach is the fact that the "base" function allows for very good engineering accuracy when a single-term approximation is used and constitutes the exact solution of the problem when the optimization parameter is taken equal to unity and the plate is simply supported at its four edges.

2. The proposed approximation

Consider the structural system shown in Figure 1 when it executes transverse vibrations at its fundamental mode.

*Lord Rayleigh suggested the procedure when using a single co-ordinate function and determining one eigenvalue. The procedure was extended rather recently when a summation of co-ordinate functions was employed, and then the higher order eigenvalues were optimized [2].

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The expression

$$W \simeq W_a = A_1 \sin \pi x/\gamma a \sin \pi y/b, \tag{1}$$

where W_a = approximate plate amplitude and $\gamma > 1$, is a valid approximation for the plate under study.

Clearly, the natural boundary conditions at $x = a$ are not satisfied but this is admissible when making use of the Rayleigh–Ritz method. If $\gamma = 1$ one has the case of a simply supported rectangular plate and equation (1) constitutes the exact fundamental mode. The parameter γ, contained in the argument of the assumed mode shape expressed in terms of a sinusoidal function, now constitutes ‘‘Rayleigh's optimization parameter''.

Substituting equation (1) in the expression of the maximum strain energy

$$U_{\max} = \frac{D}{2}\iint_{A_0}\left\{\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \tag{2a}$$

and in the maximum kinetic energy

$$T_{\max} = \frac{\rho h \omega^2}{2}\iint_{A_0} W^2\,\mathrm{d}x\,\mathrm{d}y, \tag{2b}$$

where A_0 is the area of the plate planform, one obtains from the minimization condition

$$\frac{\partial J}{\partial A_1}[W] = \frac{\partial}{\partial A_1}(U_{\max} - T_{\max}) = 0 \tag{3}$$

and, after straightforward algebraic manipulations,

$$\Omega_1 = \sqrt{\rho h/D}\,\omega_1 a^2 = \pi^2\, N(\gamma)/M(\gamma), \tag{4}$$

where

$$N(\gamma) = \{[(1/\gamma)^2 + (a/b)^2]^2(1 - (\gamma/2\pi)\sin(2\pi/\gamma)) + (2/\pi)(1-\mu)(1/\gamma)(a/b)^2 \sin(2\pi/\gamma)\}^{1/2}, \tag{5a}$$

$$M(\gamma) = (1 - (\gamma/2\pi)\sin(2\pi/\gamma))^{1/2}. \tag{5b}$$

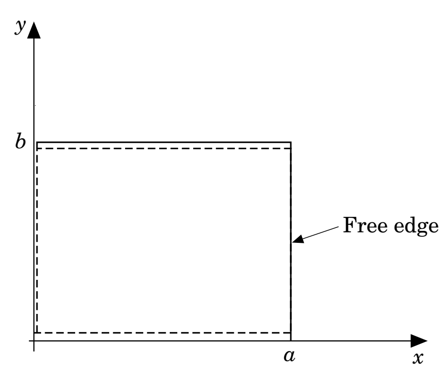

Figure 1. The vibrating structural system under study.

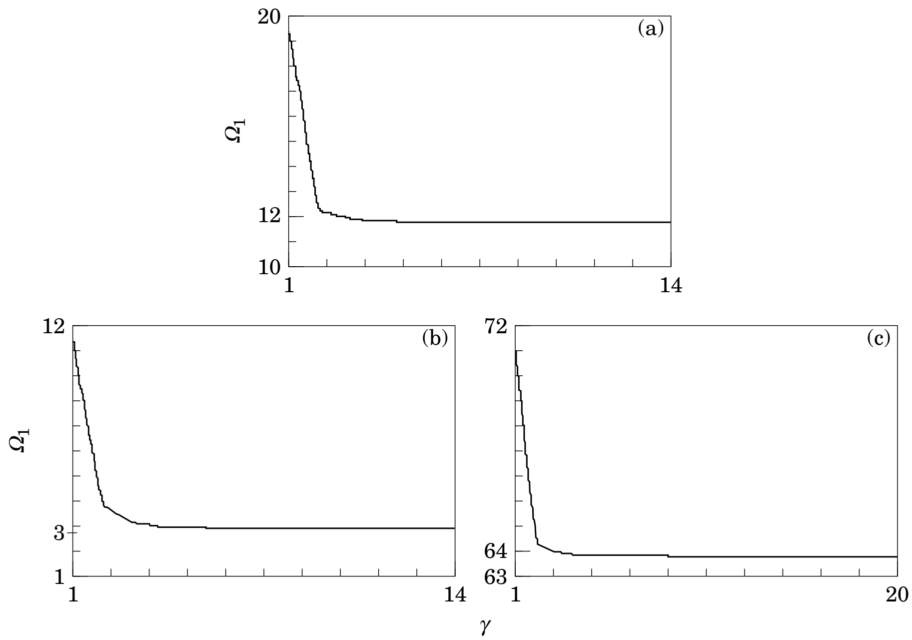

Figure 2. The variation of the fundamental frequency with respect to γ (equation (4)): (a) $a/b = 1{\cdot}0$: (b) $a/b = 0{\cdot}4$; (c) $a/b = 2{\cdot}5$.

Since

$$\Omega_1 = \Omega_1(\gamma),$$

by requiring

$$\mathrm{d}\Omega_1/\mathrm{d}\gamma = 0 \tag{6}$$

one obtains an optimized value of the fundamental frequency coefficient Ω_1.

For $\gamma = 1$, equations (4) and (5) yield the exact fundamental frequency coefficient of a simply supported rectangular plate.

If one now uses a summation of sinusoidal terms, one expresses W_α as

$$W_\alpha = \sin(\pi y/b) \sum_{i=1}^{N} A_i \sin(\pi x/\gamma_i a), \tag{7}$$

where, for the "base function" $\sin(\pi x/\gamma_1 a) \sin \pi y/b$, $\gamma_1 > 1$. When approximating the fundamental mode shape, the remaining values of the γ_i's will be smaller that unity, since each one of the additional terms constitutes an improvement over the assumed fundamental shape.

3. NUMERICAL RESULTS

The Poisson ratio has been taken to be equal to 0·30 in all calculations performed in the present study.

Figure 2 deals with the cases of three rectangular plates: $a/b = 1$, 0·4 and 2·5 when a single co-ordinate function is used. In all cases, Ω_1 is plotted as a function of γ; see equation (4).

TABLE 1

A comparison of the fundamental frequency coefficients, $\Omega_1 = \sqrt{(\rho h/D)}\omega_1 a^2$, and the convergence of the method

$a/b = 2{\cdot}5$	$a/b = 3/2$	$a/b = 1$	$a/b = 2/3$	$a/b = 0{\cdot}4$
63·2875†	24·0097†	11·685†	6·0937†	3·008 †
63·7505 ($\gamma = 50$)	24·2158 ($\gamma = 40$)	11·784 ($\gamma = 60$)	6·1371 ($\gamma = 70$)	3·0209 ($\gamma = 70$)
63·6018 $\gamma_1 = 51$ $\gamma_2 = 0{\cdot}62$	24·1114 $\gamma_1 = 52$ $\gamma_2 = 0{\cdot}63$	11·7197 $\gamma_1 = 50$ $\gamma_2 = 0{\cdot}649$	6·1049 $\gamma_1 = 60$ $\gamma_2 = 0{\cdot}674$	3·0106 $\gamma_1 = 63$ $\gamma_2 = 0{\cdot}69$
63·4748 $\gamma_1 = 70$ $\gamma_2 = 0{\cdot}48$ $\gamma_3 = 0{\cdot}45$	24·0531 $\gamma_1 = 75$ $\gamma_2 = 0{\cdot}50$ $\gamma_3 = 0{\cdot}46$	11·6956 $\gamma_1 = 75$ $\gamma_2 = 0{\cdot}51$ $\gamma_3 = 0{\cdot}46$	6·0968 $\gamma_1 = 80$ $\gamma_2 = 0{\cdot}58$ $\gamma_3 = 0{\cdot}48$	3·00862 $\gamma_1 = 80$ $\gamma_2 = 0{\cdot}58$ $\gamma_3 = 0{\cdot}49$

† From reference [9].

One immediately notices the fact that Ω_1 tends to asymptotic values as γ acquires large values. These asymptotic values are excellent approximations to the value of the fundamental frequency coefficient, as can be inferred form Table 1.

In Table 1 are depicted values of Ω_1 obtained using one-, two- and three-term approximations, respectively. The first row contains the very accurate results obtained by Leissa [9]. The maximum difference is of the order of 0·3% for $a/b = 2{\cdot}5$, while for $a/b = 0{\cdot}4$ the approximate eigenvalue practically coincides with the exact result, when three approximating terms are used.

Consideration of non-uniform thickness, orthotropy, etc., does not present any formal difficulties.

On the other hand, the procedure can be extended to other types of boundary conditions. For instance, if the plate is clamped at $x = 0$ and simply supported at $y = 0$, b, a suitable one-term approximation is

$$W_\alpha = A_1 \sin^2(\pi x/\gamma a)\sin(\pi y/b), \tag{8}$$

where, again, $\gamma > 1$.

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